Chapter 5
Strong Duality

With the help of perturbation that we applied in Chapter 4, we can now prove the duality theorem. Recall that we are given a linear program

$$\min\{c^T x \colon x \in \mathbb{R}^n,\ Ax = b,\ x \geqslant 0\}, \tag{41}$$

called the *primal* and its *dual*

$$\max\{b^T y \colon y \in \mathbb{R}^m,\ A^T y \leqslant c\}. \tag{42}$$

The theorem of weak duality tells us that $c^T x^* \geqslant b^T y^*$ if x^* and y^* are primal and dual feasible solutions respectively. The strong duality theorem tell us that if there exist feasible primal and dual solutions, then there exist feasible primal and dual solutions which have the same objective value.

Theorem 5.1. *If the primal linear program has an optimal solution, then so does the dual linear program and the objective values coincide.*

Proof. The simplex method terminates on the perturbed problem (37)

$$\min\{c^T x \colon x \in \mathbb{R}^n,\ Ax = b',\ x \geqslant 0\},$$

and outputs an optimal basis B which is also an optimal basis of the primal. The reduced cost are nonnegative, i.e.,

$$c^T - c_B^T A_B^{-1} A \geqslant 0.$$

The vector $y^* = {A_B^{-1}}^T c_B$ is a feasible dual solution which has the same objective function value as the basic feasible solution x^* of the primal which is defined as $x_B^* = A_B^{-1} b$ and $x_{\overline{B}}^* = 0$, see proof of Lemma 3.5. ☐

We can formulate dual linear programs also if the linear program is not in equation standard form. Consider for example a linear program

$$\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b,\ x \geqslant 0\}. \tag{43}$$

We transform this into standard form via slack variables $z \geqslant 0$

$$\min\{-c^T x + 0^T z \colon x \in \mathbb{R}^n, z \in \mathbb{R}^m, Ax + z = b, x, z \geqslant 0\}.$$

The dual of this linear program in equation standard form is

$$\max\{b^T y \colon [A \mid I_m]^T y \leqslant \tbinom{-c}{0}\}$$

This can be re-formulated as

$$\max\{b^T y \colon A^T y \leqslant -c, y \leqslant 0\}.$$

Again, this is the same as

$$\min\{b^T(-y) \colon A^T(-y) \geqslant c, -y \geqslant 0\}$$

and this finally is equivalent to

$$\min\{b^T y \colon A^T y \geqslant c, y \geqslant 0\}. \tag{44}$$

The procedure above can be described as follows. We transform a linear program into a linear program in equation standard form and construct its dual linear program. This dual is then transformed into an equivalent linear program again which is conveniently described.

Let us perform such operations on the dual linear program

$$\max\{b^T y \colon y \in \mathbb{R}^m, A^T y \leqslant c\}$$

of the primal $\min\{c^T x \colon x \in \mathbb{R}^n, Ax = b, x \geqslant 0\}$. We transform it into equation standard form

$$\begin{aligned} \min -b^T y^+ + b^T y^- + 0^T z & \\ A^T y^+ - A^T y^- + z &= c \\ y^+, y^-, z^+ &\geqslant 0. \end{aligned}$$

The dual linear program of this is

$$\begin{aligned} \max c^T x & \\ Ax &\leqslant -b \\ -Ax &\leqslant b \\ x &\leqslant 0. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \min c^T(-x) & \\ A(-x) &\geqslant b \\ A(-x) &\leqslant b \\ -x &\geqslant 0 \end{aligned}$$

which is equivalent to the primal linear program

$$\begin{aligned} \min c^T x & \\ Ax &= b \\ x &\geqslant 0. \end{aligned}$$

Loosely formulated one could say that "The dual of the dual is the primal". But this, of course, is not to be understood as a mathematical statement. In any case we can state the following corollary.

Corollary 5.1. *If the dual linear program has an optimal solution, then so does the primal linear program and the objective values coincide.*

A proof of the duality theorem via Farkas' lemma

Remember Farkas' lemma (Theorem 2.9) which states that $Ax = b, x \geqslant 0$ has a solution if and only if for all $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geqslant 0$ one also has $\lambda^T b \geqslant 0$. In fact the duality theorem follows from this. First, we derive another variant of Farkas' lemma.

Theorem 5.2 (Second variant of Farkas' lemma). *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The system $Ax \leqslant b$ has a solution if and only if for all $\lambda \geqslant 0$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.*

Proof. Necessity is clear: If x^* is a feasible solution, $\lambda \geqslant 0$ and $\lambda^T A = 0$, then $\lambda^T Ax^* \leqslant \lambda^T b$ implies $0 \leqslant \lambda^T b$.

On the other hand, $Ax \leqslant b$ has a solution if and only if

$$Ax^+ - Ax^- + z = b,\ x^+, x^-, z \geqslant 0 \tag{45}$$

has a solution. So, if $Ax \leqslant b$ does not have a solution, then also (45) does not have a solution. By Farkas' lemma, there exists a $\lambda \in \mathbb{R}^m$ with $\lambda^T [A \mid -A \mid I_m] \geqslant 0$ and $\lambda^T b < 0$. For this λ one also has $\lambda^T A = 0$ and $\lambda \geqslant 0$. □

We are now ready to prove the theorem of strong duality via the second variant of Farkas' lemma. In fact we prove Corollary 5.1, which serves our purpose, since, by the discussion preceding Corollary 5.1, Theorem 5.1 and Corollary 5.1 are equivalent.

Proof (of Corollary 5.1 via Farkas' lemma). Let δ be the objective function value of an optimal solution of the dual $\max\{b^T y \colon y \in \mathbb{R}^m,\ A^T y \leqslant c\}$. For all $\varepsilon > 0$, the system $A^T y \leqslant c, -b^T y \leqslant -\delta - \varepsilon$ does not have a solution. By the second variant of Farkas' lemma, there exists a $\lambda \geqslant 0$ with $\lambda^T \binom{-b^T}{A^T} = 0$ and $\lambda^T \binom{-\delta-\varepsilon}{c} < 0$. Write λ as $\lambda = \binom{\lambda_1}{\lambda'}$ with $\lambda' \in \mathbb{R}^n$. If λ_1 were zero, we could apply the second variant of Farkas' lemma to the system $A^T y \leqslant c$ and λ', since we know that $A^T y \leqslant c$ has a solution. Therefore, we can conclude $\lambda_1 > 0$. Furthermore, by scaling, we can assume $\lambda_1 = 1$. One has $\lambda'^T A^T = b^T$ and $\lambda'^T c < \delta + \varepsilon$. The first equation implies that λ' is a feasible solution of the primal (recall $\lambda' \geqslant 0$). The second equation shows that the objective function value of λ' is less than $\delta + \varepsilon$. This means that the optimum value of the primal linear program is also δ, since the primal has an optimal solution (see Corollary 3.1) and ε can be chosen arbitrarily small. □

Exercises

1. Formulate the dual linear program of

$$\begin{aligned} \max 2x_1 + 3x_2 - 7x_3 & \\ x_1 + 3x_2 + 2x_3 &= 4 \\ x_1 + x_2 &\leqslant 8 \\ x_1 - x_3 &\geqslant -15 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

2. Consider the following linear program

$$\begin{aligned} \max x_1 + x_2 & \\ 2x_1 + x_2 &\leqslant 6 \\ x_1 + 2x_2 &\leqslant 8 \\ 3x_1 + 4x_2 &\leqslant 22 \\ x_1 + 5x_2 &\leqslant 23 \end{aligned}$$

   Show that $(4/3, 10/3)$ is an optimal solution by providing a suitable feasible dual solution.