Midterm exam

Last name: First name:

Exercise:	1	2	3	4	5	6	7	8	Σ
max points:	8	5	7	6	5	11	10	8	60
achieved points:									

Check whether the exam is complete: it should have 10 pages (Exercises 1–8). Write your name on the title page. Solutions have to be written below the exercises. Solutions must be comprehensible. In case of lack of space, additional paper can be asked from the exam supervision.

Use neither pencil nor red colored pen!

Duration: 120 min

Exercise 1: (Multiple Choice, points $\{-1,0,1\}$ each)
No justifications needed. Mark 'yes' or 'no'. **Wrong answers cause negative points!**

a) The linear program $\min\{c^T x \mid Ax = b,\ x \geq \mathbf{0}\}$ (A full row rank) can have an infinite number of basic solutions ($A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$).	○ yes	○ no
b) A set $C \subseteq \mathbb{R}^n$ is convex if and only if $\lambda x + (1-\lambda)y \in C$ for any $x, y \in C$ and $\lambda \in \mathbb{R}$.	○ yes	○ no
c) A set $C \subseteq \mathbb{R}^n$ is a cone if it is convex and for each $x \in C$ and $\lambda \in \mathbb{R}$ with $\lambda \geq 0$ we have $\lambda x \in C$.	○ yes	○ no
d) One has $$\min\{c^T x \mid Ax = b,\ x \geq \mathbf{0}\} = \max\{b^T y \mid A^T y \leq c\}$$ given that both linear programs are feasible ($A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$).	○ yes	○ no
e) For every matrix $A \in \mathbb{R}^{m\times n}$ and vector $b \in \mathbb{R}^m$, the system $Ax = b, x \geq \mathbf{0}$ has a solution if and only if there is a $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geq \mathbf{0}$ and $\lambda^T b \geq 0$.	○ yes	○ no
f) Every basic solution of the linear program $\min\{c^T x \mid Ax = b,\ x \geq \mathbf{0}\}$ is feasible. ($A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$).	○ yes	○ no
g) If the linear program $\min\{c^T x \mid Ax = b,\ x \geq \mathbf{0}\}$ is feasible and bounded, then there exists an optimal basic feasible solution. ($A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, A full row rank).	○ yes	○ no
h) The lexicographic pivoting rule ensures that the simplex algorithm terminates.	○ yes	○ no

Exercise 2: (5 points)
A manager of an oil refinery has 6 million barrels of crude oil A and 3 million barrels of crude oil B allocated for production during the coming month.
These resources can be used to make either gasoline, which sells for 40$ per barrel, or home heating oil, which sells for 35$ per barrel. There are three production processes with the following characteristics:

	Process 1	Process 2	Process 3
Input crude A	3	3	5
Input crude B	5	2	3
Output gasoline	4	2	3
Output heating oil	3	2	4
Cost	51$	32$	41$

All quantities are in barrels. For example, with the first process, 3 barrels of crude A and 5 barrels of crude B are used to produce 4 barrels of gasoline and 3 barrels of heating oil at a cost of 51 $.
Formulate a linear programming problem that would help the manager maximize net revenue over the next month.

Solution:

Use reverse side if you need more space

Exercise 3: (7 points)
Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Let $A = \{x_1, \dots, x_5\}$. Find two disjoint subsets $A_1,\ A_2 \subseteq A$ such that

$$\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2) \neq \emptyset.$$

Prove that your choice is correct by giving a vector that is contained in $\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2)$.
Hint: Recall the proof of Radon's lemma

Solution:

Use reverse side if you need more space

Exercise 4: (6 points)
Given a set $X \subseteq \mathbb{R}^n$, show that the set

$$\operatorname{cone}(X) := \left\{ \sum_{i=1}^{t} \lambda_i x_i \ : \ t \in \mathbb{N}, x_i \in X, \lambda_i \geq 0 \ \forall i = 1, \ldots, t \right\}$$

is a cone.

Solution:

Use reverse side if you need more space

Exercise 5: (5 points)
Transform the following linear program to *equation standard form*

$$
\begin{array}{lrcrcrcl}
\max & 3x_1 & + & 3x_2 & - & 3x_3 & & \\
& x_1 & - & x_2 & - & x_3 & \leq & 0 \\
& 3x_1 & + & 5x_2 & & & \geq & 7 \\
& x_1 & & & & & \leq & 0 \\
& x_2 & & & & & \geq & 0
\end{array}
$$

Solution:

Use reverse side if you need more space

Exercise 6: (11 points)

Consider the following linear program:

$$
\begin{array}{lrcrcrcrcrcrcrcl}
\min & -x_1 & - & 6x_2 & - & 4x_3 & - & 3x_4 & & & & & & & & \\
 & 3x_1 & + & 6x_2 & + & 6x_3 & + & 3x_4 & + & x_5 & & & & & = & 6 \\
 & 2x_1 & + & 6x_2 & + & 4x_3 & & & & & + & x_6 & & & = & 5 \\
 & -x_1 & + & 4x_2 & + & 4x_3 & + & 2x_4 & & & & & + & x_7 & = & 5 \\
 & & & & & & & & & & & & & x_1,x_2,x_3,x_4,x_5,x_6,x_7 & \geq & 0
\end{array}
$$

Find an optimal solution to the linear program using the simplex method.

For each iteration of the simplex method, give the tableau and the corresponding basis. In each iteration, if there is more than one variable with negative reduced costs, choose the smallest index to enter the basis.

Hint: If done correctly, the indices entering the basis will be 1, 2, 3, 4 *(in this order)*

Solution:

Use the next page if you need more space

Solution:

Use reverse side if you need more space

Exercise 7: (10 points)
Consider the following LP:

$$\begin{array}{llllllll} \max & 2y_1 & + & 2y_2 & + & 4y_3 & & \\ & y_1 & - & 2y_2 & + & 2y_3 & \leq & -1 \\ & 3y_1 & - & 2y_2 & + & 4y_3 & \leq & -3 \\ & y_1 & & & & & \leq & 0 \\ & y_2 & & & & & \leq & 0 \\ & y_3 & & & & & \leq & 0 \end{array}$$

a) Formulate a dual of this linear program

b) Solve the dual using the simplex method.

For each iteration of the simplex method, give the tableau and the corresponding basis. In each iteration, if there is more than one variable with negative reduced costs, choose the smallest index to enter the basis. If there are multiple choices for indices to leave the basis, choose the highest index.

c) Give the optimal objective value of the primal.

Solution:

Use the next page if you need more space

Solution:

Use reverse side if you need more space

Exercise 8: (8 points)

During execution of the simplex method, you end up with the following tableaux. In each case mark the correct answers and give a short explanation (1 sentence).

a)

x_1	x_2	x_3	x_4	x_5	
-3	-1	0	0	0	0
3	0	1	0	0	1
-1	-2	0	1	0	2
2	-1	0	0	1	3

1) The current solution is feasible.	○ yes ○ no	Reason:
2) The current solution is optimal.	○ yes ○ no	Reason:
3) The linear program is unbounded.	○ yes ○ no	Reason:

b)

x_1	x_2	x_3	x_4	x_5	
0	-1	0	0	0	0
3	1	1	0	0	1
-1	0	0	1	0	2
2	0	0	0	1	3

1) The current solution is degenerate.	○ yes ○ no	Reason:
2) The current solution is optimal.	○ yes ○ no	Reason:
3) The linear program is unbounded.	○ yes ○ no	Reason:

c)

x_1	x_2	x_3	x_4	x_5	
-3	-1	0	0	0	0
3	9	1	0	0	0
-1	-2	0	1	0	2
2	-1	0	0	1	3

1) The current solution is degenerate.	○ yes ○ no	Reason:
2) The current solution is optimal.	○ yes ○ no	Reason:
3) The linear program is unbounded.	○ yes ○ no	Reason: