Chapter 6
The two-phase simplex method

We now deal with the first question raised at the end of Chapter 3. How do we find an initial basic feasible solution with which the simplex algorithm is started? *Phase one* of the simplex method deals with the computation of an initial feasible basis, which is then handed over to *phase two*, the simplex method as we described it so far.

Phase one

Suppose we have to solve a linear program

$$\begin{aligned} \min c^T x & \\ Ax &= b \\ x &\geqslant 0. \end{aligned} \tag{46}$$

By multiplying some rows with -1 if necessary, we can achive that the right-hand-side b satisfies $b \geqslant 0$. From this, we construct a linear program from which an initial basic solution is readily available

$$\begin{aligned} \min z_1 + \cdots + z_m & \\ Ax + z &= b \\ x, z &\geqslant 0. \end{aligned} \tag{47}$$

The set $B = \{n+1, \dots, n+m\}$ is a feasible basis with basic feasible solution (x^*, z^*) defined by $x^* = 0$ and $z^* = b$. The next lemma is immediate.

Lemma 6.1. *The linear program* (46) *is infeasible if and only if the optimum value of the linear program* (47) *is larger than zero.*

Proof. If x^* is a feasible solution of (46), then $(x^*, 0)$ is a feasible solution of (47) with objective function value equal to zero. Thus if (46) is feasible, then the optimum value of (47) is zero (it cannot be smaller, since the z_i are nonnegative).

If now, on the other hand, the optimum value of (47) is zero, we can show that (46) is feasible. Since then, if (x^*, z^*) is an optimum solution of (47), then $z^* = 0$ and thus x^* is a feasible solution of (46). □

Phase one of the simplex method consists of solving the linear program (47) with the initial feasible basis $B = \{n+1, \dots, n+m\}$. If the objective value of the computed solution of (47) is larger than zero, we assert that (46) is infeasible.

Otherwise, the goal is now to extract an initial feasible basis of this solution. Let $B' \subseteq \{1, \dots, m+n\}$ be the basis returned by the simplex algorithm and let $(x^*, 0)$ be the associated basic feasible solution. If $B' \subseteq \{1, \dots, m\}$, then this B' is already a feasible basis of (46). It could however be that $B' \cap \{n+1, \dots, n+m\} \neq \emptyset$. We could augment $B' \cap \{1, \dots, m\}$ to a basis B of $Ax = b, x \geqslant 0$, provided that A has full row-rank.

Instead we describe an alternative method which uses tableau mechanism to *drive the artificial variables out* of the basis B'. We describe it now. Let $B' = \{B_1, \dots, B_m\}$ and suppose that $B_\ell \in \{n+1, \dots, n+m\}$. We inspect the tableau of the basis B' of (47)

$$\begin{array}{c|c} * & * \\ \hline A_{B'}^{-1} A & x_{B'}^* . \end{array} \tag{48}$$

If the ℓ-th row of $A_{B'}^{-1} A$ is zero, then the ℓ-th row of the system $A_{B'}^{-1} Ax = A_{B'}^{-1} b$ can be deleted from the tableau and $B' := B' - \{B_\ell\}$.

Otherwise there exists an entry in this row, which is unequal to zero. Let this entry be in column j. We now let j enter the basis and let B_ℓ leave the basis by the appropriate pivot operations. Observe that you may have to multiply the corresponding row by -1, if the pivot entry would be negative. We continue these steps until there are no artificial variables left in the basis.

An example

The following example is taken from [1, p. 114].

Consider the linear program

$$\begin{array}{rrrrrrrrrl} \min & x_1 & + & x_2 & + & x_3 & & & & \\ & x_1 & + & 2x_2 & + & 3x_3 & & & = & 3 \\ & -x_1 & + & 2x_2 & + & 6x_3 & & & = & 2 \\ & & - & 4x_2 & - & 9x_3 & & & = & -5 \\ & & & & & 3x_3 & + & x_4 & = & 1 \\ & & & & x_1, & x_2, & x_3, & x_4 & \geqslant & 0 \end{array}$$

We form the auxiliary linear program to initialize phase one of the simplex algorithm.

$$
\begin{array}{llllllllll}
\min & & & & x_5 + x_6 + x_7 + x_8 & & & & \\
x_1 & + 2x_2 & + 3x_3 & & + x_5 & & & & = 3 \\
-x_1 & + 2x_2 & + 6x_3 & & & + x_6 & & & = 2 \\
 & 4x_2 & + 9x_3 & & & & + x_7 & & = 5 \\
 & & 3x_3 & + x_4 & & & & + x_8 & = 1 \\
\multicolumn{9}{l}{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geqslant 0.}
\end{array}
$$

By evaluating the reduced cost of the basis $B = \{5,6,7,8\}$ we arrive at the tableau

0	-8	-21	-1	0	0	0	0	-11
1	2	3	0	1	0	0	0	3
-1	2	6	0	0	1	0	0	2
0	4	9	0	0	0	1	0	5
0	0	3	1	0	0	0	1	1

We let 4 enter the basis and 8 leaves the basis. The tableau for $B = \{4,5,6,7\}$ is

0	-8	-18	0	0	0	0	1	-10
1	2	3	0	1	0	0	0	3
-1	2	6	0	0	1	0	0	2
0	4	9	0	0	0	1	0	5
0	0	3	1	0	0	0	1	1

Next, 3 enters the basis and 4 leaves again. We obtain the tableau of $B = \{3,5,6,7\}$.

0	-8	0	6	0	0	0	7	-4
0	0	1	1/3	0	0	0	1/3	1/3
1	2	0	-1	1	0	0	-1	2
-1	2	0	-2	0	1	0	-2	0
0	4	0	-3	0	0	1	-3	2

Next, 2 enters the basis and 6 leaves the basis. We obtain the tableau of $B = \{2,3,5,7\}$

-4	0	0	-2	0	4	0	-1	-4
-1/2	1	0	-1	0	1/2	0	-1	0
0	0	1	1/3	0	0	0	1/3	1/3
2	0	0	1	1	-1	0	1	2
2	0	0	1	0	-2	1	1	2

Next, 1 enters the basis and 5 leaves. The tableau of $B = \{1,2,3,7\}$ is

0	0	0	0	2	2	0	1	0
1	0	0	1/2	1/2	-1/2	0	1/2	1
0	1	0	-3/4	1/4	1/4	0	-3/4	1/2
0	0	1	1/3	0	0	0	1/3	1/3
0	0	0	0	-1	-1	1	0	0

Now we see that the 4-th row of the matrix $A_B^{-1}A$ is zero. Thus, we can delete the last constraint. We arrive at the tableau

0	0	0	0	0
1	0	0	1/2	1
0	1	0	-3/4	1/2
0	0	1	1/3	1/3

which is yields a feasible basic solution $x_1 = 1, x_2 = 1/2, x_3 = 1/3$. Phase two of the simplex algorithm is initiated with the tableau belonging to the basis $\{1,2,3\}$.

0	0	0	-1/12	-11/6
1	0	0	1/2	1
0	1	0	-3/4	1/2
0	0	1	1/3	1/3

After one more iteration one arrives at the tableau corresponding of the basis $B = \{1,2,4\}$.

0	0	1/4	0	-7/4
1	0	-3/2	0	1/2
0	1	9/4	0	5/4
0	0	3	1	1

The dual simplex method

We briefly discuss also the dual simplex method.

Definition 6.1 (Dual feasible basis). A basis B which yields nonnegative reduced cost $\overline{c}^T = c^T - c_B^T A_B^{-1} A \geqslant 0$ is called a *dual feasible basis*.

In the dual simplex method, we are given a tableau

$$\begin{array}{c|c} c^T - c_B^T A_B^{-1} A & -c_B^T x_B^* \\ \hline A_B^{-1} A & x_B^*. \end{array} \tag{49}$$

together with a dual feasible basis B defining the tableau. Suppose the basic solution x^* with $x_B^* = A_B^{-1} b$ and $x_{\overline{B}}^* = 0$ is not feasible, or in other words, the basis B is not (primal) feasible, however dual feasible. We want to pivot to a new basis B' which remains dual feasible, whose objective function value $c_{B'}^T x_{B'}^*$ is larger compared to $c_B^T x_B^*$. Remember that this value is equal to the objective value of the corresponding dual solution and that the dual linear program $\max\{b^T y \colon A^T y \leqslant c,\, x \in \mathbb{R}^n\}$ is a maximization problem.

We now describe the pivot step which yields an improved dual feasible solution if the linear program does not have any degenerate basic solutions. By applying perturbation, or a suitable pivot rule, we can thereby achieve that the dual simplex method also eventually terminates.

Suppose that $x^*(B_i)$ is less than zero. We choose B_i to *exit* the basis B and search for a $j \in \{1,\dots,n\}$ that should enter the basis B to form the new basis B'. This is in contrast to the primal simplex method, where we are choosing an entering variable first and then determine an exiting variable.

Let $a_1,\dots,a_n,x^*(B_i)$ be the i-th row of the system matrix of the tableau (49).

Lemma 6.2. *If $a_i \geqslant 0$ for each $i = 1,\dots,n$, then the linear program is infeasible and the dual linear program is unbounded.*

Proof. The proof is straightforward with linear programming duality. Clearly, there cannot be a feasible solution $\widetilde{x} \geqslant 0$ of the primal, since with this $\widetilde{x}$ one has $\sum_{i=1}^n a_i \widetilde{x}_i = x^*(B_i) < 0$ which is impossible, since the a_i are all greater than or equal to 0. The theorem of strong duality then implies that the dual linear program is unbounded. □

Now let j be an index with $a_j < 0$. If we let j enter the basis, then we must transform the tableau in such a way that the j-th column of the tableau becomes the i-th unit vector. We want to ensure that the tableau is still dual feasible.

To this end, we consider the i-th row of the system matrix.

$$a_1,\dots,a_j,\dots,a_n,x^*(B_i)$$

and multiply it with $1/a_j$ to obtain.

$$a_1/a_j,\dots,a_{j-1}/a_j,1,a_{j+1}/a_j,\dots,a_n/a_j,x^*(B_i)/a_j$$

If we perform now elementary row operations on the tableau to have the i-th unit vector in the j-th column, then we would want the new reduced costs to remain nonnegative. The k-th reduced cost is

$$\overline{c}_k - \overline{c}_j a_k/a_j.$$

If a_k was positive, then the new reduced cost of k is clearly positive as well. If $a_k < 0$, then the new reduced cost of k is nonnegative if and only if

$$\overline{c}_k/a_k \leqslant \overline{c}_j/a_j.$$

Let $K = \{j\colon a_j < 0\}$ be the indices corresponding to negative entries of the i-th row of the system matrix of the tableau. If $j \in K$ is an index with $a_j < 0$ such that

$$\overline{c}_k/a_k \leqslant \overline{c}_j/a_j, \text{ for all } k \in K,$$

then j can enter the basis and the dual objective value has strictly increased if there are no degenerate basic solutions.

This yields *one iteration of the dual simplex method* which operates on a dual feasible basis B:

1. If $x_B^* \geqslant 0$, then output the optimal basis B.
2. Else let i be an index with $x^*(B_i) < 0$. If the i-th row of the system matrix contains only nonnegative entries, the assert that the linear program is infeasible and that the dual linear program is unbounded.
3. Let $K = \{i : a_i < 0\}$ and let $j \in K$ be an index with
$$\overline{c}_j / a_j = \max\{\overline{c}_k / a_k : k \in K\}.$$
4. The index i leaves the basis and j enters the basis.
5. Update the tableau.

Notice that the update step strictly reduces the upper-right entry in the tableau if $\overline{c}_j > 0$. Again, by a perturbation argument, we can achieve that such a degeneracy never occurs and the dual simplex method terminates.

Example 6.1 (See p. 162 in [1]).

1	1	0	0	0	
-1	-2	1	0	-2	$B = \{3,4\}$
-1	0	0	1	-1	

1/2	0	1/2	0	-1	
1/2	1	-1/2	0	1	$B = \{2,4\}$
-1	0	0	1	-1	

0	0	1/2	1/2	-3/2	
1	0	0	-1	1	$B = \{1,2\}$
0	1	-1/2	1/2	1/2	

Exercises

1. Solve the following linear program using the two-phase simplex method:
$$\begin{aligned} \min\ & 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 \\ \text{s.t.}\quad & x_1 + x_2 + 4x_4 + x_5 = 2 \\ & x_1 + 2x_2 + -3x_4 + x_5 = 2 \\ & x_1 - 4x_2 + 3x_3 = 1 \\ & x_1, x_2, x_3, x_4 \geqslant 0 \end{aligned}$$
During the first phase, let the following indices enter the basis in this order: $1,2,5$. Use the lexicographic pivoting rule to decide which index will leave the basis in each step.
2. Solve the following simplex tableau using the dual simplex method.

x_1	x_2	x_3	x_4	x_5	
3	2	1	0	0	0
-2	2	-1	1	0	-3
-2	-1	1	0	1	-1

3. Consider the following simplex tableau:

x_1	x_2	x_3	x_4	x_5	
δ	-2	0	0	0	-10
-1	η	1	0	0	4
α	-4	0	1	0	1
γ	3	0	0	1	β

The current basic variables are x_3, x_4, x_5. The entries $\alpha, \beta, \gamma, \delta, \eta$ in the tableau are unknown parameters.
For each one of the following statements, find some parameter values that will make the statement true.

a. The current solution is feasible but not optimal.
b. The current solution is optimal.
c. The optimal cost is $-\infty$.

References

1. D. Bertsimas and J. N. Tsitsiklis. *Introduction to linear optimization.* Athena Scientfic, Belmont, Mass., 1997.