Chapter 8
Polyhedra and Integer Programming

In this section we give definitions and fundamental facts about polyhedra. An excellent reference for this topic is the book by Schrijver [4]. A *polyhedron* P is a set of vectors of the form $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$, for some matrix $A \in \mathbb{R}^{m\times n}$ and some vector $b \in \mathbb{R}^m$. We write $P(A,b)$. The polyhedron is *rational* if both A and b can be chosen to be rational.

Recall that a finite set $V \subseteq \mathbb{R}^n$ is *affinely independent* if for each $v \in V$ one has $v \notin \text{affine.hull}(V \setminus \{v\})$. This is equivalent to $(V - v) \setminus \{0\}$ being linearly independent for each $v \in V$. The *dimension* of V is the size of the largest subset of V which is affinely independent minus one:

$$\dim(V) = \max\{|U| - 1 \mid U \subseteq V \text{ is affinely independent}\}$$

Example 1. • $\dim(\mathbb{R}^n) = n$
- $\dim(\{x\}) = 0$ for every $x \in \mathbb{R}^n$
- $\dim(\emptyset) = -1$

Notice that $V \subseteq \mathbb{R}^n$ is affinely independent if and only if $(V - v) \setminus \{0\}$ is linearly independent for each $v \in V$.

Definition 1. An inequality $a^T x \leqslant \beta$ is called an *implicit equality* of $Ax \leqslant b$ if each $x^* \in P(A,b)$ satisfies $a^T x^* = \beta$. We denote the subsystem consisting of implicit equalities of $Ax \leqslant b$ by $A^= x \leqslant b^=$ and the subsystem consisting of the other inequalities by $A^{\leqslant} x \leqslant b^{\leqslant}$. A constraint is *redundant* if its removal from $Ax \leqslant b$ does not change the set of feasible solution of $Ax \leqslant b$.

In the following, a vector x satisfies $Ax < b$ if and only if $a_i^T x < b_i$ for all $1 \leqslant i \leqslant m$, where $a_1,\dots,a_m$ are the rows of A.

Lemma 1. *Let $P(A,b)$ be a non-empty polyhedron. Then there exists an $x \in P(A,b)$ with $A^{\leqslant} x < b^{\leqslant}$.*

Proof. Suppose that the inequalities in $A^{\leqslant} x \leqslant b^{\leqslant}$ are $a_1^T x \leqslant \beta_1, \dots, a_k^T x \leqslant \beta_k$. For each $1 \leqslant i \leqslant k$ there exists an $x_i \in P$ with $a_i^T x_i < \beta_i$. Thus the point $x = 1/k(x_1 + \cdots + x_k)$ is a point of $P(A,b)$ satisfying $A^{\leqslant} x < b^{\leqslant}$.

Lemma 2. *Let $Ax \leqslant b$ be a system of inequalities. One has*

$$\text{affine.hull}(P(A,b)) = \{x \in \mathbb{R}^n \mid A^= x = b^=\} = \{x \in \mathbb{R}^n \mid A^= x \leqslant b^=\}.$$

Proof. Let $x_1, \dots, x_t \in P(A,b)$ and suppose that $a^T x \leqslant \beta$ is an implicit equality. Then since $a^T x_i = \beta$ one has $a^T(\sum_{j=1}^t \lambda_i x_i) = \beta$. Therefore the inclusions $\subseteq$ follow.

Suppose now that x_0 satisfies $A^= x \leqslant b^=$. Let $x_1 \in P(A,b)$ with $A^{\leqslant} x_1 < b^{\leqslant}$. If $x_0 = x_1$ then $x_0 \in P(A,b) \subseteq \text{affine.hull}(P(A,b))$. Otherwise the line segment between x_0 and x_1 contains more than one point in P and thus $x_0 \in \text{affine.hull}(P)$.

Decomposition theorem for polyhedra

A nonempty set $C \subseteq \mathbb{R}^n$ is a *cone* if $\lambda x + \mu y \in C$ for each $x, y \in C$ and $\lambda, \mu \in \mathbb{R}_{\geqslant 0}$. A cone C is *polyhedral* if $C = \{x \in \mathbb{R}^n \mid Ax \leqslant 0\}$. A cone *generated by* vectors $x_1, \dots, x_m \in \mathbb{R}^n$ is a set of the form $C = \{\sum_{i=1}^m \lambda_i x_i \mid \lambda_i \in \mathbb{R}_{\geqslant 0}, i = 1, \dots, m\}$. A point $x = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i \in \mathbb{R}_{\geqslant 0}$, $i = 1, \dots, m$ is called a *conic combination* of the $x_1, \dots, x_m$. The set of conic combinations of X is denoted by $\text{cone}(X)$.

Theorem 1 (Farkas-Minkowsi-Weyl theorem). *A convex cone is polyhedral if and only if it is finitely generated.*

Proof. Suppose that $a_1, \dots, a_m$ span $\mathbb{R}^n$ and consider the cone $C = \{\sum_{i=1}^m \lambda_i a_i \mid \lambda_i \geqslant 0, i = 1, \dots, m\}$. Let $b \notin C$. Then the system $A\lambda = b$, $\lambda \geqslant 0$ has no solution. By Theorem **??** (Farkas' lemma), this implies that there exists a $y \in \mathbb{R}^n$ such that $A^T y \leqslant 0$ and $b^T y > 0$.

Suppose that the columns of A which correspond to inequalities in $A^T y \leqslant 0$ that are satisfied by y with equality have rank $< n-1$. Denote these columns by $a_{i_1}, \dots, a_{i_k}$. Then there exists a $v \neq 0$ which is orthogonal to each of these columns and to b, i.e., $a_{i_j}^T v = 0$ for each $j = 1, \dots, k$ and $b^T v = 0$. There also exists a column a^* of A which is not in the set $\{a_{i_1}, \dots, a_{i_k}\}$ such that $(a^*)^T v > 0$ since the columns of A span $\mathbb{R}^n$. Therefore there exists an $\epsilon > 0$ such that

i) $A^T(y + \epsilon \cdot v) \leqslant 0$
ii) The subspace generated by the columns of A which correspond to inequalities of $A^T x \leqslant 0$ which are satisfied by $y + \epsilon \cdot v$ with equality strictly contains $\langle a_{i_1}, \dots, a_{i_k} \rangle$.

Notice that we have $b^T y = b^T(y + \epsilon \cdot v) > 0$.

Continuing this way, we obtain a solution of the form $y+u$ of $A^T x \leqslant 0$ such that one has $n-1$ linearly independent columns of A whose corresponding inequality in $A^T x \leqslant 0$ are satisfied with equality. Thus we see that each b which does not belong to C can be separated from C with an inequality of the form $c^T x \leqslant 0$ which is uniquely defined by $n-1$ linearly independent vectors from the set $a_1, \dots, a_m$. This shows that C is polyhedral.

Suppose now that $a_1,\dots,a_m$ do not span $\mathbb{R}^n$. Then there exist linearly independent vectors $d_1,\dots,d_k$ such that each d_i is orthogonal to each of the $a_1,\dots,a_m$ and $a_1,\dots,a_m,d_1,\dots,d_k$ spans $\mathbb{R}^n$. The cone generated by $a_1,\dots,a_m,d_1,\dots,d_k$ is polyhedral and thus of the form $Ax \leqslant 0$ with some matrix $A \in \mathbb{R}^{m\times n}$. Suppose that $\langle a_1,\dots,a_m\rangle = \{x \in \mathbb{R}^n \mid Ux = 0\}$. Now $C = \{x \in \mathbb{R}^n \mid Ax \leqslant 0, Ux = 0\}$ and C is polyhedral.

Now suppose that $C = \{x \in \mathbb{R}^n \mid a_1^T x \leqslant 0,\dots,a_m^T x \leqslant 0\}$. The cone

$$C' := \operatorname{cone}(a_1,\dots,a_m) = \{\sum_{i=1}^m \lambda_i a_i \mid \lambda_i \geqslant 0,\ i=1,\dots,m\}$$

is polyhedral and thus of the form $C' = \{x \in \mathbb{R}^n \mid b_1^T x \leqslant 0,\dots,b_k^T x \leqslant 0\}$. Clearly, $\operatorname{cone}(b_1,\dots,b_k) \subseteq C$ since $b_i^T a_j \leqslant 0$. Suppose now that $y \in C\setminus\operatorname{cone}(b_1,\dots,b_k)$. Then, since $\operatorname{cone}(b_1,\dots,b_k)$ is polyhedral, there exists a $w \in \mathbb{R}^n$ with $w^T y > 0$ and $w^T b_i \leqslant 0$ for each $i = 1,\dots,k$. From the latter we conclude that $w \in C'$. From $y \in C$ and $w \in C'$ we conclude $w^T y \leqslant 0$, which is a contradiction.

A set of vectors $Q = \operatorname{conv}(X)$, where $X \subseteq \mathbb{R}^n$ is finite is called a *polytope*.

Theorem 2 (Decomposition theorem for polyhedra). *A set $P \subseteq \mathbb{R}^n$ is a polyhedron if and only if $P = Q + C$ for some polytope Q and a polyhedral cone C.*

Proof. Suppose $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ is a polyhedron. Consider the polyhedral cone

$$\left\{\begin{pmatrix} x \\ \lambda \end{pmatrix} \mid x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}_{\geqslant 0};\ Ax - \lambda b \leqslant 0\right\} \tag{8.1}$$

is generated by finitely many vectors $\begin{pmatrix} x_i \\ \lambda_i \end{pmatrix}$, $i = 1,\dots,m$. By scaling with a positive number we may assume that each $\lambda_i \in \{0,1\}$. Let Q be the convex hull of the x_i with $\lambda_i = 1$ and let C be the cone generated by the x_i with $\lambda_i = 0$. A point $x \in \mathbb{R}^n$ is in P if and only if $\begin{pmatrix} x \\ 1 \end{pmatrix}$ belongs to (8.1) and thus if and only if

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \in \operatorname{cone}\left\{\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix},\dots,\begin{pmatrix} x_m \\ \lambda_m \end{pmatrix}\right\}.$$

Therefore $P = Q + C$.

Suppose now that $P = Q+C$ for some polytope Q and a polyhedral cone C with $Q = \operatorname{conv}(x_1,\dots,x_m)$ and $C = \operatorname{cone}(y_1,\dots,y_t)$. A vector x_0 is in P if and only if

$$\begin{pmatrix} x_0 \\ 1 \end{pmatrix} \in \operatorname{cone}\left\{\begin{pmatrix} x_1 \\ 1 \end{pmatrix},\dots,\begin{pmatrix} x_m \\ 1 \end{pmatrix},\begin{pmatrix} y_1 \\ 0 \end{pmatrix},\dots,\begin{pmatrix} y_t \\ 0 \end{pmatrix}\right\} \tag{8.2}$$

By Theorem 1 (8.2) is equal to

$$\left\{\begin{pmatrix} x \\ \lambda \end{pmatrix} \mid Ax - \lambda b \leqslant 0\right\} \tag{8.3}$$

for some matrix A and vector b. Thus $x_0 \in P$ if and only if $Ax_0 \leqslant b$ and thus P is a polyhedron.

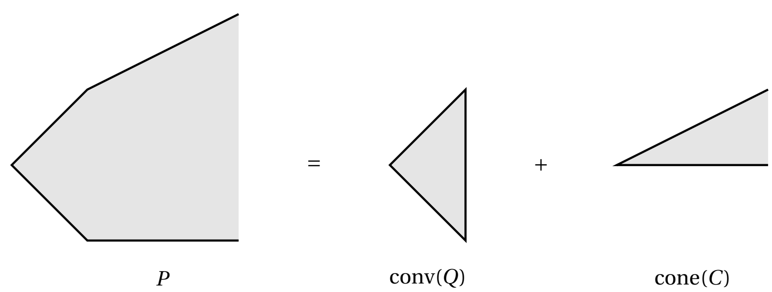

Fig. 8.1 A polyhedron and its decomposition into Q and C

Let $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$. The *characteristic cone* is $\text{char.cone}(P) = \{y \mid y + x \in P \text{ for all } x \in P\} = \{y \mid Ay \leqslant 0\}$. One has

i) $y \in \text{char.cone}(P)$ if and only if there exists an $x \in P$ such that $x + \lambda y \in P$ for all $\lambda \geqslant 0$
ii) $P + \text{char.cone}(P) = P$
iii) P is bounded if and only if $\text{char.cone}(P) = \{0\}$.
iv) If the decomposition of P is $P = Q + C$, then $C = \text{char.cone}(P)$.

The *lineality space* of P is defined as $\text{char.cone}(P) \cap -\text{char.cone}(P)$. A polyhedron is *pointed*, if its lineality space is $\{0\}$.

Faces

An inequality $c^T x \leqslant \delta$ is called *valid* for P if each $x \in P$ satisfies $c^T x \leqslant \delta$. If in addition $(c^T x = \delta) \cap P \neq \emptyset$, then $c^T x \leqslant \delta$ is a *supporting inequality* and $c^T x = \delta$ is a supporting hyperplane.

A set $F \subseteq \mathbb{R}^n$ is called a *face* of P if there exists a valid inequality $c^T x \leqslant \delta$ for P with $F = P \cap (c^T x = \delta)$.

Lemma 3. *A set $\emptyset \neq F \subseteq \mathbb{R}^n$ is a face of P if and only if $F = \{x \in P \mid A'x = b'\}$ for a subset $A'x \leqslant b'$ of $Ax \leqslant b$.*

Proof. Suppose that $F = \{x \in P \mid A'x = b'\}$. Consider the vector $c = 1^T A'$ and $\delta = 1^T b'$. The inequality $c^T x \leqslant \delta$ is valid for P. It is satisfied with equality by each $x \in F$. If $x' \in P \setminus F$, then there exists an inequality $a^T x \leqslant \beta$ of $A'x \leqslant b'$ such that $a^T x' < \beta$ and consequently $c^T x' < \delta$.

On the other hand, if $c^T x \leqslant \delta$ defines the face F, then by the linear programming duality

$$\max\{c^T x \mid Ax \leqslant b\} = \min\{b^T \lambda \mid A^T \lambda = c, \lambda \geqslant 0\}$$

there exists a $\lambda \in \mathbb{R}^m_{\geqslant 0}$ such that $c = \lambda^T A$ and $\delta = \lambda^T b$. Let $A'x \leqslant b'$ be the subsystem of $Ax \leqslant b$ which corresponds to strictly positive entries in $Ax \leqslant b$. One has $F = \{x \in P \mid A'x = b'\}$.

A *facet* of P is an inclusion-wise maximal face F of P with $F \neq P$. An inequality $a^T x \leqslant \beta$ of $Ax \leqslant b$ is called *redundant* if $P(A,b) = P(A',b')$, where $A'x \leqslant b'$ is the system stemming from $Ax \leqslant b$ by deleting $a^T x \leqslant \beta$. A system $Ax \leqslant b$ is irredundant if $Ax \leqslant b$ does not contain a redundant inequality.

Lemma 4. *Let $Ax \leqslant b$ be an irredundant system. Then a set $F \subseteq P$ is a facet if and only if it is of the form $F = \{x \in P \mid a^T x = \beta\}$ for an inequality $a^T x \leqslant \beta$ of $A^{\leqslant} x \leqslant b^{\leqslant}$.*

Proof. Let F be a facet of P. Then $F = \{x \in P \mid c^T x \leqslant \delta\}$ for a valid inequality $c^T x \leqslant \delta$ of P. There exists a $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $c = \lambda^T A$ and $\delta = \lambda^T b$. There exists an inequality $a^T x \leqslant \beta$ of $A^{\leqslant} x \leqslant b^{\leqslant}$ whose corresponding entry in λ is strictly positive. Clearly $F \subseteq \{x \in P \mid a^T x = \beta\} \subset P$. Since F is an inclusion-wise maximal face one has $F = \{x \in P \mid a^T x = \beta\}$.

Let F be of the form $F = \{x \in P \mid a^T x = \beta\}$ for an inequality $a^T x \leqslant \beta$ of $A^{\leqslant} x \leqslant b^{\leqslant}$. Clearly $F \neq \emptyset$ since the system $Ax \leqslant b$ is irredundant. If F is not a facet, then $F \subseteq F' = \{x \in P \mid a'^T x = \beta'\}$ with another inequality $a'^T x \leqslant \beta'$ of $A^{\leqslant} x \leqslant b^{\leqslant}$. Let $x^* \in \mathbb{R}^n$ be a point with $a^T x^* > \beta$ and which satisfies all other inequalities of $Ax \leqslant b$. Such an x^* exists, since $Ax \leqslant b$ is irredundant. Let $\widetilde{x} \in P$ with $A^{\leqslant} \widetilde{x} < b^{\leqslant}$. There exists a point $\overline{x}$ on the line-segment $\overline{\widetilde{x} x^*}$ with $a^T \overline{x} = \beta$. This point is then also in F' and thus $a'^T x = \beta'$ follows. This shows that $a'^T x^* > \beta'$ and thus $a^T x \leqslant \beta$ can be removed from the system. This is a contradiction to $Ax \leqslant b$ being irredundant.

Lemma 5. *A face F of $P(A,b)$ is inclusion-wise minimal if and only if it is of the form $F = \{x \in \mathbb{R}^n \mid A'x = b'\}$ for some subsystem $A'x \leqslant b'$ of $Ax \leqslant b$.*

Proof. Let F be a minimal face of P and let $A'x \leqslant b'$ a the subsystem of inequalities of $Ax \leqslant b$ with $F = \{x \in P \mid A'x = b'\}$. Suppose that $F \subset \{x \in \mathbb{R}^n \mid A'x = b'\}$ and let $x_1 \in \mathbb{R}^n \setminus P$ satisfy $A'x_1 = b'$ and $x_2 \in F$. There exists "a first" inequality $a^T x \leqslant \beta$ of $Ax \leqslant b$ which is "hit" by the line-segment $\overline{x_2 x_1}$. Let $x^* = \overline{x_2 x_1} \cap (a^T x = \beta)$. Then $x^* \in F$ and thus $F \cap (a^T x = \beta) \neq \emptyset$. But $F \supset F \cap (a^T x = \beta)$ since $a^T x \leqslant \beta$ is not an inequality of $A'x \leqslant b'$. This is a contradiction to the minimality of F.

Suppose that F is a face with $F = \{x \in \mathbb{R}^n \mid A'x = b'\} = \{x \in P \mid A'x = b'\}$ for a subsystem $A'x \leqslant b'$ of $Ax \leqslant b$. Suppose that there exists a face $\widetilde{F}$ of P with $\emptyset \subset \widetilde{F} \subset F$. By Lemma 3 $\widetilde{F} = \{x \in P \mid A'x = b', A^* x = b^*\}$, where $A^* x \leqslant b^*$ is a subsystem of $Ax \leqslant b$ which contains an inequality $a^T x \leqslant \beta$ such that there exists an $x_1, x_2 \in F$ with $a^T x_1 < \beta$ and $a^T x_2 \leqslant \beta$. The line $\ell(x_1, x_2) = \{x_1 + \lambda(x_2 - x_1) \mid \lambda \in \mathbb{R}\}$ is contained in F but is not contained in $a^T x \leqslant \beta$. This shows that F is not contained in P which is a contradiction.

Exercise 5 asks for a proof of the following corollary.

Corollary 1. *Let* F_1 *and* F_2 *be two inclusion-wise minimal faces of* $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$*, then* $\dim(F_1) = \dim(F_2)$.

We say that a polyhedron contains a line $\ell(x_1, x_2)$ with $x_1 \neq x_2 \in P$ if $\ell(x_1, x_2) = \{x_1 + \lambda(x_2 - x_1) \mid \lambda \in \mathbb{R}\} \subseteq P$. A *vertex* of P is a 0-dimensional face of P. An *edge* of P is a 1-dimensional face of P.

Example 2. Consider a linear program $\min\{c^T x \colon Ax = b, x \geqslant 0\}$. A basic feasible solution defined by the basis $B \subseteq \{1, \ldots, n\}$ is a vertex of the polyhedron $P = \{x \in \mathbb{R}^n \colon Ax = b, x \geqslant 0\}$. This can be seen as follows. The inequality $a^T x \geqslant 0$ is valid for P, where $a_B = \mathbf{0}$ and $a_{\overline{B}} = \mathbf{1}$. The inequality is satisfied with equality by a point $x^* \in P$ if and only if $x^*_{\overline{B}} = \mathbf{0}$. Since the columns of A_B are linearly independent, as B is a basis, the unique point which satisfies $a^T x \geqslant 0$ with equality is the basic feasible solution

In exercise you are asked to show that the simplex method can be geometrically interpreted as a walk on the graph $G = (V, E)$, where V is the set of basic feasible solutions and $uv \in E$ if and only if $\operatorname{conv}\{u, v\}$ is a 1-dimensional face of the polyhedron defined by the linear program.

Integer Programming

An *integer program* is a problem of the form

$$\begin{gathered} \max c^T x \\ Ax \leqslant b \\ x \in \mathbb{Z}^n, \end{gathered}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

The difference to linear programming is the *integrality constraint* $x \in \mathbb{Z}^n$. This powerful constraint allows to model discrete choices but, at the same time, makes an integer program much more difficult to solve than a linear program. In fact one can show that integer programming is NP-hard, which means that it is *in theory* computationally intractable. However, integer programming has nowadays become an important tool to solve difficult industrial optimization problems efficiently. In this chapter, we characterize some integer programs which are easy to solve, since the *linear programming relaxation* $\max\{c^T x \colon Ax \leqslant b\}$ yields already an optimal integer solution. The following observation is crucial.

Theorem 3. *Suppose that the optimum solution* x^* *of the linear programming relaxation* $\max\{c^T x \colon Ax \leqslant b\}$ *is integral, i.e.,* $x^* \in \mathbb{Z}^n$*, then* x^* *is also an optimal solution to the integer programming problem* $\max\{c^T x \colon Ax \leqslant b, x \in \mathbb{Z}^n\}$

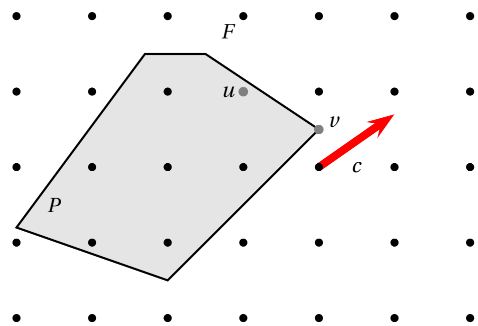

Fig. 8.2 This picture illustrates a polyhedron P an objective function vector c and optimal points u, v of the integer program and the relaxation respectively.

Before we present an example for the power of integer programming we recall the definition of an undirected graph.

Definition 2 (Undirected graph, matching). An *undirected graph* is a tuple $G = (V, E)$ where V is a finite set, called the *vertices* and $E \subseteq \binom{V}{2}$ is the set of *edges* of G. A *matching* of G is a subset $M \subseteq E$ such that for all $e_1 \neq e_2 \in M$ one has $e_1 \cap e_2 = \emptyset$.

We are interested in the solution of the following problem, which is called *maximum weight matching* problem. Given a graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}$, compute a matching with maximum weight $w(M) = \sum_{e \in M} w(e)$.

For a vertex $v \in V$, the set $\delta(v) = \{e \in E \colon v \in e\}$ denotes the *incident* edges to v.

The maximum weight matching problem can now be modeled as an integer program as follows.

$$\begin{aligned} \max \textstyle\sum_{e\in E} w(e)x(e) \\ v \in V : \textstyle\sum_{e \in \delta(v)} x(e) \leqslant 1 \\ e \in E : 0 \leqslant x(e) \\ x \in \mathbb{Z}^{|E|}. \end{aligned}$$

Clearly, if an integer vector $x \in \mathbb{Z}^n$ satisfies the constraints above, then this vector is the *incidence vector* of a matching of G. In other words, the integral solutions to the constraints above are the vectors $\{\chi^M \colon M \text{ matching of } G\}$, where $\chi^M(e) = 1$ if $e \in M$ and $\chi^M(e) = 0$ otherwise.

Integral Polyhedra

In this section we derive sufficient conditions on an integer program to be solved easily by an algorithm for linear programming. A central notion is the one of an integral polyhedron. A rational polyhedron P is called *integral* if each minimal face of P contains an integer point.

Theorem 4. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ be a rational nonempty polyhedron with vertices. P is integral if and only if for all integral vectors $c \in \mathbb{Z}^n$ with $\max\{c^T x \mid x \in P\} < \infty$ one has $\max\{c^T x \mid x \in P\} \in \mathbb{Z}$.*

Proof. Let P be integral and $c \in \mathbb{Z}^n$ with $\max\{c^T x \mid x \in P\} = \delta < \infty$. Since the face $F = \{x \in P \mid c^T x = \delta\}$ contains an integer point it follows that $\delta \in \mathbb{Z}$.

On the other hand let x^* be a vertex of P and assume that $x^*(i) \notin \mathbb{Z}$. There exists a subsystem $A'x \leqslant b'$ of $Ax \leqslant b$ with $A' \in \mathbb{R}^{n\times n}$, A' nonsingular and $A'x^* = b'$. Let $a_1, \ldots, a_n$ be the rows of A'. Since A' is invertible, there exists an integer vector $c \in \operatorname{cone}(a_1, \ldots, a_n) \cap \mathbb{Z}^n$ such that $c \pm e_i \in \operatorname{cone}(a_1, \ldots, a_n)$. The point x^* maximizes both $c^T x$ and $(c + e_i)^T x$. Clearly not both numbers $c^T x^*$ and $(c + e_i)^T x^*$ can be integral, which is a contradiction.

Lemma 6. *Let $A \in \mathbb{Z}^{n\times n}$ be an integral and invertible matrix. One has $A^{-1}b \in \mathbb{Z}^n$ for each $b \in \mathbb{Z}^n$ if and only if $\det(A) = \pm 1$.*

Proof. Recall Cramer's rule which says $A^{-1} = 1/\det(A)\widetilde{A}$, where $\widetilde{A}$ is the adjoint matrix of A. Clearly $\widetilde{A}$ is integral. If $\det(A) = \pm 1$, then A^{-1} is an integer matrix.

If $A^{-1}b$ is integral for each $b \in \mathbb{Z}^n$, then A^{-1} is an integer matrix. We have $1 = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$. Since A and A^{-1} are integral it follows that $\det(A)$ and $\det(A^{-1})$ are integers. The only divisors of one in the integers are ± 1.

A matrix $A \in \mathbb{Z}^{m\times n}$ with $m \leqslant n$ is called *unimodular* if each $m \times m$ sub-matrix has determinant $0, \pm 1$.

Theorem 5. *Let $A \in \mathbb{Z}^{m\times n}$ be an integral matrix of full row-rank. The polyhedron defined by $Ax = b, x \geqslant 0$ is integral for each $b \in \mathbb{Z}^m$ if and only if A is unimodular.*

Proof. Suppose that A is unimodular and b is integral. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geqslant 0\}$ does not contain a line and thus has vertices. A vertex x^* is of the form $x^*_B = A_B^{-1}b$ and $x^*_{\overline{B}} = 0$, where $B \subseteq \{1, \ldots, n\}$ is a basis. Since A_B is unimodular one has $x^* \in \mathbb{Z}^n$.

If A is not unimodular, then there exists a basis B with $\det(A_B) \neq \pm 1$. By Lemma 6 there exists an integral $b \in \mathbb{Z}^n$ with $(A_B)^{-1}b \notin \mathbb{Z}^m$. Let λ be the maximal absolute value of a component of $A_B^{-1}b$. Then $b' = \lceil \lambda \rceil A_B \mathbf{1} + b$ is an integral vector with $A_B^{-1}b' = \lceil \lambda \rceil \mathbf{1} + A_B^{-1}b \geqslant 0$ and $A_B^{-1}b' \notin \mathbb{Z}^m$. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b', x \geqslant 0\}$ has thus a fractional (non-integer) vertex.

An integral matrix $A \in \{0, \pm 1\}^{m\times n}$ is called *totally unimodular* if each of its square sub-matrices has determinant $0, \pm 1$.

Theorem 6 (Hoffman-Kruskal Theorem). *Let $A \in \mathbb{Z}^{m\times n}$ be an integral matrix. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b, x \geqslant 0\}$ is integral for each integral $b \in \mathbb{Z}^m$ if and only if A is totally unimodular.*

Proof. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b, x \geqslant 0\}$ is integral if and only if the polyhedron $Q = \{z \in \mathbb{R}^{n+m} \mid (A|I)z = b, z \geqslant 0\}$ is integral. The assertion thus follows from Theorem 5.

If an integral polyhedron has vertices, then an optimal vertex solution of a linear program over this polyhedron is integral.

Applications of total unimodularity

Bipartite matching

A graph is *bipartite*, if V has a partition into sets A and B such that each edge uv satisfies $u \in A$ and $v \in B$.

A *matching* of G is a subset $M \subseteq E$ such that $e_1 \cap e_2 = \emptyset$ holds for each $e_1 \neq e_2 \in M$. Let $c : E \longrightarrow \mathbb{R}$ be a weight function. The weight of a matching is defined as $c(M) = \sum_{e \in M} c(e)$. The *weighted matching problem* is defined as follows. Given a graph $G = (V, E)$ and edge-weights $c : E \longrightarrow \mathbb{R}$, compute a matching M of G with $c(M)$ maximal.

We now define an *integer program* for this problem and show that, for bipartite graphs, an optimal vertex of the corresponding linear program is integral.

The idea is as follows. We have decision variables $x(e)$ for each edge $e \in E$. We want to model the characteristic vectors $\chi^M \in \{0,1\}^E$ of matchings, where $\chi^M(e) = 1$ if $e \in M$ and $\chi^M(e) = 0$ otherwise. This is achieved with the following set of constraints.

$$\begin{aligned} \textstyle\sum_{e \in \delta(v)} x(e) &\leqslant 1, \ \forall v \in V \\ x(e) &\geqslant 0, \ \forall e \in E. \end{aligned} \tag{8.4}$$

Clearly, the set of vectors $x \in \mathbb{Z}^E$ which satisfy the system (8.4) are exactly the characteristic vectors of matchings of G. The matrix $A \in \{0,1\}^{V \times E}$ which is defined as

$$A(v,e) = \begin{cases} 1 & \text{if } v \in e, \\ 0 & \text{otherwise} \end{cases}$$

is called *node-edge incidence matrix* of G.

Lemma 7. *If G is bipartite, the node-edge incidence matrix of G is totally unimodular.*

Lemma 7 implies that each vertex of the polytope P defined by the inequalities (8.4) is integral. Thus an optimal vertex of the linear program $\max\{c^T x \mid x \in P\}$ corresponds to a maximum weight matching.

Proof (Proof of Lemma 7). Let $G = (V, E)$ be a bipartite graph with bi-partition $V = V_1 \cup V_2$.

Let A' be a $k \times k$ sub-matrix of A. We are interested in the determinant of A. Clearly, we can assume that A does not contain a column which contains only one 1, since we simply consider the sub-matrix A'' of A', which emerges from developing the determinant of A' along this column. The determinant of A' would be $\pm 1 \cdot \det(A'')$.

Thus we can assume that each column contains exactly two ones. Now we can order the rows of A' such that the first rows correspond to vertices of V_1 and then follow the rows corresponding to vertices in V_2. This re-ordering only affects the sign of the determinant. By summing up the rows of A' in V_1 we obtain exactly the same row-vector as we get by summing up the rows of A' corresponding to V_2. This shows that $\det(A') = 0$.

Flows

Let $G = (V, A)$ be a directed graph, see chapter 9. The *node-edge incidence matrix of a directed graph* is a matrix $A \in \{0, \pm 1\}^{V \times E}$ with

$$A(v,a) = \begin{cases} 1 & \text{if } v \text{ is the starting-node of } a, \\ -1 & \text{if } v \text{ is the end-node of } a, \\ 0 & \text{otherwise.} \end{cases} \tag{8.5}$$

A *feasible flow* f of G with capacities u and in-out-flow b is then a solution $f \in \mathbb{R}^A$ to the system $Af = b$, $0 \leqslant f \leqslant u$.

Lemma 8. *The node-edge incidence matrix A of a directed graph is totally unimodular.*

Proof. Let A' be a $k \times k$ sub-matrix of A. Again, we can assume that in each column we have exactly one 1 and one -1. Otherwise, we develop the determinant along a column which does not have this property. But then, the A' is singular, since adding up all rows of A' yields the 0-vector.

A consequence is that, if the b-vector and the capacities u are integral and an optimal flow exists, then there exists an integer optimal flow.

Further applications of polyhedral theory

Doubly stochastic matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is *doubly stochastic* if it satisfies the following linear constraints

$$\begin{aligned} \textstyle\sum_{i=1}^n A(i,j) &= 1, \ \forall j = 1, \dots, n \\ \textstyle\sum_{j=1}^n A(i,j) &= 1, \ \forall i = 1, \dots, n \\ A(i,j) &\geqslant 0, \ \forall 1 \leqslant i, j \leqslant n. \end{aligned} \tag{8.6}$$

A permutation matrix is a matrix which contains exactly one 1 per row and column, where the other entries are all 0.

Theorem 7. *A matrix $A \in \mathbb{R}^{n \times n}$ is doubly stochastic if and only if A is a convex combination of permutation matrices.*

Proof. Since a permutation matrix satisfies the constraints (8.6), then so does a convex combination of these constraints.

On the other hand it is enough to show that each vertex of the polytope defined by the system (8.6) is integral and thus a permutation matrix. However, the matrix defining the system (8.6) is the node-edge incidence matrix of the complete bipartite graph having $2n$ vertices. Since such a matrix is totally unimodular, the theorem follows.

The matching polytope

We now come to a deeper theorem concerning the convex hull of matchings. We mentioned several times in the course that the maximum weight matching problem can be solved in polynomial time. We are now going to show a theorem of Edmonds [1] which provides a complete description of the matching polytope and present the proof by Lovász [3].

Before we proceed let us inspect the symmetric difference $M_1 \Delta M_2$ of two matchings of a graph G. If a vertex is adjacent to two edges of $M_1 \cup M_2$, then one of the two edges belongs to M_1 and one belongs to M_2. Also, a vertex can never be adjacent to three edges in $M_1 \cup M_2$. Edges which are both in M_1 and M_2 do not appear in the symmetric difference. We therefore have the following lemma.

Lemma 9. *The symmetric difference $M_1 \Delta M_2$ of two matchings decomposes into node-disjoint paths and cycles, where the edges on these paths and cycles alternate between M_1 and M_2.*

The *Matching polytope* $P(G)$ of an undirected graph $G = (V, E)$ is the convex hull of incidence vectors χ^M of matchings M of G.

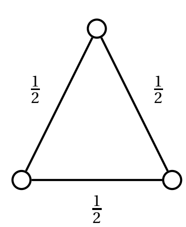

Fig. 8.3 Triangle

The incidence vectors of matchings are exactly the 0/1-vectors that satisfy the following system of equations.

$$\begin{aligned} \textstyle\sum_{e\in\delta(v)} x(e) &\leqslant 1 \;\forall v\in V \\ x(e) &\geqslant 0 \;\forall e\in E. \end{aligned} \tag{8.7}$$

However the triangle (Figure 8.3) shows that the corresponding polytope is not integral. The objective function $\max 1^T x$ has value 1.5. However, one can show that a maximum weight matching of an undirected graph can be computed in polynomial time which is a result of Edmonds [2].

The following (Figure 8.4) is an illustration of an Edmonds inequality. Suppose that U is an odd subset of the nodes V of G and let M be a matching of G. The number of edges of M with both endpoints in U is bounded from above by $\lfloor |U|/2\rfloor$.

Thus the following inequality is valid for the integer points of the polyhedron defined by (8.7).

$$\sum_{e\in E(U)} x(e) \leqslant \lfloor |U|/2\rfloor, \qquad \text{for each } U\subseteq V, \quad |U|\equiv 1 \pmod 2. \tag{8.8}$$

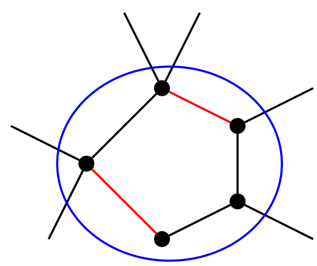

Fig. 8.4 Edmonds inequality.

The goal of this lecture is a proof of the following theorem.

Theorem 8 (Edmonds 65). *The matching polytope is described by the following inequalities:*

i) $x(e) \geqslant 0$ *for each* $e\in E$,
ii) $\sum_{e\in\delta(v)} x(e) \leqslant 1$ *for each* $v\in V$,
iii) $\sum_{e\in E(U)} x(e) \leqslant \lfloor |U|/2\rfloor$ *for each* $U\subseteq V$

Lemma 10. *Let* $G=(V,E)$ *be connected and let* $w: E\longrightarrow \mathbb{R}_{>0}$ *be a weight-function. Denote the set of maximum weight matchings of* G *w.r.t.* w *by* $\mathscr{M}(w)$*. Then one of the following statements must be true:*

i) $\exists v\in V$ *such that* $\delta(v)\cap M \neq \emptyset$ *for each* $M\in\mathscr{M}(w)$
ii) $|M| = \lfloor |V|/2\rfloor$ *for each* $M\in\mathscr{M}(w)$ *and* $|V|$ *is odd.*

Proof. Suppose both i) and ii) do not hold. Then there exists $M \in \mathscr{M}(w)$ leaving two exposed nodes u and v. Choose M such that the minimum distance between two exposed nodes u, v is minimized.

Now let t be on shortest path from u to v. The vertex t cannot be exposed.

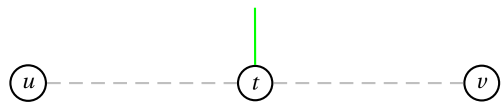

Fig. 8.5 Shortest path between u and v.

Let $M' \in \mathscr{M}(w)$ leave t exposed. Both u and v are covered by M' because the distance to u or v from t is smaller than the distance of u to v.

Consider the symmetric difference $M \triangle M'$ which decomposes into node disjoint paths and cycles. The nodes u, v and t have degree one in $M \triangle M'$. Let P be a path with endpoint t in $M \triangle M'$

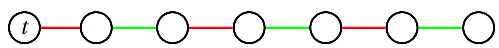

Fig. 8.6 Swapping colors.

If we swap colors on P, see Figure 8.6, we obtain matchings $\widetilde{M}$ and $\widetilde{M'}$ with $w(M) + w(M') = w(\widetilde{M}) + w(\widetilde{M'})$ and thus $\widetilde{M} \in \mathscr{M}(w)$.

The node t is exposed in $\widetilde{M}$ and u or v is exposed in $\widetilde{M}$. This is a contradiction to u and v being shortest distance exposed vertices

Proof (Proof of Theorem 8).

Let $w^T x \leqslant \beta$ be a *facet* of $P(G)$, we need to show that this facet it is of the form

i) $x(e) \geqslant 0$ for some $e \in E$
ii) $\sum_{e \in \delta(v)} x(e) \leqslant 1$ for some $v \in V$
iii) $\sum_{e \in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor$ for some $U \in P_{odd}$

To do so, we use the following method: One of the inequalities i), ii), iii) is satisfied with equality by each χ^M, $M \in \mathscr{M}(w)$. This establishes the claim since the matching polytope is full-dimensional and a facet is a maximal face.

If $w(e) < 0$ for some $e \in E$, then each $M \in \mathscr{M}(w)$ satisfies $e \notin M$ and thus satisfies $x(e) \geqslant 0$ with equality.

Thus we can assume that $w \geqslant 0$.

Let $G^* = (V^*, E^*)$ be the graph induced by edges e with $w(e) > 0$. Each $M \in \mathscr{M}(w)$ contains maximum weight matching $M^* = M \cap E^*$ of G^* w.r.t. w^*.

If G^* is not *connected* , suppose that $V^* = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$ and $V_1, V_2 \neq \emptyset$ and there is no edge connecting V_1 and V_2, then $w^T x \leqslant \beta$ can be written as the sum of $w_1^T x \leqslant \beta_1$ and $w_2^T x \leqslant \beta_2$, where β_i is the maximum weight of a matching in V_i w.r.t. w_i, $i = 1, 2$, see Figure 8.7. This would also contradict the fact that $w^T x \leqslant \beta$ is a facet, since it would follow from the previous inequalities and thus would be a redundant inequality.

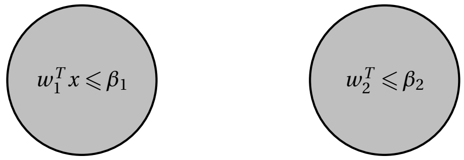

Fig. 8.7 G^* is connected.

Now we can use Lemma 10 for G^*.

i) $\exists v$ such that $\delta(v) \cap M = \emptyset$ for each $M \in \mathscr{M}(w)$. This means that each M in $\mathscr{M}(w)$ satisfies

$$\sum_{e \in \delta(v)} x(e) \leqslant 1 \quad \text{with equality}$$

ii) $|M \cap E^*| = \lfloor |V^*|/2 \rfloor$ for each $M \in \mathscr{M}(w)$ and $|V^*|$ is odd. This means that each M in $\mathscr{M}(w)$ satisfies

$$\sum_{e \in E(V^*)} x(e) \leqslant \lfloor |V^*|/2 \rfloor \quad \text{with equality}$$

Exercises

1. Each nonempty polyhedron $P \subseteq \mathbb{R}^n$ can be represented as $P = L + Q$, where $L \subseteq \mathbb{R}^n$ is a linear space and $Q \subseteq \mathbb{R}^n$ is a pointed polyhedron.
2. Let $P \subset \mathbb{R}^n$ be a polytope and $f : \mathbb{R}^n \to \mathbb{R}^m$ a linear map.
   i) Show that $f(P)$ is a polytope.
   ii) Let $y \in \mathbb{R}^m$ be a vertex of $f(P)$. Show that there is a vertex $x \in \mathbb{R}^n$ of P such that $f(x) = y$.
3. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and consider the polyhedron $P = P(A, b)$. Show that $\dim(P) = n - \operatorname{rank}(A^=)$.
4. i) Show that the dimension of each minimal face of a polyhedron P is equal to $n - \operatorname{rank}(A)$.
   ii) Show that a polyhedron has a vertex if and only if the polyhedron does not contain a line.
5. Show that the affine dimension of the minimal faces of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is invariant.
6. In this exercise you can assume that a linear program $\max\{c^T x \mid Ax \leqslant b\}$ can be solved in polynomial time. Suppose that $P(A, b)$ has vertices and that the linear program is bounded. Show how to compute an optimal *vertex* solution of the linear program in polynomial time.

7. Let $P = \{x \in \mathbb{R}^n \colon\ Ax = b,\ x \geqslant 0\}$ be a polyhedron, where $A \in \mathbb{R}^{m\times n}$ has full row-rank. Let B_1, B_2 be two bases such that $|B_1 \cap B_2| = m-1$ and suppose that the associated basic solutions x_1^* and x_2^* are feasible. Show that, if $x_1 \neq x_2$, then $\text{conv}\{x_1^*, x_2^*\}$ is a 1-dimensional face of P.

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