Prof. Friedrich Eisenbrand
Martin Niemeier

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# Introduction to Discrete Optimization

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Solutions 12

**Exercise 1**

A caterer requires $r_j \in \mathbb{N}$ fresh napkins on each of $n$ successive days, $j = 1, \ldots, n$. He can meet his requirements either by purchasing new napkins or by using napkins previously laundered. Moreover, the laundry has two kinds of service: quick service requires $p$ days and slow service requires $q$ days. Suppose a new napkin costs $a$ cents, quick laundering costs $b$ cents and slow laundering costs $c$ cents. How should the caterer, starting with no napkins, meet his requirements with minimum costs?

Explain how to formulate this as a minimum cost flow problem.

**Solution**

We will model this as a minimum cost circulation problem, i.e. the external flow will be zero for each node.

For each $j = 1, \ldots, n$, we introduce two nodes $j_1$ and $j_2$ and an arc $(j_1, j_2)$ of capacity $r_j$ and costs $-M$, where $M$ is a big positive number which will be determined later.

By sending flow from $j_1$ to $j_2$ we model the usage of napkins on day $j$. Incoming flow at $j_1$ will simulate fresh napkins, while outgoing flow at $j_2$ simulates dirty napkins.

We now model the option to buy napkins: We add a "supernode" $s$ and arcs $(s, j_1)$ of cost $a$ and infinite capacity for each $j = 1, \ldots, n$.

To model the quick laundering service, for each $j = 1, \ldots, n-p$ we add an arc $(j_2, (j+p)_1)$ of infinite capacity and costs $b$. Similarly, to model the slow laundering service, for each $j = 1, \ldots, n-q$ we add an arc $(j_2, (j+q)_1)$ of infinite capacity and costs $c$.

It might be reasonable to pass on clean or dirty napkins from one day to another without using it. To model this, we add arcs $(j_1, (j+1)_1)$ and $(j_2, (j+1)_2)$ for each $j = 1, \ldots, n-1$ of costs 0 and infinite capacity.

Finally we add arcs $(n_1, s)$ and $(n_2, s)$ of infinite capacity and zero costs. This models the "leftovers" to be thrown away in the end.

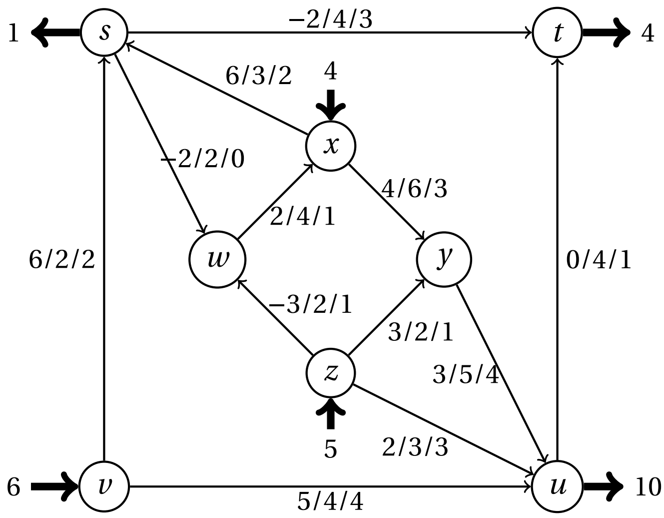
The value $M$ has to be big enough such that every optimal solution to the min cost circulation problem uses each arc $(j_1, j_2)$ with full capacity of $r_j$. Observe that $M := a + 1$ is sufficient, because whenever we have a flow $f$ such that $f(j_1, j_2) < r_j$, then there is a negative cycle $(s, j_1, j_2, (j+1)_2 \ldots, n_2, s)$ of costs $-1$ in the corresponding residual network asserting that $f$ is not optimal.

Now it is straightforward to see that every possible strategy of the caterer with total cost of $C$ corresponds to a feasible flow of value $C - n \cdot M$, and every feasible flow of value $C$ corresponds to a valid strategy for the caterer with total cost of $C + n \cdot M$. Thus our formulation can be used to find the optimal strategy for the caterer.

**Exercise 2**

Consider the following flow network with external flow.

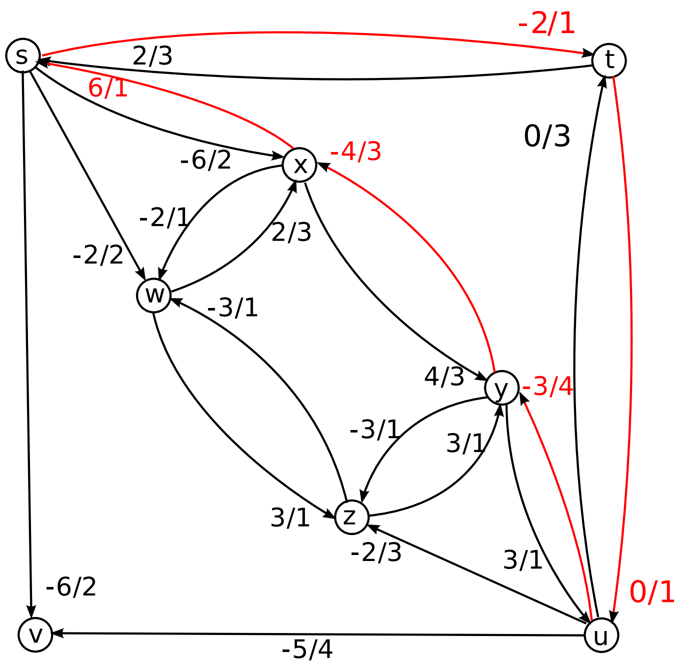
Every arc label is of the form $c(a)/u(a)/x(a)$, where $c$ denote the costs, $u$ the capacities and $x$ a feasible flow.



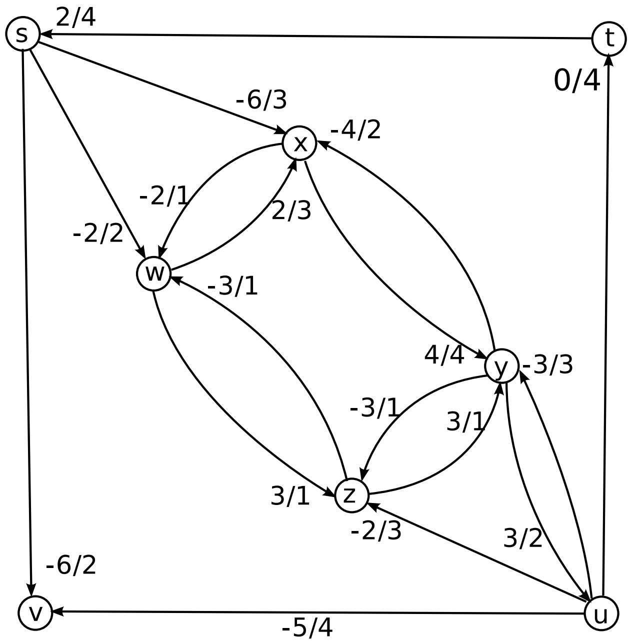

Draw the corresponding residual network and perform one augmentation of the cycle cancelling algorithm, i.e. choose a negative cycle, augment flow along the cycle and give the resulting residual network.

**Solution**

The residual network corresponding to the flow network and flow looks as follows:




A cycle of length −3 is marked red. The minimum capacity of the arcs of the cycle is 1. Thus we augment a flow of 1 along this negative cycle and get a new residual network for the modified flow:




**Exercise 3**
Let $D = (V, A)$ be a directed graph with capacities $u\colon A \to \mathbb{Q}_{\geq 0}$, costs $c\colon A \to \mathbb{Q}$ and external flow $b\colon V \to \mathbb{Q}$, and assume that we have an optimal solution $f\colon A \to \mathbb{Q}_{\geq 0}$ for the MCNFP.

Let $a' \in A$, and define a new capacity function

$$u' : A \to \mathbb{Q}_{\geq 0}, a \mapsto \begin{cases} u(a) + 1, & \text{if } a = a' \\ u(a), & \text{else.} \end{cases}$$

Show how to find an optimal solution for the MCNFPs defined by $u'$ efficiently using the solution $f$.

*Hint: Use a shortest path algorithm*

**Solution**
Let $f$ be an optimal solution for the MCNFP defined by $D, u, c$ and $b$ and let $D_u(f)$ denote the residual network corresponding to $D$, $u$ and $f$. We distinguish two cases.

In case 1 we have $f(a') < u(a')$. Then the residual network $D_u(f)$ has the same arcs as the residual network $D_{u'}(f)$. Since $f$ is optimal, $D_u(f)$ does not have a negative cycle, and thus neither has $D_{u'}(f)$. Hence $f$ is optimal for $D, u', c$ and $b$ as well and we are done.

In case 2 we have $f(a') = u(a')$. Observe that if $D_{u'}(f)$ contains a negative cycle, then it must use the arc $a'$ since it is the only arc that is not present in $D_u(f)$. Let $a' = (t, s)$. We compute a shortest $s-t$-path $P$ in $D_{u'}(f)$ of value $w$. Observe that $P \cup \{a'\}$ is a cycle in $D_{u'}(f)$. Again we distinguish two cases: If $w + c(a') \geq 0$, we claim that $D_{u'}(f)$ does not contain a negative cycle, since a negative cycle $C$ would then yield the $s-t$-path $C - a'$ of length strictly less than $w$, a contradiction. Thus $f$ is optimal and we are done.

If $w + c(a') < 0$, we augment one unit of flow along the cycle $P \cup \{a'\}$. Call the resulting flow $f'$. We claim that $f'$ is optimal. We need to show that $D_{u'}(f')$ does not contain a negative cycle. Assume the contrary, i.e. let $C$ be a negative cycle in $D_{u'}(f')$.

$C$ does not use $a'$, since $a'$ is not present in $D_{u'}(f')$. Since $D_u(f)$ does not contain a negative cycle, the cycle $C$ uses at least one reverse arc of the cycle $P \cup \{a'\}$ we augmented.

Let $C\Delta(P \cup \{a'\})$ be the the set of arcs of $C \cup (P \cup \{a'\})$ with pairs of reverse arcs removed.

Observe that $C\Delta(P \cup \{a'\})$ is a union of cycles. Every arc of $C\Delta(P \cup \{a'\})$ is also present in $D_u(f)$, except for the arc $a'$. Thus all cycles in $C\Delta(P \cup \{a'\})$ except for the cycle containing $a'$ has nonnegative length. Let $C'$ be this cycle. We have

$$c(P \cup \{a'\}) > c(C) + c(P \cup \{a'\}) \geq c(C\Delta(P \cup \{a'\})) \geq c(C').$$

The first inequality is due to the fact that $C$ is a negative cycle, and the last inequality is due to the fact that all cycles in $C\Delta(P \cup \{a'\})$ despite $C'$ are nonnegative.

This shows that $C'$ contains an $s-t$-path in $D_u(f)$ that is shorter than $P$, a contradiction.