Chapter 3
The development of the simplex method

In this chapter, we describe an algorithm which solves linear programming problems $\min\{c^T x\colon Ax = b, x \geqslant 0\}$ in equation standard form, where $A \in \mathbb{R}^{m\times n}$ has full row-rank, i.e., the rows of A are linearly independent. Our presentation follows closely the one in [1].

Equation standard form

We have seen in Chapter 1 that each linear program can be described in inequality standard form, i.e., as $\max\{c^T x\colon Ax \leqslant b\}$. The simplex algorithm is conveniently described for linear programs in *equation standard form* $\min\{c^T x\colon Ax = b, x \geqslant 0\}$.

The next example shows how to transform a linear program in inequality standard form in equation standard form. Consider the linear program

$$\begin{array}{ll} \text{maximize} & 3x_1 - 2x_2 \\ \text{subject to} & 2x_1 - x_2 \leqslant 4 \\ & x_1 + 3x_2 \leqslant 5 \\ & -x_1 \leqslant 0. \end{array} \tag{16}$$

Our goal is to transform this linear program into one of the form $\min\{c^T x\colon Ax = b, x \geqslant 0\}$. The objective function can be re-written as $\min -3x_1 + 2x_2$. Furthermore, we observe that the last inequality of (16) can be re-written as $x_1 \geqslant 0$. The variable x_2 is not bounded from below. The trick is to split x_2 into $x_2 = x_2^+ - x_2^-$, where $x_2^+, x_2^- \geqslant 0$. The linear program then becomes

$$\begin{array}{lll} \text{minimize} & -3x_1 + 2x_2^+ - 2x_2^- & \\ \text{subject to} & 2x_1 - x_2^+ + x_2^- & \leqslant 4 \\ & x_1 + 3x_2^+ - 3x_2^- & \leqslant 5 \\ & x_1, x_2^+, x_2^- & \geqslant 0. \end{array} \tag{17}$$

Next we introduce two new *slack variables* $z_1, z_2 \geqslant 0$, which model $z_1 = 4-(2x_1 - x_2^+ + x_2^-)$ and $z_2 = 5-(x_1+3x_2^+ - 3x_2^-)$ and obtain a linear program in equation standard form which is equivalent to (16)

$$\begin{array}{lll} \text{minimize} & -3x_1+2x_2^+-2x_2^- & \\ \text{subject to} & 2x_1 - x_2^+ + x_2^- + z_1 & = 4 \\ & x_1+3x_2^+ - 3x_2^- + z_2 & = 5 \\ & x_1, x_2^+, x_2^-, z_1, z_2 & \geqslant 0. \end{array}$$

In matrix notation, we can describe the transformation procedure from above as follows. The input is a linear program in inequality standard form $\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b\}$ with $A \in \mathbb{R}^{m\times n}$. From this, one creates the equivalent linear program

$$\begin{array}{ll} \min & -c^T x^+ + c^T x^- \\ & Ax^+ - Ax^- + I_m z = b \\ & x^+, x^-, z \geqslant 0 \\ & x^+, x^- \in \mathbb{R}^n, z \in \mathbb{R}^m, \end{array}$$

where $I_m \in \mathbb{R}^{m\times m}$ is the $m\times m$ identity matrix, i.e. for $1 \leqslant i,j \leqslant m$ one has

$$I_m(i,j) = \begin{cases} 1, & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

The full row-rank assumption

We recall some definitions and facts from linear algebra. For $A \in \mathbb{R}^{m\times n}$ the *row-rank* of A is the maximum number of linearly independent rows. The *column-rank* of A is the maximum number of linearly independent columns. The row-rank and column rank of A are equal. This number is the *rank* of A and is denoted by $\operatorname{rank}(A)$.

Definition 3.1. A matrix $A \in \mathbb{R}^{m\times n}$ has *full row-rank*, if $\operatorname{rank}(A) = m$. Similarly, A has *full column-rank*, if $\operatorname{rank}(A) = n$.

We now show that we furthermore can assume that the matrix A in the linear program

$$\min\{c^T x \colon x \in \mathbb{R}^n,\ Ax = b,\ x \geqslant 0\} \tag{18}$$

has full row-rank, i.e., the rows of A are linearly independent. To see this, suppose that $A \in \mathbb{R}^{m\times n}$ does not have full row-rank. Then there is a row a_j of A which is in the span of the other rows, i.e., one has

$$a_j = \sum_{\substack{i=1\\ i\neq j}}^m \lambda_i a_i \text{ with suitable numbers } \lambda_i \in \mathbb{R}, i \in \{1,\dots,m\}\setminus\{j\}.$$

If $\sum_{\substack{i=1\\i\neq j}}^m \lambda_i b(i) = b(j)$, then one has for all $x \in \mathbb{R}^n$ with $a_i^T x = b(i), i = 1,\dots,m, i \neq j$ also $a_j^T x = b(j)$ which means that the j-th equation in $Ax = b$ can be removed. If $\sum_{\substack{i=1\\i\neq j}}^m \lambda_i b(i) \neq b(j)$, then there does not exist an $x \in \mathbb{R}^n$ with $Ax = b$ and the LP (18) is infeasible.

Example 3.1. Consider the linear program

$$\begin{aligned} \min x_1 + 2\,x_2 + x_3 + 4x_4 & \\ x_1 + x_2 + 2x_3 + 3x_4 &= 5 \\ x_1 + 2x_2 + x_3 + 4x_4 &= 7 \\ 3x_1 + 4x_2 + 5x_3 + 10x_4 &= 17 \\ x_1, x_2, x_3, x_4 \geqslant 0. & \end{aligned} \tag{19}$$

The third equation is can be obtained by multiplying the first equation with 2 and adding it to the second equation. This linear program is not of full row-rank and it is equivalent to the linear program

$$\begin{aligned} \min x_1 + 2\,x_2 + x_3 + 4x_4 & \\ x_1 + x_2 + 2x_3 + 3x_4 &= 5 \\ x_1 + 2x_2 + x_3 + 4x_4 &= 7 \\ x_1, x_2, x_3, x_4 \geqslant 0. & \end{aligned} \tag{20}$$

which is of full row-rank. The linear program

$$\begin{aligned} \min x_1 + 2\,x_2 + x_3 + 4x_4 & \\ x_1 + x_2 + 2x_3 + 3x_4 &= 5 \\ x_1 + 2x_2 + x_3 + 4x_4 &= 7 \\ 3x_1 + 4x_2 + 5x_3 + 10x_4 &= 15 \\ x_1, x_2, x_3, x_4 \geqslant 0. & \end{aligned}$$

is not of full row-rank but is infeasible, since, as before, the third row of the constraint matrix is 2 times the first row plus the second row, however $2 \cdot 5 + 7 = 17 \neq 15$.

Assumption 1 *We assume in the following that our linear program which we want to solve is in equation standard form* (18)*, or in* standard form *for short, with $A \in \mathbb{R}^{m\times n}$ having full row-rank.*

Bases and basic solutions

We use the following notation. For an index set $I = \{i_1,\dots,i_k\} \subseteq \{1,\dots,n\}$, with $i_1 < i_2 < \dots < i_k$, A_I denotes the $m \times k$ matrix whose j-th column is i_j-th column a^{i_j} of A for $j = 1,\dots,k$. Similarly $c_I \in \mathbb{R}^k$ denotes the vector whose j-th component is $c(i_j)$, $j = 1,\dots,k$.

Example 3.2. Let $A = \left(\begin{smallmatrix} 2 & 3 & 1 \\ 4 & 5 & 6 \end{smallmatrix}\right)$ and $I = \{2,3\}$, then $A_I = \left(\begin{smallmatrix} 3 & 1 \\ 5 & 6 \end{smallmatrix}\right)$.

The proof of the next lemma is very similar to the proof of Carathéodory's theorem, Theorem 2.6. It will help us to determine an optimal solution to (18).

Lemma 3.1. *Let x^* be a feasible solution of the linear program* (18) *and let $J = \{i : x^*(i) > 0\}$ be the index set corresponding to those components of x^* which are strictly positive. There exists a procedure that either asserts that the linear program is unbounded or computes a solution $\widetilde{x}$ of* (18) *such that*

i) the index set $\widetilde{J} = \{i : \widetilde{x}(i) > 0\}$ is contained in J, $\widetilde{J} \subseteq J$,
ii) $A_{\widetilde{J}}$ has full column-rank,
iii) $c^T x^* \geqslant c^T \widetilde{x}$.

Proof. Let x^* be a feasible solution and suppose that the columns of A_J are linearly dependent. The idea is now to compute a new solution x' with $J' = \{j : x'(j) > 0\} \subset J$ and $c^T x' \leqslant c^T x^*$ or to assert that the linear program is unbounded. After at most n repetitions of this procedure, one has found a solution $\widetilde{x}$ which satisfies the condition of the theorem or one has asserted that the linear program is unbounded.

Let $\overline{J}$ be the index set $\overline{J} = \{1,\dots,n\} \setminus J$. Since A_J does not have full column rank there exists a $d \in \mathbb{R}^n$ with $d \neq 0$, $A d = 0$ and $d(j) = 0$ for each $j \in \overline{J}$. Consider the points

$$x^* \pm \lambda\, d \text{ for } \lambda \in \mathbb{R}. \tag{21}$$

Since $A(x^* \pm \lambda d) = b$ for each $\lambda \in \mathbb{R}$, the only way to make a point (21) infeasible is to violate the lower bounds $x^* \pm \lambda d \geqslant 0$. Since $x^*_J > 0$ and $d_{\overline{J}} = 0$ there exists a sufficiently small $\varepsilon^* > 0$ such that for each $\lambda \in \mathbb{R}$ with $0 \leqslant \lambda \leqslant \varepsilon^*$ the point $x^* \pm \lambda d$ is feasible.

Suppose that $c^T d \neq 0$. By multiplying d with -1 we can assume that $c^T d < 0$. If $d \geqslant 0$, then $x^* + \lambda\, d \geqslant 0$ for each $\lambda \geqslant 0$ and since $c^T(x^* + \lambda\, d) = c^T x^* + \lambda c^T d$, we can make the objective value of $x^* + \lambda\, d$ as small as we wish. Thus we assert that the linear program is unbounded. Otherwise, let $K = \{i : d(i) < 0\}$ be the index set corresponding to the negative components of d. How large can we choose $\lambda > 0$ such $x^* + \lambda d$ is still feasible? For $\lambda > 0$, the point $x^* + \lambda\, d$ is feasible if and only if

$$\lambda \leqslant \min\{-x^*(k)/d(k) : k \in K\}. \tag{22}$$

Let $\lambda_{\max} = \min\{-x^*(i)/d(i) : i \in K\}$ and let $k^* \in K$ be an index, where the minimum is attained. The point $x' = x^* + \lambda_{\max} d$ is feasible, $c^T x' < c^T x^*$ and since $x'(k^*) = 0$ one has with $J' = \{i : x'(i) > 0\}$ also $J' \subset J$.

Suppose now that $c^T d = 0$. By multiplying d with -1 if necessary, we can assume that d has strictly negative components. We proceed as above with the index set K and $\lambda_{\max}$ to construct a feasible $x' = x^* + \lambda_{\max} d$ with $c^T x' \leqslant c^T x^*$ and $J' = \{i : x'(i) > 0\} \subset J$. □

Example 3.3. Consider the linear program (20) and the feasible solution $x^* = (1,1,0,1)$. The set J from the proof above is $J = \{1,2,4\}$. The vector $d^T = (-2,-1,0,1)$

satisfies $Ad = 0$ and $d_{\overline{J}} = 0$. A largest $\lambda > 0$ such that $(1,1,0,1) + \lambda(-2,-1,0,1) \geqslant 0$ is $\lambda_{\max} = 1/2$ and the new solution is $(1,1,0,1)+(1/2)\cdot(-2,-1,0,1) = (0,0.5,0,1.5)$. This is a basic feasible solution and the procedure described in the proof of Lemma 3.1 ends.

Definition 3.2 (Basis, basic solution, basic feasible solution). A set $B \subseteq \{1,\dots,n\}$ with $|B| = m$ and A_B non-singular is called *basis*. A vector $x^* \in \mathbb{R}^n$ is a *basic solution*, if there exists a basis B with $x^*_B = A_B^{-1} b$ and $x^*_{\overline{B}} = 0$, where $\overline{B} = \{1,\dots,n\} \setminus B$. We say that x^* is associated to the basis B. If additionally $x^* \geqslant 0$ holds, then x^* is a *basic feasible solution.*

Lemma 3.2. *Let $x^* \in \mathbb{R}^n$ be a feasible solution and let $J = \{i \colon x^*(i) \neq 0\}$. If the columns of A_J are linearly independent, then x^* is a basic feasible solution.*

Proof. Augment J to index set $B \supseteq J$ such that A_B is non-singular. One has $A_B x^*_B = b$ and $x^*(j) = 0$ for all $j \notin B$, thus x^* is a basic solution. □

In exercise 2 you are to prove that there could be several bases which are associated to a basic feasible solution x^*.

Exercise 11 shows that there exist optimization problems $\min\{c^T x \colon x \in K\}$ where $K \subseteq \mathbb{R}^n$ is closed and bounded which are bounded but do not have optimum solutions. A corollary of Lemma 3.1 is that this cannot happen for linear programs.

Corollary 3.1. *A bounded and feasible linear program has an optimal basic feasible solution.*

Proof. Lemma 3.1 together with Lemma 3.2 shows that for each feasible x^*, there exists a basic feasible solution with an objective function value which is at most the objective function value of x^*. Since the number of basic feasible solutions is finite, a bounded linear program has an optimal basic feasible solution.

A naive algorithm for linear programming

Suppose that we have a bounded linear program $\min\{c^T x \mid x \in \mathbb{R}^n, Ax = b, x \geqslant 0\}$ in standard form. We can already describe a finite algorithm which computes an optimal solution, if the linear program is feasible. Lemma 3.1 tells us that, if there exists an optimal solution, then there exists also an optimal basic feasible solution.

The algorithm maintains a value which reflects the *best objective value* observed so far. In the beginning this is ∞.

We now enumerate all index sets $B \in \binom{n}{m}$ and test, whether A_B is non-singular with Gaussian elimination. If this is the case, we compute $x^*_B = A_B^{-1} b$ and check whether $x^*_B \geqslant 0$. If this is also the case, we check whether $c_B^T x^*_B$ is smaller than the previously best observed objective value and update this best objective function value if this is again the case.

This algorithm is however very inefficient. The enumeration of all index sets $B \in \binom{2m}{m}$ requires already at least 2^m steps. If m is large this is enormous, see exercise 3. The simplex algorithm is based on a smarter principle. It *improves* a basic feasible solution, by replacing it by another basic feasible solution which is in the neighborhood of the previous one. We will describe this next.

Simplex method: First steps

Consider the linear programming problem

$$\begin{array}{lccccccccc} \min & 2x_1 & + & 3x_2 & + & 4x_3 & + & 4x_4 & & \\ & x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 3 \\ & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \\ & & & & & & x_1, x_2, x_3, x_4 \geqslant 0. \end{array} \tag{23}$$

The first two columns of the constraint matrix A are linearly independent. Let us now compute $x^* \in \mathbb{R}^n$ such that $x_B^* = A_B^{-1} \cdot \binom{3}{2}$. For this, we subtract the second row of the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 3 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \end{array}$$

from the first, to obtain

$$\begin{array}{ccccccccc} & & x_2 & + & 2x_3 & + & 3x_4 & = & 1 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \end{array}$$

Then we subtract the first row from the second, to obtain

$$\begin{array}{ccccccccc} & & x_2 & + & 2x_3 & + & 3x_4 & = & 1 \\ x_1 & + & & + & -1x_3 & + & -2x_4 & = & 1 \end{array}$$

Finally we swap the first and second row, and obtain the system

$$\begin{array}{ccccccccc} x_1 & + & & + & -1x_3 & + & -2x_4 & = & 1 \\ & & x_2 & + & 2x_3 & + & 3x_4 & = & 1 \end{array}$$

Definition 3.3 (Elementary row-operations). Let $A \in \mathbb{R}^{m \times n}$ be a matrix. An *elementary row operation* on A is the addition of a multiple of one row, to another row.

The following lemma is known from linear algebra.

Lemma 3.3. *Let $A, A' \in \mathbb{R}^{m \times n}$ and $b, b' \in \mathbb{R}^m$. If the matrix $\left[A'|b'\right]$ results from $[A|b]$ by a finite series of elementary row operations, then $\{x \in \mathbb{R}^n \colon Ax = b\} = \{x \in \mathbb{R}^n \colon A'x = b'\}$.*

In other words, we have re-written our linear program from above as

$$\begin{array}{rcrcrcrcl} \min\ 2x_1 & + & 3x_2 & + & 4x_3 & + & 4x_4 & & \\ x_1 & + & & + & -1x_3 & + & -2x_4 & = & 1 \\ & & x_2 & + & 2x_3 & + & 3x_4 & = & 1 \\ & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4 \geqslant 0.} \end{array}$$

From this representation, we can easily read off the basic feasible solution $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$ associated to $B = \{1,2\}$. We now consider the system

$$\begin{array}{rcrcrcrcl} 2x_1 & + & 3x_2 & + & 4x_3 & + & 4x_4 & = & 0 \\ \hline x_1 & + & & + & -1x_3 & + & -2x_4 & = & 1. \\ & & x_2 & + & 2x_3 & + & 3x_4 & = & 1 \end{array}$$

If we subtract 2-times the second row and 3-times the third row from the first row, we obtain the system

$$\begin{array}{rcrcrcrcr} & + & & + & 0x_3 & + & -1x_4 & = & -5 \\ \hline 1x_1 & + & 0x_2 & + & -1x_3 & + & -2x_4 & = & 1 \\ 0x_1 & + & 1x_2 & + & 2x_3 & + & 3x_4 & = & 1 \end{array}$$

We have eliminated the basic variables in the first row. The entry behind the equality sign of the first row is the negative of the objective value of the basic feasible solution $(1,1,0,0)$. We write this in a more compact form as follows

$$\begin{array}{cccc|l} 0 & 0 & 0 & -1 & -5 \\ \hline 1 & 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{array} \tag{24}$$

The general interpretation of this approach is as follows. We start with

$$\begin{array}{c|c} c^T & 0 \\ \hline A & b \end{array}$$

and, given a basis B of A compute

$$\begin{array}{c|c} c^T - c_B^T A_B^{-1} A & -c_B^T x_B^* \\ \hline A_B^{-1} A & x_B^*. \end{array} \tag{25}$$

Definition 3.4 (Tableau, reduced costs). The matrix (4) is called the *tableau* of the basis B. The tableau is *feasible* if $x_B^* \geqslant 0$. The first row of the tableau is the 0-th row. The remaining m rows constitute the *system-matrix* of the tableau.

The vector $\overline{c}_B^T = c^T - c_B^T A_B^{-1} A$ is called the *reduced cost vector* of B. For $i \in \{1,\dots,n\}$ $\overline{c}_B(i)$ is the reduced cost of the variable x_i.

Now back to our concrete example. Is the solution $(1,1,0,0)$ optimal? The fact that the reduced cost of x_4 is negative suggests to make a *basis change*. We want to include x_4 into the basis, since then the elimination of the variable x_4 in the 0-th row will make the element in the top-right corner larger and, remembering that this entry is the negative of the objective function value, we would have obtained a better solution. If x_4 should enter the basis, then which variable should leave, x_1 or x_2? If x_1 should leave the basis, then we need to apply elementary row-operations to turn the gray column below into the first unit vector and if x_2 should leave the basis, then this column should become the second unit vector.

$$\begin{array}{cccc|l} 0 & 0 & 0 & -1 & -5 \\ \hline 1 & 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{array} \tag{26}$$

If x_1 leaves the basis, then the tableau becomes infeasible. This is because the first entry of the gray column is negative. We decide therefore to make the following basis change $B = \{1,2\} \to B' = \{1,4\}$. To compute the new tableau for B' we multiply the last row by 1/3 and add 2-times the last row to the second and add the last row to the first row of the tableau, to obtain

$$\begin{array}{cccc|c} 0 & 1/3 & 2/3 & 0 & -14/3 \\ \hline 1 & 2/3 & 1/3 & 0 & 5/3 \\ 0 & 1/3 & 2/3 & 1 & 1/3 \end{array} \tag{27}$$

The next lemma shows that the above is indeed the tableau of $B = \{1,4\}$.

Lemma 3.4. *Let $B \subset \{1,\ldots,n\}$ be an index set and consider a matrix*

$$\begin{array}{c|c} d^T & \beta \\ \hline Q & q \end{array} \tag{28}$$

where

i) $[Q|q]$ can be obtained from $[A|b]$ via a series of elementary row operations
ii) $[d^T|\beta] = [c^T|0] + v^T$, where v^T is in the row-span of $[A|b]$
iii) $d_B = 0$
iv) $Q_B = I_m$,

then B is a basis and (28) *is the tableau of B.*

The dual linear program

Our guess is that the tableau (27) is optimal. To confirm this suspicion, we introduce the concept of the dual linear program.

Definition 3.5 (Dual of a linear program in standard form). Let $\min\{c^T x \colon x \in \mathbb{R}^n, Ax = b, x \geqslant 0\}$ be a linear program in standard form. The linear program $\max\{b^T y \colon y \in \mathbb{R}^m, A^T y \leqslant c\}$ is the *dual linear program* of $\min\{c^T x \colon x \in \mathbb{R}^n, Ax =$

$b,\ x \geqslant 0\}$. The linear program $\min\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax = b,\ x \geqslant 0\}$ is the *primal linear program.*

Example 3.4. The dual linear program of (23) is

$$\begin{array}{rcrcl} \max\, 3y_1 & + & 2y_2 & & \\ 4y_1 & + & y_2 & \leqslant & 4 \\ 3y_1 & + & y_2 & \leqslant & 4 \\ 2y_1 & + & y_2 & \leqslant & 3 \\ y_1 & + & y_2 & \leqslant & 2 \end{array} \tag{29}$$

In our example of the simplex method above, we have arrived at a basis $B = \{1,4\}$ whose reduced cost vector did not have a negative entry. The next theorem reveals that such a basis, if it is feasible, also yields an optimal solution.

Theorem 3.1 (Weak duality). *Let x^* and y^* be feasible solutions of the primal and dual respectively, then $c^T x^* \geqslant b^T y^*$.*

Proof. We have

$$b^T y^* = (x^{*T} A^T) y^* = x^{*T}(A^T y^*) \leqslant x^{*T} c = c^T y^* \tag{30}$$

where the inequality in (30) follows from the fact that $x^* \geqslant 0$ and $A^T y^* \leqslant c$. □

Lemma 3.5. *Let x^* be a basic solution associated to the basis B. If the reduced cost vector of B is non-negative and if $x_B^* \geqslant 0$, then x^* is an optimal basic feasible solution.*

Proof. We show that $y^* = {A_B^{-1}}^T c_B$ is a feasible solution of the dual with objective value $b^T y^* = c^T x^*$. The assertion then follows from weak duality. We have

$$\begin{aligned} A^T y^* &= A^T {A_B^{-1}}^T c_B \\ &\leqslant c, \end{aligned}$$

since $c^T - c_B^T A_B^{-1} A \geqslant 0$. This means that y^* is a feasible dual solution. Its objective value is

$$\begin{aligned} b^T y^* &= {y^*}^T A x^* \\ &= c_B^T A_B^{-1} A x^* \\ &= c_B^T x_B^* \\ &= c^T x^* \end{aligned}$$

and the assertion follows. □

This shows that we have found in (27) an optimal basic feasible solution $(5/3, 0, 0, 1/3)$.

Definition 3.6 (Optimal basis). A basis B with $\overline{c}^T = c^T - c_B^T A_B^{-1} A \geqslant 0$ and with $A_B^{-1} b \geqslant 0$ is called an *optimal basis.*

We have seen above that an optimal basis B yields an optimal basic feasible solution x^* with $x^*_B = A_B^{-1}b$ and $x^*_{\overline{B}} = 0$.

A larger example

Before we describe the simplex method in full generality, we inspect another example. We consider the linear program

$$\begin{array}{rl} \min & -5x_1 - 3x_2 - 2x_3 \\ & 2x_1 + 3x_2 + x_3 \leqslant 5 \\ & 4x_1 + x_2 + 2x_3 \leqslant 9 \\ & 3x_1 + 4x_2 + 2x_3 \leqslant 10 \\ & x_1, x_2, x_3 \geqslant 0 \end{array} \tag{31}$$

We add variables x_4, x_5 and x_6 and transform this linear program in standard form

$$\begin{array}{rl} \min & -5x_1 - 3x_2 - 2x_3 \\ & 2x_1 + 3x_2 + x_3 + x_4 = 5 \\ & 4x_1 + x_2 + 2x_3 + x_5 = 9 \\ & 3x_1 + 4x_2 + 2x_3 + x_6 = 10 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geqslant 0 \end{array} \tag{32}$$

We start with the feasible basis $B = \{4,5,6\}$ and the corresponding tableau, where the red-columns correspond to basis elements

-5	-4	-3	0	0	0	0
2	3	1	1	0	0	5
4	1	2	0	1	0	9
3	4	2	0	0	1	10

The reduced cost of x_1 is negative, we decide to let x_1 enter the basis. We now want to transform the gray column into a unit vector such that the tableau is still feasible. The red columns are the basis-columns.

-5	-4	-3	0	0	0	0
2	3	1	1	0	0	5
4	1	2	0	1	0	9
3	4	2	0	0	1	10

(33)

We now need to decide, which element should leave $B = \{4,5,6\}$. If it should be 4, then the gray column should be transformed into $(1,0,0)^T$. Let us see what happens if we apply elementary row-operations on the system matrix of the tableau such that the first column becomes $(1,0,0)^T$.

0	-4	-3	0	0	0	25/2
1	3/2	1/2	1/2	0	0	5/2
0	-5	0	-2	1	0	-1
0	2	1/2	-3/2	0	1	1/2

The result is the tableau of $\{1,5,6\}$, the objective function value has improved but the basic solution of $\{1,5,6\}$ is not feasible, since $x^*(4) = -1$.

What went wrong? Let us abstract from the specific situation and suppose that we have the following situation. We have a feasible basis $B = \{B_1,\dots,B_m\}$ and a tableau of the basis B. The reduced cost of j are negative and we want to add j to B. The picture below represents this situation, where the j-th column is the gray column. This column is also called the *pivot column*.

$$
\begin{array}{ccc|l}
 & \overline{c}(j) & & -c^T x^* \\
\hline
 & u(1) & & x^*(B_1) \\
 & \vdots & & \vdots \\
 & u(i) & & x^*(B_i) \\
 & \vdots & & \vdots \\
 & u(m) & & x^*(B_m)
\end{array}
$$

If we now decide to let B_i leave the basis, then we transform the gray column into the i-th unit vector with elementary row-operations and obtain

$$
\begin{array}{ccc|c}
 & \overline{c}(j) & & -c^T x^* \\
\hline
 & 0 & & x^*(B_1) - u(1)\cdot\frac{x^*(B_i)}{u(i)} \\
 & \vdots & & \vdots \\
 & 0 & & x^*(B_{i-1}) - u(i-1)\cdot\frac{x^*(B_i)}{u(i)} \\
 & 1 & & x^*(B_i)/u(i) \\
 & 0 & & x^*(B_{i+1}) - u(i+1)\cdot\frac{x^*(B_i)}{u(i)} \\
 & & & \vdots \\
 & 0 & & x^*(B_m) - u(m)\cdot\frac{x^*(B_i)}{u(i)}
\end{array}
$$

Then we eliminate the $\overline{c}(j)$ by adding $-\overline{c}(j)$ times the i-th row of the system matrix to the zeroth row to obtain

$$
\begin{array}{ccc|c}
 & \overline{c}(j) & & -c^T x^* - \overline{c}(j)\cdot x^*(B_i)/u(j) \\
\hline
 & 0 & & x^*(B_1) - u(1)\cdot\frac{x^*(B_i)}{u(i)} \\
 & \vdots & & \vdots \\
 & 0 & & x^*(B_{i-1}) - u(i-1)\cdot\frac{x^*(B_i)}{u(i)} \\
 & 1 & & x^*(B_i)/u(i) \\
 & 0 & & x^*(B_{i+1}) - u(i+1)\cdot\frac{x^*(B_i)}{u(i)} \\
 & \vdots & & \vdots \\
 & 0 & & x^*(B_m) - u(m)\cdot\frac{x^*(B_i)}{u(i)}
\end{array}
$$

Let us deduce a set of conditions, under which these sequence of operations are allowed and such that the obtained tableau is feasible.

a) $u(i) > 0$,
since if $u(i) \leqslant 0$, then we either cannot divide by $u(i)$ or the basis is not feasible since $x^*(B_i)/u(i) < 0$.

b) $x^*(B_i)/u(i) = \min\{x^*(B_j)/u(j) \colon B_j \in B,\, u(j) > 0\}$,
since the condition $x^*(B_j) - u(j) \cdot \frac{x^*(B_i)}{u(i)} \geqslant 0$ is satisfied for each $B_j \in B$ if and only if the condition holds.

Looking back at our example (33) we see that condition b) was violated. With $B_1 = 4, B_2 = 5, B_3 = 6$ we see that $x^*(B_1)/u(1) = 5/2$ and this is larger than $x^*(B_2)/u(2) = 9/4$, which is the minimum of the ratios. We therefore let $B_2 = 5$ leave the basis and obtain

0	-11/4	-1/2	0	5/4	0	45/4
0	5/2	0	1	-1/2	0	1/2
1	1/4	1/2	0	1/4	0	9/4
0	13/4	1/2	0	-3/4	1	13/4

To arrive then at the feasible tableau of the basis $B' = \{1, 4, 6\}$ we still need to swap the first and second row of the system matrix:

0	-11/4	-1/2	0	5/4	0	45/4
1	1/4	1/2	0	1/4	0	9/4
0	5/2	0	1	-1/2	0	1/2
0	13/4	1/2	0	-3/4	1	13/4

One iteration of the simplex method

We start with a feasible basis $B \subseteq \{1, \ldots, n\}$ and an associated tableau

$$\begin{array}{c|c} \overline{c}^T & -c_B^T x_B^* \\ \hline A_B^{-1} A & x_B^*. \end{array} \tag{34}$$

The simplex algorithm starts with a feasible tableau and keeps iterating the procedure below.

1. If $\overline{c} \geqslant 0$, then **output optimal basis** B
2. Otherwise, let j be an index with $\overline{c}(j) < 0$ and let $u = A_B^{-1} A^j$ be the j-th column of the system matrix of the tableau. If $u \leqslant 0$, then **output** $-\infty$, the linear program is unbounded.
3. Otherwise compute the ratios $x^*(B_i)/u(i)$ for all $i = 1, \ldots, m$ with $u(i) > 0$. Let i^* be an index where this minimum is attained. The index B_{i^*} leaves the basis and j enters the basis, i.e., the new basis is $B' = B \setminus \{i^*\} \cup \{j\}$.

4. Update the tableau to

$$\begin{array}{c|c} c^T - c_{B'}^T A_{B'}^{-1} A & -c_{B'}^T x_{B'}^* \\ \hline A_{B'}^{-1} A & x_{B'}^*. \end{array}$$

We need to justify that the we correctly assert in step 2 that the linear program is unbounded if $u \leqslant 0$.

Theorem 3.2. *If $\overline{c}(j) < 0$ and $u \leqslant 0$, then the linear program is unbounded.*

Proof. Consider the vector d with $d_B = -u$ and $d(j) = 1$, which is zero at all other components. Since $Ad = 0$ and since $d \geqslant 0$, we have that for each feasible x^* and for each $\lambda \geqslant 0$, also the point $x^* + \lambda d$ is feasible. What is the objective function value of d?

$$\begin{aligned} c^T d &= c(j) - c_B^T A_B^{-1} A \\ &= \overline{c}(j) \\ &< 0. \end{aligned}$$

This means that the objective function value of $x^* + \lambda d$ which is $c^T x^* + \lambda c^T d$ can be made as small as we wish by increasing λ, without becoming infeasible. The linear program is unbounded. □

Let us continue our example. We decide to bring 2 into the basis, since $\overline{c}(2) = -11/4 < 0$.

0	-11/4	-1/2	0	5/4	0	45/4
1	1/4	1/2	0	1/4	0	9/4
0	5/2	0	1	-1/2	0	1/2
0	13/4	1/2	0	-3/4	1	13/4

We compute the ratios: $9, 1/5, 1$ and decide that the second element of $\{1,4,6\}$ should leave the basis, i.e., we want to have the second unit vector in the pivot column.

0	0	-1/2	11/10	7/10	0	59/5
1	0	1/2	-1/10	3/10	0	11/5
0	1	0	2/5	-1/5	0	1/5
0	0	1/2	-13/10	-1/10	1	13/5

After one more iteration we obtain an obtimal feasible tableau

1	0	0	1	1	0	14
0	1	0	2/5	-1/5	0	1/5
2	0	1	-1/5	3/5	0	22/5
-1	0	0	-6/5	-2/5	1	2/5

What next?

The two main questions that we are next going to deal with are the following.

I) How do we find an initially feasible basis?
II) If we start from a feasible basis, does the simplex method terminate?

Exercises

1. Transform the following linear program into equation standard form

$$\begin{array}{rrrrl} \max & 3x_1 + & 4x_2 & + 2x_3 & \\ & & x_2 & + 2x_3 & \geqslant 4 \\ & 2x_1 + & x_2 & + 3x_3 & \leqslant 10 \\ & & x_2 & & \leqslant 0 \end{array}$$

2. Provide an example of a basic solution that can be associated to two different bases.
3. This exercise should give you a sense of exponential growth as one experiences it with the running time of the naive linear programming algorithm. Have a look at this program. What is its running time? Experiment with different inputs and determine the largest exponent for which the program needs less running time than a day on your computer. Make an estimation on how many rows and variables a linear program could have such that the naive algorithm for linear programming finishes within a day on your computer. Estimate how many rows and variables a linear needs to have such that the naive algorithm does not terminate on your computer before our sun will stop to shine.

```
#include <iostream>
#include <cmath>

using namespace std;

int main()
{

  double n;

  cout << "\n Enter the exponent" ;
  cin >> n ;
  long i,j;
  j=0;
  for (i=1; i<= pow(2.0,n); i++)
```

```
      j = j+i;
    cout << j;
  }
```

4. Prove Lemma 3.4.

References

1. D. Bertsimas and J. N. Tsitsiklis. *Introduction to linear optimization.* Athena Scientfic, Belmont, Mass., 1997.