Chapter 4
Termination, Cycling, and Degeneracy

We now deal first with the question, whether the simplex method terminates. The quick answer is no, if it is implemented in a careless way. Notice that we have left it open to choose the entering variable (there could be several columns whose reduced cost are less than zero) and we also did not specify how to choose the exiting variable (there could be several basis elements B_i with $u(B_i) > 0$ and $x^*(B_i)/u(B_i)$ minimal to choose from. *Pivoting rules* specify the alternative to choose from and there are pivoting rules, which prevent the simplex algorithm from *cycling*.

The following is an example of cycling, see also [1, p. 104].

-3/4	20	-1/2	6	0	0	0	3
1/4	-8	-1	9	1	0	0	0
1/2	-12	-1/2	3	0	1	0	0
0	0	1	0	0	0	1	1

0	-4	-7/2	33	3	0	0	3
1	-32	-4	36	4	0	0	0
0	4	3/2	-15	-2	1	0	0
0	0	1	0	0	0	1	1

0	0	-2	18	1	1	0	3
1	0	8	-84	-12	8	0	0
0	1	3/8	-15/4	-1/2	1/4	0	0
0	0	1	0	0	0	1	1

1/4	0	0	-3	-2	3	0	3
-3/64	1	0	3/16	1/16	-1/8	0	0
1/8	0	1	-21/2	-3/2	1	0	0
-1/8	0	0	21/2	3/2	-1	1	1

-1/2	16	0	0	-1	1	0	3
-5/2	56	1	0	2	-6	0	0
-1/4	16/3	0	1	1/3	-2/3	0	0
5/2	-56	0	0	-2	6	1	1

-7/4	44	1/2	0	0	-2	0	3
1/6	-4	-1/6	1	0	1/3	0	0
-5/4	28	1/2	0	1	-3	0	0
0	0	1	0	0	0	1	1

-3/4	20	-1/2	6	0	0	0	3
1/4	-8	-1	9	1	0	0	0
1/2	-12	-1/2	3	0	1	0	0
0	0	1	0	0	0	1	1

In fact we have not made any progress in the objective function value during all these pivots. Inspecting the pivot steps, we see that the problem lies in the fact that we have basic feasible solutions which are zero at some of the basic components. If this is never the case, then the simplex method terminates, as we show now.

Recall that a basis B with reduced costs $\overline{c} \geqslant 0$ is optimal, see Lemma 3.5. There is a weak converse to this lemma. Before we state it, we define the notion of degeneracy.

Definition 4.1 (Degenerate basic solution). A basic solution x^* associated to the basis B is *degenerate*, if there exists an index $i \in B$ with $x^*(i) = 0$. If $x^*(i) \neq 0$ for all $i \in B$, then x^* is called *non-degenerate*.

Lemma 4.1. *Let x^* be a non-degenerate basic feasible solution associated to the basis $B = \{B_1, \ldots, B_m\}$. If there exists an index $i \in \{1, \ldots, n\}$ with $\overline{c}(i) < 0$, then x^* is not optimal.*

Proof. Suppose that the reduced cost of variable j is negative, $\overline{c}(j) < 0$ and inspect the tableau of B

$$\begin{array}{c|c} c^T - c_B^T A_B^{-1} A & -c_B^T x_B^* \\ \hline A_B^{-1} A & x_B^*. \end{array}$$

Let $u(1), \ldots, u(m)$ be the components of the j-th column of the system matrix of the tableau. If $u(i) \leqslant 0$ for all i, then the linear program is unbounded and the assertion follows. Otherwise let $K = \{i : u(i) > 0\}$ and let $i^* \in K$ be an index where the minimum $\min\{x^*(B_i)/u(i) : i \in K\}$ is attained. The new objective function value after the pivot step is

$$c^T x^* + \overline{c}(j) x^*(B_{i^*})/u(i^*) \tag{35}$$

and since $\overline{c}(j) < 0$, this is smaller than $c^T x^*$. □

This Lemma also shows that the simplex method terminates if the basic feasible solutions are all non-degenerate.

Theorem 4.1. *If all basic feasible solutions are non-degenerate, then the simplex method terminates, regardless of the choice among the alternatives among entering and leaving variables.*

Proof. The proof of the previous lemma, in particular equation (35) shows that the objective function value strictly improves each iteration of the simplex method. Each tableau is uniquely defined by a basis B. We never re-visit a tableau, since the objective function is strictly increasing. The number of bases is bounded by $\binom{n}{m}$. The assertion follows. □

Perturbation

The termination argument for the simplex algorithm for linear programs, who do not have degenerate basic feasible solutions, is very simple and nice and in fact one can twist it into a pivot-rule for the simplex-algorithm which makes the simplex algorithm terminate.

We start with the linear program

$$\min\{c^T x \colon x \in \mathbb{R}^n, Ax = b, x \geqslant 0\} \tag{36}$$

and assume that $S \subseteq \{1,\dots,n\}$ is a feasible basis with $A_S = I_m$. This assumption can be made, since we can replace $Ax = b$ with $A_S^{-1} Ax = A_S^{-1} b$.

The idea is to transform (36) into into a linear program

$$\min\{c^T x \colon x \in \mathbb{R}^n, Ax = b', x \geqslant 0\} \tag{37}$$

such that the following conditions hold.

i) The LP (37) does not have degenerate basic solutions.
ii) If B is an infeasible basis of (36), then B is also an infeasible basis of (37).
iii) The LP (37) is feasible.

Lemma 4.2. *The simplex algorithm terminates on the linear program* (37)*. If it asserts that the linear program* (37) *is unbounded, then the linear program* (36) *is also unbounded. If it outputs an optimal basis of* (37) *then this basis is also optimal for* (36)*.*

Proof. It follows from Theorem 4.1 and the property i) that the simplex method terminates on the linear program (37). If it asserts that (37) is unmounded, then there exists a $d \geqslant 0$ with $Ad = 0$ and $c^T d < 0$. This means that the linear program (36) is also unbounded.

Suppose therefore that the simplex method on (37) terminates by outputting a basis $\widetilde{B}$. This basis $\widetilde{B}$ is a feasible basis of (36) by property ii). The reduced costs of this basis are the same for the linear programs (36) and (37) which implies that $\widetilde{B}$ is also an optimal basis of (37). □

Now we describe the construction of (37), more precisely the construction of b', which is

$$b' = b + \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^m \end{pmatrix} \tag{38}$$

for some small $\varepsilon > 0$. Since $\varepsilon > 0$ the basic solution x^* associated to S with

$$x^*_S = b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \text{ and } x^*_{\overline{S}} = 0$$

is a basic feasible solution of (37) and thus iii) follows. We show that we can choose an $\varepsilon > 0$ such that ii) holds. Suppose that B is an infeasible basis. This means that there exists an entry i of B such that $(A_B^{-1}\, b)(i) < 0$. Clearly, we can choose $\varepsilon_B > 0$ small enough such that

$$\left[A_B^{-1} \left(b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \right) \right] (i) < 0 \tag{39}$$

and the basis B is still infeasible for the linear program (37). The most interesting property to infer is property i). Let B be any basis and consider the vector

$$A_B^{-1} \left(b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \right) \tag{40}$$

which determines the components of x^*_B of the basic solution. If we treat ε as a variable, then the i-th component of this vector (40) is a polynomial in ε which we denote by $p_{iB}(\varepsilon)$. Each of the polynomials $p_{iB}(\varepsilon)$ is unequal to the zero polynomial, see exercise 2 which implies that there are only a finite (at most m) real-roots of p_{iB}. Consequently, there exists an $\varepsilon_{iB} > 0$ for each p_{iB} such that $p_{iB}(\varepsilon) \neq 0$ for each $\varepsilon \in]0, \varepsilon_{iB}[$. Thus if we choose $\varepsilon > 0$ such that it is smaller than all ε_B and all ε_{iB} from above, then the perturbed problem with

$$b' = b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

satisfies the properties i-iii) from above.

Symbolic computation with ε

The good news is that we do not have to calculate with a concrete $\varepsilon > 0$ at all. Let us understand which variable would leave the basis if we would calculate with a very small ε.

Recall that we consider the index set $K = \{i : u(i) > 0\}$, where u is the pivot column of the tableau

$$\begin{array}{c|c} c^T - c_B^T A_B^{-1} A & -c_B^T x_B^* \\ \hline A_B^{-1} A & x_B^*. \end{array}$$

The leaving variable is B_i, where $i \in K$ minimizes the expression $x^*(B_i)/u(i)$, $i \in K$. We have

$$x^*(B_i)/u(i) = [(A_B^{-1} b)(i) + (A_B^{-1} \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix})(i)]/u(i).$$

Definition 4.2 (Lexicographic order). Let $u, v \in \mathbb{R}^n$ be vectors. We have $u <_{lex} v$, if $u = v$ or if the first nonzero component of $u - v$ is strictly negative.

Since $\varepsilon > 0$ is arbitrarily small we see that we need to find i such that the vector $[(A_B^{-1} b)(i) | (A_B^{-1})_i]/u(i)$ is lexicographically minimal. This is the **lexicographic pivoting rule**:

1. If $\overline{c} \geqslant 0$, then **output optimal basis** B
2. Otherwise, let j be an index with $\overline{c}(j) < 0$ and let $u = A_B^{-1} A^j$ be the j-th column of the system matrix of the tableau. If $u \leqslant 0$, then **output** $-\infty$, the linear program is unbounded.
3. Otherwise compute the vectors $[(A_B^{-1} b)(i) | (A_B^{-1})_i]/u(i)$ for all $i = 1, \ldots, m$ with $u(i) > 0$. Let i^* be an index for which this vector is lexicographically smallest. The index B_{i^*} leaves the basis and j enters the basis, i.e., the new basis is $B' = B \setminus \{B_{i^*}\} \cup \{j\}$.
4. Update the tableau to

$$\begin{array}{c|c} c^T - c_{B'}^T A_{B'}^{-1} A & -c_{B'}^T x_{B'}^* \\ \hline A_{B'}^{-1} A & x_{B'}^*. \end{array}$$

Theorem 4.2. *The simplex method terminates, if the leaving variable is chosen according to the lexicographic pivot rule above.*

Bland's rule

The following is an alternative pivoting rule which avoids cycling, see, e.g. [2].

> During an iteration of the simplex method (Chapter 3), find a smallest j such that $\overline{c}(j)$ is less than 0. Let this j enter the basis.
>
> Out of all the indices i that may leave the basis, choose the smallest one.

Exercises

1. Provide an example of a degenerate basic feasible solution x^* and an associated basis B which is optimal but whose reduced cost vector $\overline{c}^T = c^T - c_B^T A_B^{-1} A$ does not satisfy $\overline{c} \geqslant 0$.
2. Show that $p_{iB}(\varepsilon) \neq 0$ holds for each $i \in \{1, \ldots, m\}$ and each basis B.
3. Provide an example of a valid perturbation, as described above, such that a basis B which is feasible for the original LP (36) is infeasible for the problem (37).

References

1. D. Bertsimas and J. N. Tsitsiklis. *Introduction to linear optimization*. Athena Scientfic, Belmont, Mass., 1997.
2. V. Chvátal. *Linear programming*. W. H. Freeman and Company, 1983.