# NOTES

Edited by **William Adkins**

# A Topological Puzzle

**Inta Bertuccioni**

In their interesting book *Creative Puzzles of the World* [**2**], P. van Delft and J. Botermans describe a puzzle invented by Steward Coffin that involves disentangling a knotted rope $r$ from a rigid wire $W$, shaped as in Figure 1. They also hint at a solution, but several people believe that the puzzle is impossible. I propose to show that this is indeed the case.

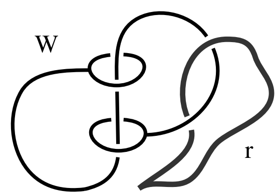


**Figure 1.**

The proof simply consists in computing the fundamental group of the complement $X$ of $W$ in $\mathbb{R}^3$ and checking that the class of $r$ in $\pi_1(X)$ is not 1. The first step means finding a presentation of $\pi_1(X)$. This could quickly be achieved using the Wirtinger presentation, but I prefer to give a more self-contained proof, using only van Kampen's theorem.

Let $W_1$ be the wire obtained from $W$ by adding to it the two straight bars $P$ and $Q$ (see Figure 2) and $W_2$ the wire obtained by adding to $W$ the two straight bars $R$ and $S$. Clearly $W = W_1 \cap W_2$. We call $W_0$ the union of $W_1$ and $W_2$ obtained by attaching to $W$ the four bars $P$, $Q$, $R$, and $S$. Let $S^3$ be the one point compactification of $\mathbb{R}^3$. We think of the extra point as the unique point at infinity and we choose it as the base point for the various fundamental groups that are going to appear. This simplifies the drawing and the deformation of loops: instead of beginning and ending at the same point of $\mathbb{R}^3$, we let them come from an infinite distance (from an arbitrary direction) and return to infinity in a possibly different direction. Let $X = S^3 \setminus W$ and $X_i = S^3 \setminus W_i$ for $i = 0, 1, 2$, so that $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. By van Kampen's theorem (see, for instance, [**1**, chap. 4, sec. 2]) $\pi_1(X)$ is the amalgamated product of $\pi_1(X_1)$ and $\pi_1(X_2)$ over $\pi_1(X_0)$. It is not difficult to see that there exists a deformation of $S^3$ that transforms $W_1$ into a wedge of plane (in particular, unknotted) circles (more precisely, solid tori), like the wire in Figure 3.

Our first task is to determine the fundamental group of its complement. The fundamental group of the complement—call it $Y_1$—of a single unknotted circle is infinite cyclic, generated by a loop that crosses the disc spanned by the circle only once. This follows, for instance, from the well-known decomposition of $S^3$ into two solid tori (see [**1**, chap. 4, sec. 6]) showing that the complement of an unknotted solid torus in $S^3$ is homeomorphic to a solid torus. Assume, by induction, that the fundamental group of the complement $Y_n$ of a wedge of $n$ circles is the free group of rank $n$. Then, given

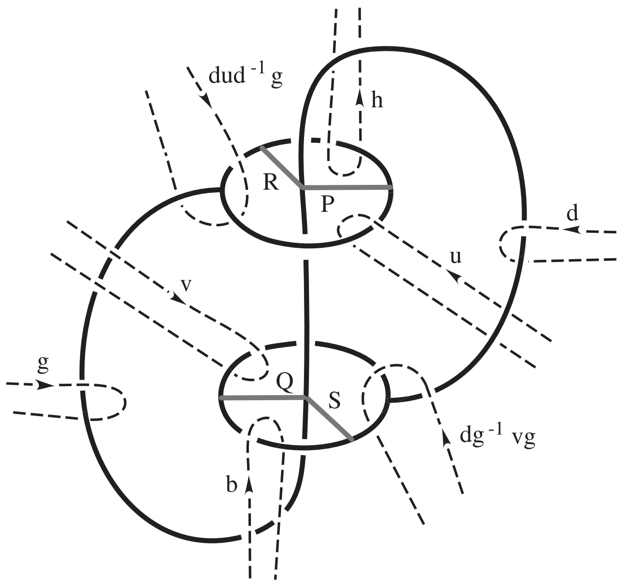


**Figure 2.**

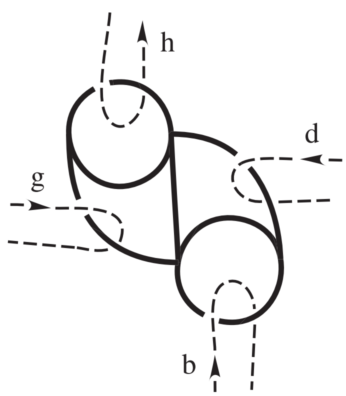


**Figure 3.**

$n+1$ circles $C_1, \ldots, C_{n+1}$, we can span a disc in $C_{n+1}$, obtaining a wedge of $n$ circles whose complement we call $Y'$. We can also span a disc in each of $C_1, \ldots, C_n$ to obtain a single circle, whose complement we call $Y''$. Clearly $Y_{n+1} = Y' \cup Y''$ and the intersection $Y' \cap Y''$ is simply connected. By van Kampen's theorem $\pi_1(Y_{n+1})$ is the free product of $\pi_1(Y')$ and $\pi_1(Y'')$ and is, therefore, free of rank $n+1$, its generators being obvious. Applying this to our case we see that $\pi_1(X_1)$ is freely generated by the (homotopy classes of the) loops $g$, $d$, $h$, and $b$. The same holds for $\pi_1(X_2)$. When we deform $W_0$ into a wedge of circles we get two more circles. Thus, to generate $\pi_1(X_0)$ we must add to $g, d, h$, and $b$ the free generators $u$ and $v$ (see Figure 2). A free presentation of $\pi_1(X)$ is therefore given by

$$\pi_1(X) = \left\langle g, d, h, b;\ \epsilon_1(u) = \epsilon_2(u), \epsilon_1(v) = \epsilon_2(v) \right\rangle,$$

where

$$\epsilon_i : \pi_1(X_0) \to \pi_1(X_i)$$

is the homomorphism induced by the inclusion. It is clear that $\epsilon_2(u) = h$ (after removal of the bar $P$, the loop $u$ can be deformed into $h$) and, similarly, that $\epsilon_2(v) = b$. After removing the bar $S$ from $W_0$, the loop $b$ can be deformed into $dg^{-1}vg$. Similarly, after removing $R$, we can deform the loop $h$ into $dud^{-1}g$. Hence

$$\pi_1(X) = \left\langle g, d, h, b;\ b = dg^{-1}bg, h = dhd^{-1}g \right\rangle = \left\langle d, h, b;\ d = \left[b, [h^{-1}, d]\right]\right\rangle.$$

It remains to show that $d$ is not trivial in $\pi_1(X)$. Mapping $d$, $h$, and $b$ to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

respectively, yields a homomorphism $\varphi : \pi_1(X) \to GL_2(\mathbb{Q})$ with $\varphi(d) \neq 1$. It follows that $d \neq 1$.

REFERENCES

1. W. S. Massey, *A Basic Course in Algebraic Topology*, Springer-Verlag, Berlin, 1991.
2. P. Van Delft and J. Botermans, *Creative Puzzles of the World*, Abrams, New York, 1975.

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# The Ascoli-Arzelà Theorem via Tychonoff's Theorem

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The purpose of this note is to point out that the Ascoli-Arzelà theorem may be derived as an immediate consequence of the Tychonoff theorem.

Suppose then that $K$ is a compact Hausdorff space. Let $C(K)$ denote the space of continuous complex-valued functions on $K$. We say that a subfamily $\mathcal{F}$ of $C(K)$ is *pointwise bounded* if for every $x$ in $K$ there exists $r(x) > 0$ such that $|f(x)| \leq r(x)$ for all $f$ in $\mathcal{F}$, and that $\mathcal{F}$ is *equicontinuous* if for each $x$ in $K$ and $\epsilon > 0$ there exists $U$, a neighborhood of $x$, such that $|f(x) - f(y)| < \epsilon$ for all $f$ in $\mathcal{F}$ whenever $y$ belongs to $U$.

**Theorem (Ascoli-Arzelà).** *If $K$ is a compact Hausdorff space then any pointwise bounded and equicontinuous sequence of functions in $C(K)$ has a subsequence converging uniformly on $K$.*

*Proof.* Suppose that $r(x) > 0$ for every $x$ in $K$ and that for each $x$ in $K$ and $\epsilon > 0$ we are given $\omega(x, \epsilon)$, a neighborhood of $x$. Let $\mathcal{F}$ denote the family of all functions $f : K \to \mathbb{C}$ such that $|f(x)| \leq r(x)$ for all $x$ in $K$ and $|f(x) - f(y)| \leq \epsilon$ whenever $y$ belongs to $\omega(x, \epsilon)$. Since any sequence in a compact metric space has a convergent subsequence, we need show only that $\mathcal{F}$ is a compact subset of $C(K)$ (in the topology of uniform convergence).

Note first that $\mathcal{F}$ lies in $C(K)$: it is clear from the definition of $\mathcal{F}$ that every element of $\mathcal{F}$ is continuous. Note as well that $\mathcal{F}$ is contained in $\prod_{x\in K} \overline{D}(0, r(x))$, where $\overline{D}(0, r(x))$ is the closed disk in the complex plane with center 0 and radius $r(x)$. Let $\mathcal{F}_1$ denote $\mathcal{F}$ with the topology of uniform convergence, and let $\mathcal{F}_2$ denote $\mathcal{F}$ with the topology it inherits as a subspace of $\prod_{x\in K} \overline{D}(0, r(x))$. We will show that $\mathcal{F}_2$ is a closed subspace of $\prod_{x\in K} \overline{D}(0, r(x))$ and that the identity map from $\mathcal{F}_2$ to $\mathcal{F}_1$ is continuous. It will then follow from the Tychonoff theorem that $\mathcal{F}_2$ (and hence $\mathcal{F}_1$) is compact.

The fact that $\mathcal{F}_2$ is a closed subset of $\prod_{x\in K} \overline{D}(0, r(x))$ is clear: if $< f_\alpha >_{\alpha\in A}$ is a net in $\mathcal{F}_2$ converging to $f$ in $\prod_{x\in K} \overline{D}(0, r(x))$ then $f_\alpha(x) \to f(x)$ for each $x$ in $K$, so $f$ belongs to $\mathcal{F}_2$.

The proof that the identity mapping from $\mathcal{F}_2$ to $\mathcal{F}_1$ is continuous is proved as in the traditional proof of the Ascoli-Arzelà Theorem: Suppose that $\epsilon > 0$. The fact that $K$ is compact shows that there exist finitely many points $x_1, \dots x_n$ of $K$ such that $K = \bigcup_{j=1}^n \omega(x_j, \epsilon)$. Now if $< f_\alpha >_{\alpha\in A}$ is a net converging to $f$ in $\mathcal{F}_2$ then there exists $\alpha_0$ in $A$ such that $|f_\alpha(x_j) - f(x_j)| < \epsilon$ for $j = 1, \dots, n$ whenever $\alpha \geq \alpha_0$. Now for any such $\alpha$ and any $x$ in $K$ we may choose $j$ in $\{1, \dots, n\}$ such that $x$ belongs to $\omega(x_j, \epsilon)$. It follows that

$$|f_\alpha(x) - f(x)| \leq |f_\alpha(x) - f_\alpha(x_j)| + |f_\alpha(x_j) - f(x_j)| + |f(x_j) - f(x)| < 3\epsilon.$$

We conclude that the supremum of $|f_\alpha - f|$ is no larger than $3\epsilon$ whenever $\alpha \geq \alpha_0$. ∎

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# Klee's Trigonometry Problem

**PL. Kannappan**

**1. INTRODUCTION.** V. L. Klee posed the following problem in this MONTHLY [**6**].

**Problem.** Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$g(x - y) = g(x)g(y) + f(x)f(y) \tag{1}$$

for $x$ and $y$ in $\mathbb{R}$, and that $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$. Prove that $f$ and $g$ satisfy

$$g(x + y) = g(x)g(y) - f(x)f(y) \tag{2}$$

and

$$f(x \pm y) = f(x)g(y) \pm g(x)f(y) \tag{3}$$

for all real $x$ and $y$.

A solution by T. S. Chihara appeared in this MONTHLY [**3**], but it unfortunately had a gap. We first determine the general solution of (1) wihout any additional conditions and obtain (2) and (3) in the process. We then give a simple and direct solution to Klee's problem with the added conditions.

**Remark.** Compare (1)–(3) with the familiar trigonometric formulas

$$\cos(x \mp y) = \cos x \cos y \pm \sin x \sin y,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$$

**2. SOLUTION OF (1) ON GROUPS.** Let $f, g : G \to \mathbb{C}$ (where $G$ is an Abelian, two divisible group, and $\mathbb{C}$ denotes the field of complex numbers) satisfy (1) for $x$ and $y$ in $G$. (A group $G$ is *two divisible* if each $x$ in $G$ can be expressed in the form $x = y + y = 2y$ for some (unique) $y$ in $G$. Thus $x/2$ is meaningful for $x$ in $G$.) We determine the general solution of (1) (see [**1**], [**2**], [**4**], [**5**], [**7**], [**9**], [**10**], and [**11**]).

First we consider constant solutions of (1). If $g \equiv c$, a constant, and $f \not\equiv 0$, then (1) gives that $f \equiv d$, a constant for which $d^2 + c^2 = c$, but this won't satisfy (2). If $f \equiv d$, a constant, then (1) becomes

$$g(x - y) = g(x)g(y) + d^2 = g(y - x).$$

In particular, $g$ is even (i.e., $g(-x) = g(x)$), so $g(x - y) = g(x + y)$ (replace $y$ by $-y$ in the foregoing identity and appeal to the evenness of $g$). Then with $x = (u + v)/2$ and $y = (u - v)/2$ for any $u$ and $v$ in $G$, we have $g(v) = g(u)$, that is, $g$ is a constant (here the two divisibility of $G$ is used). Identity (2) won't hold for such solutions unless $d = 0 = c$.

We henceforth consider only the nonconstant (nontrivial) solutions $f$ and $g$ of (1). Interchange of $x$ and $y$ in (1) yields

$$g(x - y) = g(y - x),$$

whence $g$ is even.

Replace $y$ by $-y$ in (1) to get

$$g(x + y) = g(x)g(y) + f(x)f(-y) \tag{4}$$

for $x$ and $y$ in $G$. We show that $f$ is odd (i.e., $f(-x) = -f(x)$). Change $x$ to $-x$ in (1) to obtain

$$g(x + y) = g(x)g(y) + f(-x)f(y) \tag{5}$$

for $x$ and $y$ in $G$. Thus

$$f(x)f(-y) = f(-x)f(y).$$

Since $f \not\equiv 0$,

$$f(x) = kf(-x) = k^2 f(x)$$

for all $x$ in $G$, where $k$ is a constant such that $k^2 = 1$.

If $k = 1$, then $f$ is even and (1) and (4) imply that $g(x - y) = g(x + y)$ and then $g$ is constant (here again two divisibility of $G$ is used), contrary to assumption. Hence

$k = -1$, $f$ is odd, and (4) becomes

$$g(x + y) = g(x)g(y) - f(x)f(y),$$

which is (2), and $g$ satisfies the *cosine equation* (6) (add (1) and (2)):

$$g(x + y) + g(x - y) = 2g(x)g(y). \tag{6}$$

Further, applying associativity and (2), we get

$$g(x + y + z) = \big[g(x)g(y) - f(x)f(y)\big]g(z) - f(x + y)f(z)$$

and

$$g(x + y + z) = g(x)\big[g(y)g(z) - f(y)f(z)\big] - f(x)f(y + z),$$

that is,

$$\big[f(x + y) - f(y)g(x)\big]f(z) = \big[f(y + z) - f(y)g(z)\big]f(x)$$

or

$$f(x + y) - f(y)g(x) = h(y)f(x), \tag{7}$$

where

$$h(y) = \frac{1}{f(z_0)}\big[f(y + z_0) - f(y)g(z_0)\big]$$

with $f(z_0) \neq 0$.

Change $x$ to $-x$ in (7) and use the fact that $f$ is odd and $g$ even to conclude that

$$f(x - y) = -f(y)g(x) + f(x)h(y), \tag{8}$$

that is (add (7) and (8)),

$$f(x + y) + f(x - y) = 2f(x)h(y) \tag{9}$$

for $x$ and $y$ in $G$. Interchange $x$ and $y$ in (9) to see that

$$f(x + y) - f(x - y) = 2f(y)h(x) \tag{10}$$

for $x$ and $y$ in $G$. Addition of (9) and (10) yields

$$f(x + y) = f(x)h(y) + f(y)h(x). \tag{11}$$

Since $f \not\equiv 0$, (7) and (11) imply that $h(x) = g(x)$ for all $x$ in $G$. Hence (9) becomes

$$f(x + y) = f(x)g(y) + f(y)g(x) \tag{12}$$

for $x$ and $y$ in $G$, which is one part of (3). Replace $y$ by $-y$ in (12) to obtain

$$f(x - y) = f(x)g(y) - f(y)g(x), \tag{13}$$

which is the other part of (3).

Now we are in a position to determine $f$ and $g$. Since $g$ satisfies the cosine equation (6), we must have

$$g(x) = \frac{E(x) + E^*(x)}{2} \tag{14}$$

for $x$ in $G$, where $E : G \to \mathbb{C}^*$ (the nonzero complex numbers) is a homomorphism satisfying the exponential equation $E(x + y) = E(x)E(y)$ and $E^* = 1/E$ [**4**].

Inserting the representation (14) for $g$ into (1) results, after a straightforward computation, in

$$f(x)f(y) = \frac{E(x) - E^*(x)}{2} \cdot \frac{E(y) - E^*(y)}{2}$$

for $x$ and $y$ in $G$. Since $f \not\equiv 0$,

$$f(x) = b\big(E(x) - E^*(x)\big) \tag{15}$$

for $x$ in $G$, with $b^2 = 1/4$. Thus we have proved the following theorem:

**Theorem 1.** *Suppose that* $f, g : G \to \mathbb{C}$ *are nonconstant solutions of* (1)*, where* $G$ *is a two divisible Abelian group. Then* $f$ *and* $g$ *satisfy* (2) *and* (3)*. Moreover, these functions are given by* (14) *and* (15) *for some homomorphism* $E : G \to \mathbb{C}^*$.

Theorem 1 has the following corollary (see [**4**], [**1**], and [**7**]):

**Corollary 1.** *Let* $f, g : \mathbb{R} \to \mathbb{C}$ *be nonconstant solutions of* (1) *with* $g$ *continuous. Then* $f$ *is also continuous, and* $g(x) = \cos(bx)$ *and* $f(x) = k\sin(bx)$*, where* $k^2 = 1/4$ *and* $b$ *is a complex constant.*

**Remark.** As a special case, Theorem 1 provides a solution to Klee's problem and shows that the additional conditions $f(t) = 1$ and $g(t) = 0$ are superfluous.

## 3. SELF-CONTAINED SOLUTION TO KLEE'S PROBLEM.

We return briefly to the original problem posed by Klee and provide a solution that makes no appeal to [**4**].

Let $f, g : \mathbb{R} \to \mathbb{R}$ be solutions of (1). The assumption that $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$ forces $f$ and $g$ to be nonconstant. As earlier, (1) implies that $g$ is even.

Substituting $y = t$ in (1) gives

$$g(x - t) = f(x), \;\; f(x + t) = g(x), \tag{16}$$

so $f(-x) = g(x + t)$ for all real $x$. Now by (16) and (1),

$$\begin{aligned} f(x - y) &= g(x - y - t) \\ &= g(x)g(y + t) + f(x)f(y + t) \\ &= g(x)f(-y) + f(x)g(y) \end{aligned} \tag{17}$$

for $x$ and $y$ in $\mathbb{R}$. Change $y$ to $-y$ in (17) to obtain the "+" half of (3). To verify both the other half of (3) and (2) it is enough to show that $f$ is odd.

Arguing as before, we find that (4) holds, and from it we again infer that

$$f(x)f(-y) = f(y)f(-x)$$

for all real $x$ and $y$.

Now setting $y = t$ and $x = -t$ yields $f(-t)^2 = 1$, so $f(-t) = \pm 1$. The choice $f(-t) = 1$ leads to the conclusion that $f$ is even and $g$ is constant, which is not the case. Thus $f$ is odd and (17) and (4) become the "$-$" half of (3) and (2), respectively. This furnishes a solution to Klee's problem. Note that we have used the conditions $f(t) = 1$, $g(t) = 0$ a couple of times.

As remarked at the end of the solution of E1079, the usual formula for $\cos(x \pm y)$ and $\sin(x \pm y)$ follow purely algebraically from the formula for $\cos(x - y)$.

**ACKNOWLEDGMENT** This paper is dedicated to Professor Z. Moszner on the occasion of his 73rd birthday.

REFERENCES

1. J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.
2. J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989.
3. T.S. Chihara, Solution to E1079, this MONTHLY **61** (1954) 197.
4. PL. Kannappan, The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups, *Proc. Amer. Math. Soc.* **19** (1968) 69–74.
5. ———, *Theory of Functional Equations*, Matscience Report, no. 48, Institute of Math. Sciences, Chennai, India, 1969.
6. V. L. Klee, E1079, this MONTHLY **60** (1953) 479.
7. S. Kurepa, On some functional equations in Banach spaces, *Studia Math.* **19** (1960) 149–158.
8. S. Parameswaran, Trigonometry retold, *Math. Gazette* **42** (1958) 81–83.
9. A.L. Rukhin, The solution of the functional equation of d'Alembert's type for commutative groups, *Internat. J. Math. Sci.* **5** (1982) 315–355.
10. L. Vietoris, Zur Kennzeichnung des Sinus und verwandter Funktionen durch Funktionalgleichungen, *J. Reine Angew. Math.* **186** (1944) 1–15.
11. W.H. Wilson, On certain related functional equations, *Bull. Amer. Math. Soc.* **26** (1919–1920) 300–312.

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# Green's Theorem and the Fundamental Theorem of Algebra

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One proof of the fundamental theorem of algebra uses Liouville's theorem, which follows from Cauchy's theorem, which in turn can be derived from Green's theorem; see for instance, the beautiful book [**1**]. The purpose of this note is to show that Green's theorem is sufficient. The proof does not use any topology or analytic function theory.

Let $p(w)$ be a polynomial in $w = x + iy$ of degree $n \geq 1$ with complex coefficients. We show that $p(w)$ must have a zero. As usual we proceed by contradiction, assuming that $p(w)$ is not zero for any $w$. Then $f(x, y) := \ln|p(x + iy)|$ is a smooth function of $(x, y)$ in $\mathbb{R}^2$. For $r > 0$, let $D_r$ be the disk of radius $r$ centered at the origin, and let $C_r$ be the boundary of $D_r$ oriented counterclockwise. We will work out both sides of Green's formula

$$\int_{C_r} P\,dx + Q\,dy = \iint_{D_r} (\partial_x Q - \partial_y P)\,dx\,dy, \tag{1}$$

with $P = -\partial_y f$ and $Q = \partial_x f$. We first work out the right-hand side.

In fact, we show that the right-hand side is zero. Note that $\partial_x Q - \partial_y P = \partial_x^2 f + \partial_y^2 f$. Since $f(x, y) = (1/2)\ln|p(x + iy)|^2$, we have

$$\partial_x f = \frac{1}{2}\frac{\partial_x|p(x+iy)|^2}{|p(x+iy)|^2}, \quad \partial_y f = \frac{1}{2}\frac{\partial_y|p(x+iy)|^2}{|p(x+iy)|^2}. \tag{2}$$

Thus,

$$\partial_x^2 f = \frac{1}{2}\frac{\partial_x^2|p|^2}{|p|^2} - \frac{1}{2}\frac{(\partial_x|p|^2)^2}{|p|^4}, \quad \partial_y^2 f = \frac{1}{2}\frac{\partial_y^2|p|^2}{|p|^2} - \frac{1}{2}\frac{(\partial_y|p|^2)^2}{|p|^4}. \tag{3}$$

We claim that $\partial_y p = i\partial_x p$ and $\partial_y \overline{p} = -i\partial_x \overline{p}$. Indeed, by linearity of derivatives, it suffices to prove these identities for $q = (x + iy)^k$. In this case, $q = \sum_{l=0}^{k} \binom{k}{l} x^{k-l}\, i^l\, y^l$ by the binomial theorem. Hence, $\partial_y q = \sum_{l=1}^{k} \binom{k}{l} l\, x^{k-l}\, i^l\, y^{l-1}$. Setting $l = j + 1$ and noting that

$$\binom{k}{j+1}(j+1) = \binom{k}{j}(k-j)$$

give

$$\partial_y q = \sum_{j=0}^{k-1} \binom{k}{j}(k-j)\, x^{k-j-1}\, i^{j+1}\, y^j,$$

which is just $i\partial_x q$. The second of these identities is proved similarly or can be proved by taking the complex conjugate of the first identity. Now employing these identities and Leibniz's rule on $|p|^2 = p\,\overline{p}$, the reader can verify with a straightforward computation using the equations for $\partial_x^2 f$ and $\partial_y^2 f$ found in (3) that $\partial_y^2 f = -\partial_x^2 f$. Thus, $\partial_x Q - \partial_y P = 0$ and so the right-hand side of (1) is zero.

We now estimate the left-hand side of (1). Without loss of generality, we assume that the leading coefficient of $p(w)$ is one. Then $p(x + iy)$ is of the form $(x + iy)^n$ plus a polynomial in $x + iy$ of degree at most $n - 1$. Hence, we can write

$$|p(x+iy)|^2 = p(x+iy)\,\overline{p(x+iy)} = (x^2 + y^2)^n + \widetilde{p}(x, y), \tag{4}$$

where $\widetilde{p}(x, y)$ is a polynomial in the variables $x$ and $y$ of degree at most $2n - 1$. Taking the partials of $|p(x + iy)|^2 = (x^2 + y^2)^n + \widetilde{p}(x, y)$ with respect to $x$ and $y$

and plugging the results into (2), we see that

$$P = -\partial_y f = -\frac{ny}{x^2 + y^2} + \widetilde{P}, \qquad Q = \partial_x f = \frac{nx}{x^2 + y^2} + \widetilde{Q},$$

where

$$\widetilde{P} = -\frac{(x^2 + y^2)\,\partial_y \widetilde{p}(x, y) - 2ny\,\widetilde{p}(x, y)}{2|p(x + iy)|^2\,(x^2 + y^2)},$$
$$\widetilde{Q} = \frac{(x^2 + y^2)\,\partial_x \widetilde{p}(x, y) - 2nx\,\widetilde{p}(x, y)}{2|p(x + iy)|^2\,(x^2 + y^2)}.$$

Using the curve $c_r(t) := (r\cos t, r\sin t)$, which traces out $C_r$ for $0 \leq t \leq 2\pi$, a direct computation gives

$$\int_{C_r} \frac{-ny}{x^2 + y^2}\,dx + \frac{nx}{x^2 + y^2}\,dy = 2\pi n.$$

Thus,

$$\int_{C_r} P\,dx + Q\,dy = 2\pi n + g(r), \tag{5}$$

where

$$g(r) = \int_{C_r} \widetilde{P}\,dx + \widetilde{Q}\,dy.$$

We analyze $g(r)$ as follows. Since $\widetilde{p}(x, y)$ is a polynomial in $x$ and $y$ of degree at most $2n - 1$, $\partial_x \widetilde{p}(x, y)$ and $\partial_y \widetilde{p}(x, y)$ are polynomials in $x$ and $y$ of degree at most $2n - 2$. Hence, the numerators of $\widetilde{P}$ and $\widetilde{Q}$ are polynomials in $x$ and $y$ of degree at most $2n$. As a result, these numerators are each bounded, in absolute value, by a constant times $(x^2 + y^2)^n$. Since $\widetilde{P}$ and $\widetilde{Q}$ contain $|p(x + iy)|^2\,(x^2 + y^2)$ in their denominators, in view of (4) it follows that $|\widetilde{P}|$ and $|\widetilde{Q}|$ are each bounded by a constant times $(x^2 + y^2)^{-1}$. These estimates on $|\widetilde{P}|$ and $|\widetilde{Q}|$ imply that $|g(r)|$ is bounded by a constant times $r^{-1}$. Since the right-hand side of (1) was shown to be zero, letting $r \to \infty$ in (5) gives the contradiction $0 = 2\pi n$. Thus, our original assumption that $p(w)$ has no zero must be false.

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REFERENCES

1. B. Fine and G. Rosenberger, *The Fundamental Theorem of Algebra*, Springer-Verlag, New York, 1997.

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# Mean Curvature and Asymptotic Volume of Small Balls

**Dominique Hulin and Marc Troyanov**

The aim of this note is to prove an asymptotic formula relating the mean curvature of a hypersurface at a given point and the volume of small balls centered at this point. For instance, consider a sphere $S_R$ of radius $R$ in Euclidean three-space $\mathbb{R}^3$. If $p$ lies on $S_R$ and $t > 0$ is small enough (i.e., $0 < t \leq R$), then

$$\frac{\operatorname{Vol}\left(B_p^+(t)\right)}{\operatorname{Vol}\left(B_p(t)\right)} = \frac{1}{2} - \frac{3}{16}\frac{1}{R}\,t,$$

where $B_p(t)$ is the ball of radius $t$ centered at $p$ and $B_p^+(t)$ is the portion of the ball lying inside the sphere $S_R$. Our goal is to show that, up to a negligible term, a similar formula holds for any hypersurface $S$ in $\mathbb{R}^n$, the factor $1/R$ being replaced with the mean curvature of the hypersurface.

We briefly recall what mean curvature is. Let $S$ be a hypersurface of class $C^2$ in Euclidean $n$-space $\mathbb{R}^n$, and assume that a unit normal vector field $N : S \to \mathbb{R}^n$ has been chosen (this is always possible locally). It is a basic fact that the normal acceleration $\langle c''(t), N(c(t))\rangle$ of a $C^2$-curve $c(t)$ on $S$ depends only on its tangent vector $V(t) = c'(t)$: indeed,

$$0 = \frac{d}{dt}\big\langle c'(t), N\big(c(t)\big)\big\rangle = \big\langle c''(t), N\big(c(t)\big)\big\rangle + \left\langle c'(t), \frac{d}{dt}N\big(c(t)\big)\right\rangle,$$

so $\langle c''(t), N(c(t))\rangle = -\langle V, D_V N\rangle$ (with the notation $D_V N = d(N(c(t))/dt)$.

The *second fundamental form* of the hypersurface $S$ at a point $p$ of $S$ is the bilinear form defined on the tangent space $T_pS$ by

$$\mathrm{II}_p(v, w) = -\langle v, D_w N\rangle.$$

The previous calculation shows that $\mathrm{II}(c'(t), c'(t))$ is the normal acceleration of the curve. It is not difficult to check that $\mathrm{II}_p$ is a symmetric bilinear form. The *mean curvature* $H(p)$ of the hypersurface $S$ at the point $p$ is then defined as

$$H(p) = \frac{1}{n-1}\,\operatorname{Trace}(\mathrm{II}_p) = \sum_{i=1}^{n-1} \mathrm{II}(v_i, v_i),$$

where $v_1, \ldots, v_{n-1}$ is any orthonormal basis of the tangent space $T_pS$. (Observe that replacing the normal field $N$ by $-N$ changes the sign of the mean curvature.) The mean curvature and the second fundamental form of a surface in $\mathbb{R}^3$ are classical topics in geometry (see, for instance, [**1**, chap. 5]).

Let us now fix a ball $B_p(t)$ in $\mathbb{R}^n$ with radius $t > 0$ and center $p$ on $S$. When $t$ is small enough, the hypersurface $S$ separates $B_p(t)$ into two connected components $B_p^+(t)$ and $B_p^-(t)$, with the convention that $N_p$ points towards $B_p^+(t)$. We want to relate the mean curvature of $S$ at $p$ to the ratio of the volumes of $B_p^+(t)$ and $B_p(t)$.

It is clear that for small values of $t$ the volume of $B_p^+(t)$ is roughly one-half the volume of the entire ball $B_p(t)$; in fact, we have

$$\mathrm{Vol}\left(B_p^+(t)\right) = \frac{1}{2}\alpha_n t^n + O(t^{n+1}),$$

where $\alpha_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. We claim: *the next term of the Taylor expansion of this volume is given by*

$$\mathrm{Vol}\left(B_p^+(t)\right) = \frac{1}{2}\alpha_n t^n - \frac{1}{2}\left(\frac{n-1}{n+1}\alpha_{n-1}\right) H(p)\, t^{n+1} + O(t^{n+2}). \tag{1}$$

In particular, the mean curvature satisfies

$$H(p) = 2\left(\frac{n+1}{n-1}\frac{\alpha_n}{\alpha_{n-1}}\right)\lim_{t\to 0}\left[\frac{1}{t}\left(\frac{1}{2} - \frac{\mathrm{Vol}(B_p^+(t))}{\mathrm{Vol}(B_p(t))}\right)\right]. \tag{2}$$

To prove (1), we choose an orthonormal coordinate system centered at $p$ such that $N_p = (0, \dots, 0, 1)$. The hypersurface is then locally defined as a graph

$$x_n = f(x_1, \dots, x_{n-1}),$$

where $f$ is a smooth function satisfying $f(0) = 0$ and $\partial f/\partial x_i(0) = 0$ for $i = 1, \dots, n-1$. One then easily checks that the second fundamental form at $p = 0$ coincides with the Hessian of $f$ at this point. Thus for $X = (x_1, \dots, x_{n-1})$ in $\mathbb{R}^{n-1}$, we have $f(X) = \frac{1}{2}\mathrm{II}(X, X) + o(\|X\|^2)$.

It is convenient to begin with an analogue of estimate (1) in which balls are replaced by cylinders. Namely, we set

$$C(t) = \left\{x = (X, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \max(\|X\|, |x_n|) \le t\right\}.$$

If $\rho > 0$ is small enough, then $f$ is well defined on $\{X \in \mathbb{R}^n : \|X\| \le \rho\}$ and we consider the intersection $C^+(t) = \{x = (X, x_n) \in C(t) : x_n \ge f(X)\}$ of the cylinder $C(t)$ with the epigraph of $f$, where $0 \le t \le \rho$. We then have

$$\mathrm{Vol}\left(C^+(t)\right) = \frac{1}{2}\,\mathrm{Vol}\left(C(t)\right) - \int_{B^{n-1}(t)} f(X)\, dX,$$

where $B^{n-1}(t) = \{X \in \mathbb{R}^{n-1} : \|X\| \le t\}$. Now observe that for $i, j = 1, \dots, n-1$ such that $i \ne j$ we have

$$\int_{B^{n-1}(t)} x_i x_j\, dX = 0,$$

while for $i = 1, \dots, n-1$

$$\int_{B^{n-1}(t)} x_i^2\, dX = \frac{1}{n-1}\int_{B^{n-1}(t)} \|X\|^2\, dX = \left(\frac{1}{n+1}\,\alpha_{n-1}\right) t^{n+1}.$$

Combining these three identities, we obtain

$$\operatorname{Vol}\big(C^+(t)\big) = \frac{1}{2}\operatorname{Vol}\big(C(t)\big) - \frac{1}{2}\left(\frac{n-1}{n+1}\,\alpha_{n-1}\right) H(p)\, t^{n+1} + O(t^{n+2}), \qquad (3)$$

since $f(X) = \frac{1}{2}\mathrm{II}(X, X) + o(\|X\|^2)$ and $H(p) = \mathrm{Trace}(\mathrm{II}_p)/(n-1)$.

We still have to verify that replacing cylinders with balls does not affect the significant term in our Taylor expansion, i.e., we need to show that

$$\left(\operatorname{Vol}\big(C^+(t)\big) - \frac{1}{2}\operatorname{Vol}\big(C(t)\big)\right) - \left(\operatorname{Vol}\big(B^+(t)\big) - \frac{1}{2}\operatorname{Vol}\big(B(t)\big)\right)$$

is at most of the order $t^{n+2}$. We in fact obtain a better estimate. For any $c$ in $\mathbb{R}$ set

$$A_c(t) = \left\{x = (X, x_n) \in C(t) \setminus B(t) : x_n \geq ct^2\right\};$$

in particular,

$$\operatorname{Vol}(A_0(t)) = \frac{1}{2}\operatorname{Vol}\big(C(t)\big) - \frac{1}{2}\operatorname{Vol}\big(B(t)\big).$$

The volume of $A_{-c}(t) \setminus A_c(t)$ is of the order $t^{n+4}$, since this set is contained in a cylinder of height $2ct^2$ whose base is an annulus of outer radius $t$ and inner radius roughly $t - c^2t^3/2$. More precisely,

$$A_{-c}(t) \setminus A_c(t) \subset \left\{x = (X, x_n) \,\middle|\, \sqrt{t^2 - c^2t^4} \leq \|X\| \leq t,\ |x_n| \leq ct^2\right\};$$

hence

$$0 < \operatorname{Vol} A_{-c}(t) - \operatorname{Vol} A_c(t) \leq 2\alpha_{n-1}ct^2\left(t^{n-1} - (\sqrt{t^2 - c^2t^4})^{n-1}\right) = O(t^{n+4}).$$

Because $A_{-c}(t) \subset A_0(t) \subset A_c(t)$, we thus obtain

$$\operatorname{Vol}(A_c(t)) = \operatorname{Vol}(A_0(t)) + O(t^{n+4}) = \frac{1}{2}\operatorname{Vol}(C(t)) - \frac{1}{2}\operatorname{Vol}(B(t)) + O(t^{n+4})$$

for all $c$. The same estimate also holds for $\operatorname{Vol} A_{-c}(t)$.

If we now choose the positive constants $c$ and $\rho > 0$ so that $|f(X)| \leq c\,\|X\|^2$ whenever $\|X\| < \rho$, then we have for $t \leq \rho$:

$$A_{-c}(t) \ \subset\ C^+(t) \setminus B^+(t) \ \subset\ A_c(t).$$

In this way we arrive at

$$\operatorname{Vol}\big(A_{-c}(t)\big) \leq \operatorname{Vol}\big(C^+(t)\big) - \operatorname{Vol}\big(B^+(t)\big) \leq \operatorname{Vol}\big(A_c(t)\big),$$

and the previous estimates implies that

$$\operatorname{Vol}\big(C^+(t)\big) - \operatorname{Vol}\big(B^+(t)\big) = \frac{1}{2}\operatorname{Vol}\big(C(t)\big) - \frac{1}{2}\operatorname{Vol}\big(B(t)\big) + O(t^{n+4}).$$

It is now clear that the estimate (1) follows from (3). ∎

The notions of the second fundamental form and the mean curvature both extend without substantial modification to the setting of hypersurfaces in Riemannian manifolds, see for instance [**2**, pp. 132–142]. We conclude this note by showing that the Taylor expansion (1) still holds in this context.

Let $S$ be a hypersurface in a Riemannian manifold $(M^n, g)$, and let $N$ be a (local) unit normal vector field defined in a neighbourhood of the point $p$ on $S$. The exponential map $\phi = \exp_p : U \subset T_pM \to M$ is well defined in a neighbourhood $U$ of 0 in $T_pM$. Moreover, when $t > 0$ is small enough, $\phi$ is a diffeomorphism from the Euclidean ball $\widetilde{B}_0(t)$ of radius $t$ and center 0 in $T_pM$ to the Riemannian ball $B_p(t)$ in $M$. Denote by $\widetilde{S}$ the Euclidean hypersurface $(\phi|_U)^{-1}(S)$ and by $\widetilde{N}$ the vector field $(\phi^{-1})_*N$.

From the fact that the 1-jets of the pull-back metric $\phi^*g$ and of the Euclidean metric $g_p$ on $T_pM$ coincide at the origin (see [**2**, Proposition 5.11, p. 78]), we infer that:

(i) the mean curvature $H(p)$ of $S$ at $p$, and $\widetilde{H}(0)$ of $\widetilde{S}$ at the origin (with respect to the normal $\widetilde{N}$) are equal;

(ii) $\mathrm{Vol}_{\mathrm{E}}(\widetilde{B}_0(t)) = \mathrm{Vol}_g(B_p(t)) + O(t^{n+2})$ (and similarly for $\widetilde{B}_0^+(t)$ and $B_p^+(t)$).

It follows that the formula (1) is still valid in the Riemannian setting without any correction. ∎

REFERENCES

1. B. O'Neill, *Elementary Differential Geometry*, Academic Press, New York 1966.
2. J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature*, Springer-Verlag, New York, 1997.

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# Timing Is Everything: The French Connection

**C. W. Groetsch**

**1. INTRODUCTION.** Marin Mersenne (1588–1648) is remembered chiefly as a one-man scientific clearing house—the Internet of his day. During the first half of the seventeenth century, when mathematical physics was taking its first tentative steps guided by Descartes and Galileo, "Mersenne did more to propagate emerging new sciences of acoustics, pneumatics, and ballistics than anyone else of his time" [**4**, p. 20]. But Mersenne was more than just a go-between and publicist. He also performed "careful experiments" [**5**, p. 186] to test some of the new theories. Galileo's parabolic model for the ballistic trajectory of a particle in a resistanceless medium was one of the "hot" new theories of the time. This model, which is featured in nearly every elementary calculus textbook, implies that the ascent time and descent time of the projectile are equal. However, while performing measurements on projectiles launched by a crossbow, Mersenne observed in 1644 (see [**6**, p. 107]) that "the time of ascent is always less than the time of descent, the difference increasing with the velocity of projection." Mersenne's experiment was foreshadowed by his remark published in 1636 that "les missiles vont plus viste en partant de l'arc ... qu'en retombant" [**7**, p. 100] and by a letter from Descartes to Mersenne in April 1643 (see [**3**, p. 657]) in which Descartes also noted that the difference in times is diminished for shots of lower altitude, i.e., with lower velocity of projection.

In this note we validate both of Mersenne's observations for the simplest model of a resisting medium, one that resists motion in proportion to the speed. For vertical flight, Brauer has validated the first of Mersenne's observations for an *arbitrary* resisting medium [**1**]. Our aim is quite modest: we provide a rigorous analytical treatment, at the level of elementary calculus, of a physical problem with significant historical roots. The elementary arguments in this note, along with the historical development of the problem, has been used as enrichment material in various undergraduate courses. Both of Mersenne's observations (as well as a geometric consequence) are established here on the basis of a single simple analytical lemma, whose easy proof can serve as an independent class exercise.

**2. THE MODEL AND A TOOL.** We begin with a review of the familiar model for the motion of a point projectile of unit mass launched from the origin into a medium that resists motion in proportion to the velocity vector. If the particle is launched with an initial speed $v$ at an angle $\theta$ in $(0, \pi/2]$ to the horizontal, then its coordinates satisfy the equations

$$\begin{aligned} \ddot{x}(t) &= -k\dot{x}(t), & \dot{x}(0) &= v\cos\theta, & x(0) &= 0, \\ \ddot{y}(t) &= -g - k\dot{y}(t), & \dot{y}(0) &= v\sin\theta, & y(0) &= 0, \end{aligned} \tag{1}$$

where $g$ is the gravitational acceleration constant and $k$ is a positive resistance constant. The dots indicate temporal derivatives.

A single integration of the equations in (1) yields

$$\begin{aligned} \dot{x}(t) &= (v\cos\theta)e^{-kt}, \\ \dot{y}(t) &= (v\sin\theta)e^{-kt} - \frac{g}{k}(1 - e^{-kt}). \end{aligned} \tag{2}$$

The *ascent time* $t_a$, characterized by $\dot{y}(t_a) = 0$, is then

$$t_a = \frac{1}{k}\ln\left(1 + \frac{vk\sin\theta}{g}\right) = \frac{1}{k}\ln c, \tag{3}$$

where here and henceforth we set $c = 1 + vk\sin\theta/g$ in the interest of tidier notation. An additional integration of (2) results in parametric equations for the trajectory:

$$\begin{aligned} x(t) &= \frac{v\cos\theta}{k}(1 - e^{-kt}), \\ y(t) &= \frac{gc}{k^2}(1 - e^{-kt}) - \frac{gt}{k}. \end{aligned} \tag{4}$$

The *flight time* $t_f$, that is, the positive root of $y$, is characterized implicitly by

$$t_f = \frac{c}{k}(1 - e^{-kt_f}). \tag{5}$$

If we set $p = kt_f/c$, then we see that $p$ satisfies

$$p = (1 - e^{-cp}). \tag{6}$$

This equation is the key to all that follows. A simple sketch (see Figure 1) suggests the following:

**Proposition.** *For $c > 1$ equation* (6) *has a unique positive solution $p$. Also, $s > p$ if and only if $s > 1 - e^{-cs}$.*

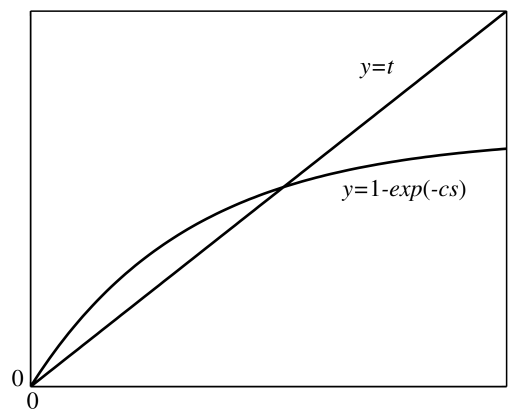


**Figure 1.** The fixed point $p$.

An analytical proof follows the picture. The function $f(s) = 1 - e^{-cs} - s$ vanishes at $s = 0$, is positive for sufficiently small positive $s$ (since $f'(0) = c - 1 > 0$), and is negative for large enough $s$. Hence, $f$ has a positive root $p$. Were $f$ to have another positive root, then $f'$ would vanish twice (since $f(0) = 0$), which is impossible because $f'$ is strictly decreasing. The same argument shows that $f(s) > 0$ for $s$ in $(0, p)$ and $f(s) < 0$ for $s$ in $(p, \infty)$.

**3. TIMING.** The *descent time* $t_d$ is the difference between the flight time and the ascent time:

$$t_d = t_f - t_a.$$

Therefore, the descent time is longer than the ascent time if and only if $t_f > 2t_a$. But by (3) this is the same as

$$p = \frac{k}{c}t_f > \frac{2k}{c}t_a = \frac{2}{c}\ln c. \tag{7}$$

According to the proposition, (7) is equivalent to

$$\frac{c}{2}(1 - e^{-2\ln c}) > \ln c. \tag{8}$$

Now by (3) $c = e^{kt_a}$, so (8) may be expressed as

$$\sinh(kt_a) > kt_a,$$

which is of course true whenever $kt_a > 0$. Therefore $t_d > t_a$, that is, the descent time exceeds the ascent time.

We now turn to Mersenne's second observation: the difference $t_d - t_a = t_f - 2t_a$ increases with the initial velocity $v$. Since

$$\frac{d}{dv}(t_f - 2t_a) = \frac{d}{dc}(t_f - 2t_a)\frac{k\sin\theta}{g},$$

it suffices to show that

$$0 < \frac{d}{dc}(t_f - 2t_a) = \frac{dt_f}{dc} - \frac{2}{kc}. \tag{9}$$

The characterization (5) implies that

$$\frac{dt_f}{dc} = \frac{1}{k}(1 - e^{-kt_f}) + ce^{-kt_f}\frac{dt_f}{dc}$$

(the fact that $t_f$ is a differentiable function of $c$ is an easy consequence of the implicit function theorem [**2**]), and hence

$$\frac{dt_f}{dc} = \frac{1 - e^{-kt_f}}{k(1 - ce^{-kt_f})}.$$

However, since $p = kt_f/c$, condition (9) is equivalent to

$$\frac{1 - e^{-cp}}{1 - ce^{-cp}} > \frac{2}{c}$$

or, taking (6) into account,

$$\frac{p}{1 - c(1 - p)} > \frac{2}{c}.$$

Equivalently,

$$p < \frac{2(c-1)}{c}. \tag{10}$$

To sum up, the increase with respect to the initial velocity of the difference between the descent and ascent times is equivalent to condition (10). But the proposition says that (10) is the same as

$$1 - e^{-2(c-1)} < \frac{2(c-1)}{c}$$

or

$$\frac{2-c}{c} < e^{-2(c-1)}, \tag{11}$$

which is clearly true if $c \geq 2$. On the other hand, if $1 < c < 2$, then on setting $d = c - 1$ and rearranging, we find that (11) is the same as

$$e^{2d} < \frac{1+d}{1-d}$$

when $0 < d < 1$. But this statement is seen to be true, for example, by comparing the series representations of the expressions on each side of the inequality. Therefore, $d/dv(t_d - t_a) > 0$, validating Mersenne's second observation.

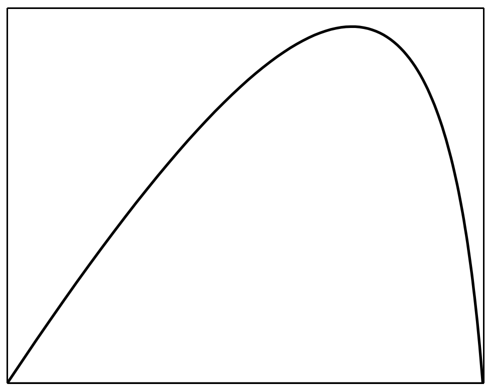

**Figure 2.**

**4. THE ARC.** Until now we have used information about the vertical coordinate of the projectile only. The timing results may therefore be recast in terms of vertical flight with initial velocity $v \sin\theta$. However, our results are also revealing of the nonvertical trajectory itself. A typical plot of the trajectory given by the parametric equations (4) (see Figure 2) shows how air resistance breaks the beautiful symmetry of Galileo's model. Given that the descent time exceeds the ascent time, it is at first sight a bit surprising to see the apex shifted to the right instead of to the left. A little thought convinces one that the physical reason for this is the exponential decay of the horizontal component of the velocity, but the shift to the right is also a mathematical consequence of our work in the previous section. The horizontal range of the shot is given by

$$\begin{aligned} x(t_f) &= \frac{v\cos\theta}{k}(1 - e^{-kt_f}) \\ &= v\cos\theta\left(\frac{t_f}{c}\right) = \frac{v\cos\theta}{k}p, \end{aligned}$$

and by (3) and (4) the apex has horizontal coordinate

$$x(t_a) = \frac{v\cos\theta}{k}\left(1 - \frac{1}{c}\right).$$

Therefore, the apex is to the right of midrange (i.e., $x(t_a) > x(t_f)/2$) if and only if

$$\left(1 - \frac{1}{c}\right) > p/2,$$

that is, if and only if

$$2\frac{(c-1)}{c} > p,$$

a fact that was established in the previous section.

REFERENCES

1. F. Brauer, What goes up must come down, eventually, this MONTHLY **108** (2001) 437–440.
2. R. Courant, *Differential and Integral Calculus*, vol. 2, Interscience, New York, 1962.
3. R. Descartes, *Oeuvres: Correspondance III*, Facsimile Edition, C. Adam and P. Tannery, eds., Centre National de la Recherche Scientifique, Paris, 1971.
4. S. Drake, Theory and practice in early modern physics, in *Trends in the History of Science*, K. Gavroglu et al., eds., Kluwer, Dordrecht, 1994, pp. 15–30.
5. ———, The scientific personality of Galileo, *Physis* **11** (1969) 181–194.
6. A. R. Hall, *Ballistics in the Seventeenth Century*, Harper and Row, New York, 1969.
7. M. Mersenne, *Harmonie Universelle*, 1636, Facsimile Edition, vol. 3, Centre National de la Recherche Scientifique, Paris, 1965.

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