CONTROL OF TRANSFER AND WEAK CLOSURE IN FUSION SYSTEMS

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Abstract. We show that $K_{\infty}$ and $K^{\infty}$ control transfer in every fusion system on a finite $p$-group when $p \geq 5$, and that they control weak closure of elements in every fusion system on a finite $p$-group when $p \geq 3$. This generalizes results of G. Glauberman concerning finite groups.

1. Introduction

Let $G$ be a finite group with Sylow $p$-subgroup $P$. The subgroup $P \cap G'$ of $P$ is called the focal subgroup of $P$ with respect to $G$. It is determined locally by the fusion of elements in $P$ under conjugation by $G$; explicitly,

$$P \cap G' = \langle x^{-1}c_g(x) \mid x \in P \text{ and } g \in G \text{ such that } c_g(x) \in P \rangle,$$

where $c_g : G \to G, x \mapsto g^{-1}xg$. The focal subgroup determines a global property of the group $G$. Indeed, $P \cap G'$ is a proper subgroup of $P$ if and only if the abelian factor group $G/G'$ has a nontrivial $p$-subgroup, which is equivalent to saying that $G$ has a nontrivial $p$-factor group. Also concerned with phenomena of fusion, an element $x \in P$ is said to be weakly closed in $P$ with respect to $H$, for some subgroup $H$ of $G$ containing $P$, if for every $g \in H$ such that $c_g(x) \in P$ we have $c_g(x) = x$.

In [4, §12–13], Glauberman defines for each finite $p$-group $P$, characteristic subgroups $K_{\infty}(P)$ and $K^{\infty}(P)$ of $P$, and shows that, denoting $K_{\infty}$ or $K^{\infty}$ by $W$,

1. when $p \geq 3$, $W$ controls weak closure of elements in $P$ with respect to $G$, that is, if $x \in P$ is weakly closed in $P$ with respect to $N_G(W(P))$, then $x$ is weakly closed in $P$ with respect to $G$;
2. when $p \geq 5$, $W$ controls $p$-transfer in $G$, that is,

$$P \cap G' = P \cap (N_G(W(P)))'.$$

In this paper, following the strategy of [7] as in our previous work [3], we generalize these results of Glauberman to arbitrary fusion systems:

Theorem 1.1. $K_{\infty}$ and $K^{\infty}$ control weak closure of elements in every fusion system on a finite $p$-group when $p \geq 3$.

Theorem 1.2. $K_{\infty}$ and $K^{\infty}$ control transfer in every fusion system on a finite $p$-group when $p \geq 5$.

As observed by Glauberman, both results fail in general for $p = 2$, as it can be seen in [4, Example 11.3], in the case of the simple group $G = \text{PSL}(2,17)$. The question about the control of transfer for $p = 3$ is still open (cf. [4, Question 16.3]).

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The above mappings \( P \mapsto K_\infty(P) \) and \( P \mapsto K_\infty(P) \) are gaining importance within the fusion system context. For instance, in [7], \( K_\infty \) and \( K_\infty \) play a central role in showing that any \( Qd(p) \)-free fusion system is induced by a finite group. Moreover, Robinson [10] uses Theorem 1.2 to obtain results on the number of irreducible characters of height zero in a \( p \)-block.

In \( \S 2 \), we define centers and control of weak closure of elements in fusion systems, and prove Theorem 1.1. In \( \S 3 \), we define focal subgroups and control of transfer in fusion systems, and state the main technical theorem (Theorem 3.1) from which Theorem 1.2 follows as a corollary. In \( \S 4 \), we consider the transfer map in fusion systems, and use it to prove some lemmas concerning focal subgroups. In \( \S 5 \), we prove Theorem 3.1. In \( \S 6 \), we generalize additional results on control of transfer and weak closure from [4] to fusion systems. We end this article with a recap in \( \S 7 \) on Glauberman’s \( K_\infty \) and \( K_\infty \) constructions. Our general terminology follows [7] and [3]; in particular, by a fusion system we always mean a saturated fusion system.

2. Control of Weak Closure in Fusion Systems

Let \( \mathcal{F} \) be a fusion system on a finite \( p \)-group \( P \). The center \( Z(\mathcal{F}) \) of \( \mathcal{F} \) is the largest subgroup \( Q \) of \( P \) such that every morphism in \( \mathcal{F} \) can be extended to a morphism in \( \mathcal{F} \) which is the identity map on \( Q \). One can easily show that \( Z(\mathcal{F}) \) is the set of all weakly closed elements in \( P \) with respect to \( \mathcal{F} \), i.e., elements \( x \in P \) such that \( \varphi(x) = x \) for all \( \varphi \in \text{Hom}_\mathcal{F}(\langle x \rangle, P) \).

Following [7], a positive characteristic \( p \)-functor is a map \( W \) sending every finite \( p \)-group \( Q \) to a characteristic subgroup \( W(Q) \) of \( Q \) such that

1. \( W(Q) \neq 1 \) if \( Q \neq 1 \);
2. if \( \varphi: Q \to R \) is an isomorphism of finite \( p \)-groups, then \( \varphi(W(Q)) = W(R) \).

For a subgroup \( Q \) of \( P \), set \( W_1(Q) = Q \) and for any positive integer \( i \), define \( W_{i+1}(Q) = W(N_P(W_i(Q))) \). We say that \( Q \) is \( (\mathcal{F}, W) \)-well-placed if \( W_i(Q) \) is \( \mathcal{F} \)-normalized for all positive integers \( i \). Note that \( W_i(Q) = W(P) \) for all sufficiently large \( i \) and that, if \( Q \) is \( (\mathcal{F}, W) \)-well-placed, so is \( W_i(Q) \) for every \( i \). Furthermore, by [3, 2.12], the set of \( (\mathcal{F}, W) \)-well-placed subgroups of \( P \) forms a conjugation family. Thus, Alperin’s fusion theorem implies that every morphism in \( \mathcal{F} \) is a composition of a finite number of restrictions of \( \mathcal{F} \)-automorphisms of \( (\mathcal{F}, W) \)-well-placed subgroups of \( P \).

Suppose further that \( Z(Q) \leq W(Q) \) for every finite \( p \)-group \( Q \). We say that \( W \) controls weak closure of elements in \( \mathcal{F} \) if

\[
Z(N_\mathcal{F}(W(P))) = Z(\mathcal{F}).
\]

The following proposition shows that control of weak closure in fusion systems is locally determined.

**Proposition 2.1.** Let \( \mathcal{F} \) be a fusion system on a finite \( p \)-group \( P \), and let \( W \) be a positive characteristic \( p \)-functor such that \( Z(Q) \leq W(Q) \) for every finite \( p \)-group \( Q \). If there exists \( x \in Z(N_\mathcal{F}(W(P))) \) such that \( x \notin Z(\mathcal{F}) \), then there exists an \( (\mathcal{F}, W) \)-well-placed subgroup \( T \) of \( P \) containing \( x \) such that \( x \in Z(N_\mathcal{F}(W(N_P(T)))) \) and \( x \notin Z(N_\mathcal{F}(T)) \).

**Proof.** We have \( Z(N_\mathcal{F}(W(P))) \leq Z(P) \), since \( W(P) \) is normal in \( P \); in particular \( x \in Z(P) \). By Alperin’s fusion theorem, there is an \( (\mathcal{F}, W) \)-well-placed subgroup
$T$ of $P$ containing $x$ and a morphism $\varphi \in \text{Aut}_F(T)$ such that $\varphi(x) \neq x$, i.e. $x \notin Z(N_F(T))$. Amongst all such $T$, choose one with $|N_P(T)|$ maximal. Note that

$$x \in T \cap Z(P) \leq N_P(T) \cap Z(P) \leq Z(N_P(T)) \leq W(N_P(T)).$$

Suppose that $x \notin Z(N_F(W(N_P(T))))$. We have

$$|N_P(T)| \geq |N_P(W(N_P(T)))| \geq |N_P(N_P(T))|,$$

where the first inequality follows from the maximality of $|N_P(T)|$. Hence, $T$ is normal in $P$ and $x \notin Z(N_F(W(N_P(T)))) = Z(N_F(W(P)))$, a contradiction. This shows that $x \in Z(N_F(W(N_P(T))))$. $\square$

Consider now the positive characteristic $p$-functors $W$ such that $Z(Q) \leq W(Q)$, for every finite $p$-group $Q$. We say that $W$ satisfies condition (C) if

$$(C) \quad O_p(G) \cap Z(G) = O_p(G) \cap Z(N_G(W(P))),$$

whenever $G$ is a finite group with Sylow $p$-subgroup $P$. Observe that this condition only depends on the subgroup structure in finite groups, and it is sufficient for the proof of the next result.

**Theorem 2.2.** Let $W$ be a positive characteristic $p$-functor such that $Z(Q) \leq W(Q)$ for every finite $p$-group $Q$. If $W$ satisfies condition (C), then $W$ controls weak closure of elements in every fusion system $F$ on a finite $p$-group $P$.

**Proof.** Suppose that the theorem is false and take a counterexample $F$ with minimal number $|F|$ of morphisms. Accordingly, there is an element $x \in Z(N_F(W(P)))$ with $x \notin Z(F)$. By Proposition 2.1, there is an $(F,W)$-well-placed subgroup $T$ of $P$ containing $x$ such that $x \in Z(N_F(W(N_P(T))))$ and $x \notin Z(N_F(T))$. If $N_F(T) < F$, then by the minimality of $|F|$, we have

$$Z(N_F(W(N_P(T)))) \leq Z(N_{N_F(T)}(W(N_P(T)))) = Z(N_F(T)),$$

contradicting the choice of $x$. Thus, $1 \neq T \leq O_p(F)$.

Set $Q = O_p(F)$ and $R = QC_P(Q)$. We show that $Q \leq R$ and hence that $Q$ is $F$-centric. Suppose that $Q < R$ and so $N_F(R) < F$. By the minimality of $|F|$, we have

$$x \in Z(N_F(W(P))) \leq Z(N_{N_F(R)}(W(P))) = Z(N_F(R)).$$

As $x \notin Q$ and every $F$-automorphism of $Q$ extends to an $F$-automorphism of $R$ (by the extension axiom) this contradicts the assumption that $x \notin Z(F)$. Therefore $Q = R = O_p(F)$ is $F$-centric.

By [1, 4.3], there exists a finite group $G$ such that $F = F_P(G)$ and such that $O_p(G) = O_p(F)$. By condition (C), we have

$$x \in O_p(G) \cap Z(N_G(W(P))) = O_p(F) \cap Z(F)$$

and so $x \in Z(F)$, a contradiction. $\square$

We now show that the positive characteristic $p$-functors $K_\infty$ and $K^\infty$ control weak closure of elements in any fusion system. We refer the reader to Section 7 for the background material. Let us also recall the following standard commutator notation. If $H$ is a subgroup of a group $G$ and $g \in G$, define $[H,g;0] = H$ and $[H,g;i+1] = [[H,g;i],g]$, for $i \geq 0$. 


Proof of Theorem 1.1. Let $W$ denote $K_{\infty}$ or $K^\infty$. By Lemma 7.2, $Z(Q) \leq W(Q)$ for every finite $p$-group $Q$. Hence, by Theorem 2.2, it will suffice to show that $W$ satisfies condition (C) when $p \geq 3$. Suppose that $G$ is a finite group, $P$ is a Sylow $p$-subgroup of $G$. We assume that $W(P) \not\leq G$. Set $Z = Z(O_p(G))$. Let $E_0$ be the set of all elements $g \in P$ such that $[X, g; p - 1] \leq Y$ for every chief factor $X/Y$ of $G$ with $X \leq Z$. Set $E = \langle E_0 \rangle$ and $L = N_G(E)$. By Theorem 7.3, $E_0$ is nonempty, and, by Theorem 7.4, we have $P \leq L < G$ and $Z \cap Z(G) = Z \cap Z(L)$. Clearly $O_p(G) \leq P \leq L$, so we have $O_p(G) \cap Z(G) = O_p(G) \cap Z(L)$. By induction on the order of $G$, we have $O_p(L) \cap Z(L) = O_p(L) \cap Z(N_L(W(P)))$. Intersecting both sides with $O_p(G)$, we get

$$O_p(G) \cap Z(L) = O_p(G) \cap Z(N_L(W(P))) \geq O_p(G) \cap Z(N_G(W(P))).$$

Thus, $O_p(G) \cap Z(G) \geq O_p(G) \cap Z(N_G(W(P)))$. The reverse inclusion is trivial. □

3. Focal Subgroups and Control of Transfer in Fusion Systems

Let $F$ be a fusion system on a finite $p$-group $P$. For $Q \leq P$, define

$$[Q, F] = \langle u^{-1} \varphi(u) \mid u \in Q, \varphi \in \text{Hom}_F(u, P) \rangle,$$

and call $[P, F]$ the $F$-focal subgroup of $P$. Note that if $F = F_P(G)$ is the fusion system on $P$ defined by the inclusion of $P$ as a Sylow $p$-subgroup of some finite group $G$, then the focal subgroup theorem reads ([6, Theorem 7.3.4]),

$$P \cap G' = [P, F].$$

Given subgroups $Q$ and $R$ of $P$ with $Q \leq R$, we say that $Q$ is weakly $F$-closed in $R$ if $\varphi(Q) = Q$ for all $\varphi \in \text{Hom}_F(Q, R)$. In particular, if $Q$ is weakly $F$-closed in $P$, then $Q \trianglelefteq P$. For short, and if there is no possible confusion, we simply say that a subgroup $Q$ is weakly $F$-closed, instead of weakly $F$-closed in $P$. It is straightforward to show that $[P, F]$ is weakly $F$-closed.

A positive characteristic $p$-functor $W$ controls transfer in $F$ if the $F$-focal subgroup equals the $N_F(W(P))$-focal subgroup, i.e., if $[P, F] = [P, N_F(W(P))].$

As for condition (C) in the previous section, we appeal now to a concept which depends only on the subgroup structure in finite groups. Namely, we say that $W$ satisfies condition (T) if

$$(T) \quad C_G(O_p(G)) \leq O_p(G) \quad \text{implies} \quad O_p(G) \cap G' = O_p(G) \cap \langle N_G(W(P)) \rangle',$

whenever $G$ is a finite group with Sylow $p$-subgroup $P$.

**Theorem 3.1.** If $W$ is a positive characteristic $p$-functor satisfying condition (T), then $W$ controls transfer in every fusion system $F$ on a finite $p$-group $P$.

We prove this theorem in §5 and get Theorem 1.2 as a corollary.

Proof of Theorem 1.2. Let $W$ denote $K_{\infty}$ or $K^\infty$. By Theorem 3.1, it will suffice to show that $W$ satisfies condition (T) when $p \geq 5$. Suppose that $G$ is a finite group, $P$ is a Sylow $p$-subgroup of $G$, and $C_G(O_p(G)) \leq O_p(G)$. Let $Q = O_p(G)$. We assume that $W(P) \not\leq G$. Let $E_0$ be the set of all elements $g \in P$ such that $[X, g; p - 1] \leq Y$ for every chief factor $X/Y$ of $G$ with $X \leq Q$. Set $E = \langle E_0 \rangle$ and $L = N_G(E)$. By Theorem 7.3, $E_0$ is nonempty, and, by Theorem 7.4, we have
P \leq L < G \text{ and } Q \cap G' = Q \cap L'. \text{ Clearly } Q \leq P \leq L, \text{ and so } Q \leq O_p(L); \text{ therefore, } C_L(O_p(L)) \leq O_p(L). \text{ By induction on the order of } G, \text{ we have }
Q \cap L' = Q \cap (O_p(L) \cap L') = Q \cap (O_p(L) \cap N_L(W(P)))' \leq Q \cap N_G(W(P))'.

Thus, Q \cap G' \leq Q \cap N_G(W(P))'. \text{ Since the opposite containment holds trivially, we get } Q \cap G' = Q \cap N_G(W(P))'.\Box

4. The Transfer Map in Fusion Systems

For a group P, a subgroup Q of P, and a group homomorphism \( \varphi: Q \to P \), let

\[ P \times_{(Q, \varphi)} P = P \times P/\sim \]

where \((x, uy) \sim (x\varphi(u), y)\) for \( x, y \in P, u \in Q\), viewed as a \( P\)-\( P \)-biset via

\[ t \cdot (x, y) = (tx, y) \quad \text{and} \quad (x, y) \cdot t = (x, yt) \]

for \( x, y, t \in P \).

The next theorem plays a crucial role in the theory of fusion systems.

**Theorem 4.1** ([2, 5.5]). Let \( \mathcal{F} \) be a fusion system on a finite \( p\)-group \( P \). There is a finite \( P\)-\( P \)-biset \( X \) with the following properties:

1. Every transitive subbiset of \( X \) is isomorphic to \( P \times_{(Q, \varphi)} P \) for some subgroup \( Q \) of \( P \) and some group homomorphism \( \varphi: Q \to P \) belonging to \( \mathcal{F} \).
2. For any \( Q \leq P \) and any \( \varphi \in \text{Hom}_\mathcal{F}(Q, P) \), the \( Q\)-\( P \)-bisets \( QX \) and \( \varphi X \) are isomorphic.
3. \(|X|/|P| \equiv 1 \) (mod \( p \)).

Let \( \mathcal{F} \) be a fusion system on a finite \( p\)-group \( P \). We call a \( P\)-\( P \)-biset \( X \) satisfying the properties of Theorem 4.1 a **\( P\)-\( P \)-biset associated with \( \mathcal{F} \)**. In the case that \( \mathcal{F} = \mathcal{F}_p(G) \) is the fusion system defined by a finite group \( G \), there is a suitable non-negative integer \( k \) such that the \( P\)-\( P \)-biset \( X = \prod^k G \) is associated with \( \mathcal{F} \). The integer \( k \) is chosen so that \(|X|/|P| \equiv 1 \) (mod \( p \)), the two other conditions of Theorem 4.1 being satisfied by any finite number of copies of the \( P\)-\( P \)-biset \( G \). We refer the reader to [2, §5] for further details.

Now, suppose that \( X = \bigsqcup_i P \times_{(Q_i, \varphi_i)} P \) and let \( A \) be an abelian group with trivial \( P\)-action. The **transfer map associated with \( X \)** is the group homomorphism

\[ t_X: H^*(P, A) \to H^*(P, A) \]

defined by

\[ t_X = \sum_i t_{Q_i}^P \circ \text{res}_{\varphi_i}, \]

where, for a subgroup \( Q \) of \( P \), the map \( t_{Q_i}^P : H^*(Q, A) \to H^*(P, A) \) is the transfer. In particular, identifying \( H^1(Q, A) \) with the set of group homomorphisms \( \text{Hom}(Q, A) \), Theorem VII.3.2 in [6] yields

\[ t_{Q_i}^P(\alpha)(x) = \sum_{t \in T} \alpha((x \cdot t)^{-1} xt), \quad \text{for all} \quad x \in P \text{ and for all} \quad \alpha \in H^1(Q, A), \]

where \( T \) is a set of left coset representatives of \( Q \) in \( P \), and where the \( \cdot \) symbol denotes the action of \( P \) on \( T \) induced by the permutation of the cosets. Thus, \( x \cdot t \in T \) and \((x \cdot t)^{-1} xt \in Q \).

By [2], we have

\[ \text{Im} t_X = H^*(P, A)^\mathcal{F} \cong \varprojlim_{\mathcal{F}} H^*(-, A), \]
where $H^*(P,A)^F$ denotes the set of elements $\alpha \in H^*(P,A)$ such that $\text{res}^F_Q(\alpha) = \text{res}_\varphi(\alpha)$ for every $Q \leq P$ and every $\varphi \in \text{Hom}_F(Q,P)$. In particular, if $F = F_P(G)$ for some finite group $G$ with Sylow $p$-subgroup $P$, we have that $\text{Im} t_X = H^*(P,A)^G$ is the set of $G$-stable elements in $H^*(P,A)$.

The following three lemmas generalize results in [4] to arbitrary fusion systems using the transfer map.

**Lemma 4.2 ([4, 4.2]).** Let $F$ be a fusion system on a finite $p$-group $P$ and let $X$ be a finite $P$-$P'$-biset associated with $F$. Set $\tau = t_X(\pi)$, where $\pi : P \to P/P'$ is the canonical surjection, and let $n = |X|/|P|$. If $u \in Z(F)$, then $\tau(u) = u^n P'$.

**Proof.** Let $X = \bigsqcup P \times (Q_i, \varphi_i) P$. Note that $|X|/|P| = \sum_i |P : Q_i|$. If $u \in Z(F)$, then — switching to multiplicative notation —

$$\tau(u) = \prod_i u_i\big((u \circ \varphi_i)(\pi)(u)\big) = \prod_i t_i^P (\pi \circ \varphi_i)(u) = \prod_i \prod_{t \in T_i} (\pi \circ \varphi_i)((u \cdot t)^{-1} ut)$$

where $T_i$ is a set of left coset representatives for $Q_i$ in $P$. Decompose each $T_i$ into $\langle u \rangle$-orbits and choose one element $t_{ij}$ from each orbit. Let $r_{ij}$ be the length of the $\langle u \rangle$-orbit containing $t_{ij}$. We then obtain $|P : Q_i| = \sum_j r_{ij}$ and since $u^{r_{ij}} \cdot t_{ij} = t_{ij}$ for all $i$ and $j$, we get

$$\tau(u) = \prod_i \prod_j \prod_{k=0}^{r_{ij} - 1} (\pi \circ \varphi_i)((u^{k+1} \cdot t_{ij})^{-1} u^{k} \cdot t_{ij}))$$

$$= \prod_i \prod_j (\pi \circ \varphi_i)(\prod_{k=0}^{r_{ij} - 1} (u^{k+1} \cdot t_{ij})^{-1} u^{k} \cdot t_{ij}))$$

$$= \prod_i \prod_j (\pi \circ \varphi_i)(t_{ij}^{-1} u^{r_{ij}} \cdot t_{ij})$$

$$= \prod_i \prod_j \pi(u^{r_{ij}})$$

$$= \prod_i \pi(u^{n_i}) = u^n P'$$

because $u \in Z(F)$. \qed

**Lemma 4.3 ([4, 4.4]).** Let $F$ be a fusion system on a finite $p$-group $P$. Then

$$[P,F] \cap Z(F) = P' \cap Z(F).$$

**Proof.** Clearly $P' \cap Z(F) \leq [P,F] \cap Z(F)$. Conversely, suppose that $z \in [P,F] \cap Z(F)$. Let $\tau$ be defined as in Lemma 4.2. By Theorem 4.1, for every subgroup $Q$ of $P$ and every morphism $\varphi : Q \to P$ in $F$, we have $\text{res}^P_Q(\tau) = \text{res}_\varphi(\tau)$, that is, $\tau(u) = (\varphi(u))$ for every $u \in Q$. Thus, $\tau(z) = P'$ as $z \in [P,F]$. On the other hand, $\tau(z) = z^n P'$ by Lemma 4.2. Thus, $z^n \in P'$. Since $n$ is prime to $p$ and $z$ is a $p$-element, it follows that $z \in P'$. \qed
Lemma 4.4 ([4, 6.7]). Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$ and let $\mathcal{G}$ be a fusion subsystem of $\mathcal{F}$ on $P$. Suppose that $Q$ is a subgroup of $P$ which is normal in $\mathcal{F}$. If $[Q, \mathcal{F}] = [Q, \mathcal{G}]$, then $[P, \mathcal{F}] \cap Q = [P, \mathcal{G}] \cap Q$.

Proof. Let $R = [P, \mathcal{F}] \cap Q$, $S = [Q, \mathcal{F}] = [Q, \mathcal{G}]$. It will suffice to show that $R \leq [P, \mathcal{G}]$. Clearly, $S \leq R$, and since $Q \triangleleft \mathcal{F}$ and $S$ is weakly $\mathcal{F}$-closed, we have $S \triangleleft \mathcal{F}$. Furthermore, $R/S \leq Z(\mathcal{F}/S)$, where the quotient fusion system is defined as in [8, 6.2]. In fact, if $x \in R$ and $\overline{\varphi} \in \text{Hom}_{\mathcal{F}/S}(\langle xS \rangle, P/S)$, then $x^{-1}\varphi(x) \in S$ for any $\varphi \in \text{Hom}_\mathcal{F}(\langle x \rangle, P)$ inducing $\overline{\varphi}$. This implies that $\overline{\varphi}(xS) = \varphi(x)S = xS$ and so $xS \in Z(R/S)$. By Lemma 4.3, $R/S \leq [P/S, \mathcal{F}/S] \cap Z(\mathcal{F}/S) = (P/S)/\cap Z(\mathcal{F}/S)$. Thus, $R \leq P/S = P'[Q, \mathcal{G}] \leq [P, \mathcal{G}]$.  

5. Proof of Theorem 3.1

To prove Theorem 3.1, we need the following two lemmas. The first shows that control of transfer in fusion systems is locally determined.

Lemma 5.1. Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$ and let $W$ be a positive characteristic $p$-functor. If $W$ controls transfer in $\mathcal{F}$ for every nontrivial $(\mathcal{F}, W)$-well-placed subgroup $Q$ of $P$, then $W$ controls transfer in $\mathcal{F}$.

Proof. For every nontrivial $(\mathcal{F}, W)$-well-placed subgroup $Q$ of $P$ we have

$$[N_P(Q), N_{\mathcal{F}}(Q)] = [N_P(Q), N_{\mathcal{F}}(W(N_P(Q)))]$$

$$\leq [N_P(W(N_P(Q))), N_{\mathcal{F}}(W(N_P(Q)))]$$

because $W(N_P(Q)) \trianglelefteq N_P(Q)$. Since $W_i(Q)$ is $(\mathcal{F}, W)$-well-placed for all $i$ and $W_i(Q) = W(P)$ for all sufficiently large $i$, we can repeat the above argument until we get

$$[N_P(Q), N_{\mathcal{F}}(Q)] \leq [P, N_{\mathcal{F}}(W(P))].$$

The lemma now follows from Alperin’s fusion theorem. 

The following result is [9, Lemma 3.7], and we include a proof for the convenience of the reader. It considerably shortens Kessar and Linckelmann’s proofs of [7, Theorems A and B] (see also [5]).

Lemma 5.2. Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$. If $Q \triangleleft \mathcal{F}$, then

$$\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle$$

where $\langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle$ denotes the subcategory of $\mathcal{F}$ on $P$ generated by $PC_{\mathcal{F}}(Q)$ and $N_{\mathcal{F}}(QC_P(Q))$.

Proof. Let $U$ be a fully $\mathcal{F}$-normalized centric radical subgroup of $P$, and take $\varphi \in \text{Aut}_{\mathcal{F}}(U)$. Note that $Q \leq U$ by [1, 1.6]. Since $Q \triangleleft \mathcal{F}$, we have $\theta = \varphi|_Q \in \text{Aut}_{\mathcal{F}}(Q)$. As $UQC_P(Q) \leq N_{\theta}$, there is $\psi \in \text{Hom}_{\mathcal{F}}(UQC_P(Q), P)$ such that $\psi|_Q = \varphi|_Q$. Then

$$\varphi = (\varphi \circ (\psi|_U)^{-1}) \circ \psi|_U.$$ 

Now, $\varphi \circ (\psi|_U)^{-1}$ is a morphism in $PC_{\mathcal{F}}(Q)$ and $\psi|_U$ is a morphism in $N_{\mathcal{F}}(QC_P(Q))$ because $\psi(QC_P(Q)) = QC_P(Q)$. Consequently, we have that $\varphi$ is a morphism in $\langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle$. By Alperin’s fusion theorem, it follows that $\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle$.  

Now we prove Theorem 3.1. The proof follows exactly the line of arguments in the proof of [7, Theorem B]. It also incorporates arguments in [4, 6.8], generalized to arbitrary fusion systems, if needed, as in Lemma 4.4.
Proof of Theorem 3.1. Suppose that the theorem is false and take a counterexample \( \mathcal{F} \) with a minimal number \(|\mathcal{F}|\) of morphisms.

- \( O_p(\mathcal{F}) \neq 1 \): Set \( Q = O_p(\mathcal{F}) \). If \( Q = 1 \), then for any nontrivial \((\mathcal{F}, W)\)-well-placed subgroup \( T \) of \( P \), we have \( N_{\mathcal{F}}(T) \not\subseteq \mathcal{F} \). By the minimality of \(|\mathcal{F}|\), it follows that \( W \) controls transfer in \( N_{\mathcal{F}}(T) \). Now, Lemma 5.1 implies that \( W \) controls transfer in \( \mathcal{F} \), a contradiction.

- \( QC_{\mathcal{F}}(Q) > Q \): Set \( R = QC_{\mathcal{F}}(Q) \). If \( R = Q \), then by [1, 4.3], there exists a finite group \( G \) such that \( \mathcal{F} = \mathcal{F}_G(G) \) and \( C_G(Q) \leq Q \); in particular, \( O_p(G) = Q \). By condition (T), we have

\[
(*) \quad Q \cap G' = Q \cap N_G(W(P))'.
\]

For every subgroup \( H \) of \( G \), let \( \overline{H} = HQ/Q \). Since \( |\mathcal{F}_G(G)| < |\mathcal{F}| \), we have \( [\overline{P}, \mathcal{F}_G(G)] = [\overline{P}, N_{\mathcal{F}_G(G)}(W(\overline{P}))] \), i.e., \( \overline{\mathcal{F}} \cap \overline{G} = \overline{\mathcal{F}} \cap N_{\mathcal{F}_G(G)}(W(\overline{P}))' \). If \( L \) is the subgroup of \( G \) containing \( Q \) such that \( \overline{L} = N_{\mathcal{F}_G(G)}(W(\overline{P})) \), then \( \overline{\mathcal{F}} \cap \overline{G} = \overline{\mathcal{F}} \cap \overline{L} \) and so \( P \cap G' \leq L'Q \). This gives

\[
P \cap G' \leq P \cap L'Q = (P \cap L')Q
\]

by Dedekind’s lemma (see [4, 6.2]). Letting \( T_1 = P \cap G', T_2 = P \cap L' \), we obtain, again by Dedekind’s lemma,

\[
P \cap G' = QT_2 \cap T_1 = (Q \cap T_1)T_2 = (Q \cap G')(Q \cap L').
\]

Hence, the containment \( N_G(W(P)) \leq L \) implies

\[
P \cap G' = (Q \cap L')(P \cap L') = P \cap L'.
\]

Since \( W(\overline{P}) \) is characteristic in \( \overline{P} \), we have \( \overline{\mathcal{F}} \leq \overline{\mathcal{F}} \). As \( O_p(G) = 1 \), we have \( L < G \), whence \( P \leq L < G \). Clearly \( Q \leq L \) and \( C_L(Q) \leq C_G(Q) \leq Q \). By the uniqueness of \( G \), it follows that \( \mathcal{F}_G(L) \not\subseteq \mathcal{F} \). Thus, the minimality of \(|\mathcal{F}|\) and the focal subgroup theorem imply

\[
P \cap L' = [P, \mathcal{F}_G(L)] = [P, N_{\mathcal{F}_G(G)}(W(P))] \leq [P, N_{\mathcal{F}}(W(P))].
\]

Therefore, \([P, \mathcal{F}] = P \cap G' = P \cap L' = [P, N_{\mathcal{F}}(W(P))]\), a contradiction.

- \( PC_{\mathcal{F}}(Q) \): Suppose that \( PC_{\mathcal{F}}(Q) \not\subseteq \mathcal{F} \). By the minimality of \(|\mathcal{F}|\), \( W \) controls transfer in \( PC_{\mathcal{F}}(Q) \). On the other hand, \( N_{\mathcal{F}}(R) \not\subseteq \mathcal{F} \) because \( R > Q = O_p(\mathcal{F}) \), and hence \( W \) also controls transfer in \( N_{\mathcal{F}}(R) \). By Lemma 5.2, it follows that \( W \) controls transfer in \( \mathcal{F} \), a contradiction. Thus, we have \( \mathcal{F} = PC_{\mathcal{F}}(Q) \).

Now let \( V \) be the inverse image of \( W(P/Q) \) in \( P \) under the canonical surjection. By [7, 3.4], we have \( N_{\mathcal{F}}(V)/Q = N_{\mathcal{F}/Q}(W(P/Q)) \). By the minimality of \(|\mathcal{F}|\), we have \([P/Q, \mathcal{F}/Q] = [P/Q, N_{\mathcal{F}/Q}(V)/Q] \), and so

\[
[P, \mathcal{F}]/([P, \mathcal{F}] \cap Q) = [P, N_{\mathcal{F}}(V)]/([P, N_{\mathcal{F}}(V)] \cap Q).
\]

Since \( \mathcal{F} = PC_{\mathcal{F}}(Q) \) and \( V \subseteq P \), we have \([Q, \mathcal{F}] = [Q, P] = [Q, N_{\mathcal{F}}(V)] \). By Lemma 4.4, we have \([P, \mathcal{F}] \cap Q = [P, N_{\mathcal{F}}(V)] \cap Q \). Thus, \([P, \mathcal{F}] = [P, N_{\mathcal{F}}(V)] \). Since \( W(P/Q) \neq 1 \), we have \( Q < V \) and so \( N_{\mathcal{F}}(V) < \mathcal{F} \). By the minimality of \(|\mathcal{F}|\), it follows that \([P, N_{\mathcal{F}}(V)] = [P, N_{\mathcal{F}}(V)] \leq [P, N_{\mathcal{F}}(W(P))] \). This shows that \([P, \mathcal{F}] = [P, N_{\mathcal{F}}(W(P))]\), a contradiction. \( \square \)
6. Additional Results

In this section, we prove some additional results on control of transfer and weak closure that generalize the statements [4, 6.3, 12.5 and 12.8]:

**Proposition 6.1 ([4, 6.9]).** Suppose that $\mathcal{F}$ is a fusion system on a nontrivial finite $p$-group $P$ such that $\text{Aut}_P(P)$ is a $p$-group (or, equivalently, $\text{N}_P(P) = \mathcal{F}_P(P)$). If there exists a positive characteristic $p$-functor that controls transfer in $\mathcal{F}$ and every quotient of $\mathcal{F}$, then $[P, \mathcal{F}] < P$.

**Proof.** Suppose that the proposition is false and take a counterexample $\mathcal{F}$ with a minimal number $|\mathcal{F}|$ of morphisms. If $W$ is a positive characteristic $p$-functor that controls transfer in $\mathcal{F}$, then $[P, \mathcal{F}] = [P, \text{N}_P(W(P))]$ and so we must have $W(P) < \mathcal{F}$. Set $Z = Z(W(P))$, $\mathcal{F} = P/Z$, and $\mathcal{F} = \mathcal{F}/Z$, where the quotient fusion system is defined as in [8, 6.2]. If $Z < P$, then $\mathcal{F} \neq 1$, $[\mathcal{F}] < |\mathcal{F}|$, and $\text{Aut}_P(\mathcal{F})$ is a $p$-group because it is a homomorphic image of $\text{Aut}_P(P)$. By the minimality of $|\mathcal{F}|$, we have $[\mathcal{F}] = [\mathcal{F}] < [\mathcal{F}]$, contradicting $[P, \mathcal{F}] = P$. So $Z = P$ and hence $P$ is abelian. By Burnside’s theorem (see [8, Theorem 3.8]), $\mathcal{F} = \text{N}_P(P) = \mathcal{F}_P(P)$ and so $1 = [P, P] = [P, \mathcal{F}] = P \neq 1$, a contradiction. \$\Box$

Corollary 6.2 is now a consequence of Theorem 1.2 and the preceding proposition.

**Corollary 6.2 ([4, 12.5]).** Let $p \geq 5$. If $\mathcal{F}$ is a fusion system on a nontrivial finite $p$-group $P$ such that $\text{Aut}_P(P)$ is a $p$-group, then $[P, \mathcal{F}] < P$.

**Proposition 6.3 ([4, 7.9]).** Let $W$ be a positive characteristic $p$-functor such that $Z(Q) \leq W(Q)$ for every finite $p$-group $Q$. Suppose that, whenever $G$ is a finite group with Sylow $p$-subgroup $P$ such that $W(P) \not\leq G$, there is $g \in P - \text{O}_p(G)$ such that $[Z(\text{O}_p(G)), g, g] = 1$. If $\mathcal{F}$ is a fusion system on a finite $p$-group $P$, then

$$Z(P)^p \cap Z(\text{N}_P(W(P))) \leq Z(\mathcal{F}).$$

**Proof.** Suppose the proposition is false and take a counterexample $\mathcal{F}$ with a minimal number $|\mathcal{F}|$ of morphisms. This gives an element $x \in Z(P)^p \cap Z(\text{N}_P(W(P)))$ such that $x \notin Z(\mathcal{F})$. By Proposition 2.1, there is an $(\mathcal{F}, W)$-well-placed subgroup $T$ of $P$ containing $x$ such that $x \in Z(\text{N}_P(W(P)))$ and $x \notin Z(\text{N}_P(T))$. As $x \in Z(P)^p \leq Z(\text{N}_P(T))^p$, the minimality of $\mathcal{F}$ implies $\mathcal{F} = \text{N}_P(T)$; in particular, $x \in T \leq \text{O}_p(\mathcal{F})$. Let $Q = \text{O}_p(\mathcal{F})$. By Lemma 5.2, we have $\mathcal{F} = (\text{PC}_P(Q), \text{N}_P(\text{QC}_P(Q)))$. Since $x \in Q \cap Z(P)$, we have $x \in Z(\text{PC}_P(Q))$ and hence $x \notin Z(\text{N}_P(\text{CP}_P(Q)))$. On the other hand,

$$x \in Z(P)^p \cap Z(\text{N}_P(\text{CP}_P(Q))(W(P)))$$

and so by the minimality of $\mathcal{F}$, we have $\mathcal{F} = \text{N}_P(\text{QC}_P(Q))$. Therefore, $Q = \text{QC}_P(Q)$ is $\mathcal{F}$-centric and hence $\mathcal{F}$ is constrained. By [1, 4.3], there exists a finite group $G$ with Sylow $p$-subgroup $P$ such that $\mathcal{F} = \mathcal{F}_P(G)$ and the result now follows from [4, Theorem 7.9]. \$\Box$

As a special case of Proposition 6.3 we get (utilizing [4, 12.3]) the following corollary.

**Corollary 6.4 ([4, 12.8]).** Let $W$ denote either $K^\infty$ or $K_\infty$. If $\mathcal{F}$ is a fusion system on a finite $p$-group $P$, then $Z(P)^p \cap Z(\text{N}_P(W(P))) \leq Z(\mathcal{F})$. 

7. Appendix: \(K^\infty\) and \(K_\infty\)

For the sake of completeness, we summarize the definitions and properties of the positive characteristic \(p\)-functors \(K^\infty\) and \(K_\infty\), as introduced in [4, §12 and 13].

Let \(P\) be a finite \(p\)-group and let \(Q \leq P\). Define \(\mathcal{M}(P; Q)\) to be the set of subgroups \(B\) of \(P\) normalized by \(Q\) and such that \(B/Z(B)\) is abelian. We identify two useful subsets. First, \(\mathcal{M}^*(P; Q)\) will denote the subset of \(\mathcal{M}(P; Q)\) containing those subgroups \(B\) for which the conjugation action of \(Q\) on \(B\) induces the trivial action on \(B/Z(B)\). The second subset, \(\mathcal{M}_*(P; Q)\), is slightly more complicated. For this subset, we choose those subgroups \(B\) of \(\mathcal{M}(P; Q)\) satisfying the following condition: if \(A \in \mathcal{M}(P; B)\) such that \(A \leq Q \cap C_P([Z(B), A])\) and \(A' \leq C_P(B)\), then the conjugation action of \(A\) on \(B\) induces the trivial action on \(B/Z(B)\).

Set \(K_{-1}(P) = P\), and for \(i \geq 0\), define inductively

\[
K_i(P) = \begin{cases} 
\langle \mathcal{M}^*(P; K_{i-1}(P)) \rangle & \text{for } i \text{ odd} \\
\langle \mathcal{M}_*(P; K_{i-1}(P)) \rangle & \text{for } i \text{ even.}
\end{cases}
\]

**Definition 7.1.** Let \(P\) be a finite \(p\)-group. We set

\[
K^\infty(P) = \bigcap_{i \geq -1, \text{ odd}} K_i(P) \\
K_\infty(P) = \langle K_i(P) \mid i \geq 0, \text{ even} \rangle.
\]

Here are the main properties of \(K^\infty(P)\) and \(K_\infty(P)\).

**Lemma 7.2.** [4, 13.1] Let \(P\) be a finite \(p\)-group and let \(W\) denote either \(K^\infty\) or \(K_\infty\).

1. \(W(P)\) is a characteristic subgroup of \(P\).
2. \(W(P)\) contains \(Z(P)\). In particular, if \(P \neq 1\), then \(W(P) \neq 1\).
3. If \(\varphi : P \to Q\) is a group isomorphism, then \(\varphi(W(P)) = W(Q)\).

Consequently, the mappings \(P \mapsto K_\infty(P)\) and \(P \mapsto K^\infty(P)\) are positive characteristic \(p\)-functors.

The next theorem is the key result — for our purposes — of Glauberman on the \(K\)-infinity subgroups.

**Theorem 7.3.** [4, 12.3] Let \(G\) be a finite group with Sylow \(p\)-subgroup \(P\) and set \(T = O_p(G)\). If \(K^\infty(P)\) or \(K_\infty(P)\) is not normal in \(G\), then there exists \(g \in P\), with \(g \notin O_p(G)\) such that:

1. \([X, g; 4] \leq Y\) for every chief factor \(X/Y\) of \(G\) such that \(X \leq T\);
2. \([Z(T), g, g] = 1\).

We also need the following result.

**Theorem 7.4.** [4, 7.2,7.3] Let \(G\) be a finite group with Sylow \(p\)-subgroup \(P\) and suppose that \(N\) is a normal \(p\)-subgroup of \(G\) and \(E_0\) is a nonempty subset of \(P\). Assume that:

1. \(\langle E_0 \rangle^g = \langle E_0 \rangle\) whenever \(g \in G\) and \(\langle E_0 \rangle^g \leq P\); and
2. \([X, g; p - 1] \leq Y\) for every \(g \in E_0\) and every chief factor \(X/Y\) of \(G\) such that \(X \leq N\).

Let \(E = \langle E_0 \rangle\) and \(L = N_G(E)\). Then \(P \leq L\), \(N \cap Z(G) = N \cap Z(L)\), \(N \cap G' = N \cap L'\), and \([N, G'] = [N, L]\).
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