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A Conformal de Rham Complex

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Abstract We introduce the notion of a *conformal de Rham complex* of a Riemannian manifold. This is a graded differential Banach algebra and it is invariant under quasiconformal maps, in particular the associated cohomology is a new quasiconformal invariant.

Keywords Quasiconformal maps · Conformal invariants · $L_{q,p}$-cohomology

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1 Introduction

A conformal map between two domains or Riemannian manifolds is a diffeomorphism preserving all angles. A quasiconformal map can be thought of as a homeomorphism which sends infinitesimal spheres to infinitesimal ellipsoids whose axes are uniformly bounded. The precise definition will be recalled below. The theory of quasiconformal mappings between plane domains and Riemann surfaces goes back to the 1920s; it lies at the heart of Teichmüller theory, i.e., the theory of deformation of Riemann surfaces. It also plays an important role in elliptic partial differential equations and in some chapters of applied mathematics, as illustrated for instance in the book [1] by Lipman Bers.

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Lavrentiev suggested studying higher dimensional quasiconformal mappings in the late 1930s, but systematic investigation began in the early 1960s by Yu. G. Reshetnyak and F. Gehring. In 1968, G. Mostow used the theory of quasiconformal mappings in the proof of his celebrated rigidity theorem. Since then, the subject has continuously attracted the attention of many mathematicians. Recent developments include some generalization to subriemannian spaces such as Carnot groups, and more abstract metric measure spaces. See [11] for a very short description of the subject.

One of the important questions in the subject is to design *quasiconformal invariants*, i.e., invariants of Riemannian manifolds which are stable under quasiconformal mappings and can thus be used to distinguish non-quasiconformally equivalent manifolds. An important class of quasiconformal invariants has been derived from the notion of the *conformal capacity* of condensers or, equivalently, of *moduli of curves families*. Another invariant is the so-called *Royden algebra*: if M is a Riemannian manifold, we define its *Royden algebra* $\mathcal{R}^n(M)$ to be the space of continuous functions $u : M \to \mathbb{R}$ such that $du \in L^n(M)$, where n is the dimension of M. This is a Banach algebra for the norm

$$\|u\|_{\mathcal{R}^n(M)} = \|u\|_{L^\infty(M)} + \|du\|_{L^n(M)}.$$

Another example of an algebraic quasiconformal invariant is given by the *Dirichlet space* $\mathcal{L}^{1,n}(M)$ of all locally integrable functions $u : M \to \mathbb{R}$ such that $du \in L^n(M)$. It is proved by S. Vodop'yanov and V. Gol'dshtein in [24] that two domains $U, V \subset \mathbb{R}^n$ are quasiconformally equivalent if and only if there exists a lattice isomorphism between $\mathcal{L}^{1,n}(U)$ and $\mathcal{L}^{1,n}(V)$. A third known algebraic invariant is the space $BMO(M)$ of functions with bounded mean oscillation. It has been proved by Martin Reimann in [16] that two quasiconformally equivalent manifolds have isomorphic BMO spaces. These algebraic invariants are important from a theoretical viewpoint, but one cannot use the Royden algebra or the Dirichlet space to quasiconformally distinguish two concrete manifolds, because these invariants are not really computable.

In the present paper, we describe a version of the de Rham complex adapted to quasiconformal geometry. This is a Banach differential graded algebra which is invariant under quasiconformal mappings by Theorem 6.5 below. We call this graded algebra the *conformal de Rham complex* and denote it by $\Omega^\bullet_{\mathrm{conf}}(M)$; it contains the Royden algebra in its center. We can then define an associated cohomology, and this *conformal cohomology* is then obviously a quasiconformal invariant with potentially interesting applications. Contrary to the Royden algebra, it is not completely hopeless to try to compute this conformal cohomology; we give in the paper some partial results in this direction. As an application we prove that the 3-dimensional Lie group SOL is not quasiconformally equivalent to the hyperbolic space $\mathbb{H}^3$.

The paper is organized as follows. We first recall some basic facts on the notion of the weak exterior derivative for locally integrable differential forms. We then define the conformal de Rham complex $\Omega^\bullet_{\mathrm{conf}}(M)$ in Sect. 3 and we prove that it is a Banach differential algebra. In Sect. 4, we prove a chain rule for forms in $\Omega^\bullet_{\mathrm{conf}}$ and maps of class $W^{1,n}$. Section 5 is a digression on conformal capacities, and the quasiconformal invariance of $\Omega^\bullet_{\mathrm{conf}}$ is proved in Sect. 6. Section 7 contains definitions and results on the cohomology of the conformal de Rham complex. Section 8 studies the notion of the *interior* conformal de Rham complex, containing a notion of differential forms "vanishing at infinity", and Sect. 9 studies the top dimensional cohomology. The last section contains some applications and some remarks on related subjects.

2 Locally Integrable Differential Forms

Recall that if (M, g) is a Riemannian manifold and $x \in M$, then the k-th exterior power of the cotangent space $\Lambda^k T_x^* M$ inherits a scalar product defined by

$$\langle \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^k, \varphi^1 \wedge \varphi^2 \wedge \cdots \wedge \varphi^k \rangle = \det(g(\theta^i, \varphi^j)),$$

where $\theta^i, \varphi^j \in \Lambda^1 T_x^* M$. A *measurable differential form* θ of degree k on M is a measurable section of the vector bundle $\Lambda^k T_x^* M \to M$. Such a form θ is said to be *locally integrable* if

$$\int_U \|\theta\| d\mathrm{vol}_g < \infty$$

for every relatively compact subset $U \Subset M$. We denote by $L^1_{loc}(M, \Lambda^k)$ the space of locally integrable differential forms of degree k on M, and by $C^r(M, \Lambda^k)$ the space of k-forms of class C^r. Finally, $C^r_0(M, \Lambda^k)$ is the space of k-forms of class C^r with compact support.

The exterior differential is a well known operator $d : C^r(M, \Lambda^k) \to C^{r-1}(M, \Lambda^{k+1})$. There is also the notion of the weak exterior differential for elements in $L^1_{loc}(M, \Lambda^k)$:

Definitions (a) One says that a form $\theta \in L^1_{loc}(M, \Lambda^{k+1})$ is the *weak exterior differential* (or the exterior differential *in the sense of currents*) of a form $\phi \in L^1_{loc}(M, \Lambda^k)$ and one writes $d\phi = \theta$ if one has

$$\int_U \theta \wedge \omega = (-1)^{k+1} \int_U \phi \wedge d\omega$$

for any oriented open subset $U \subset M$ and any smooth form $\omega \in C^\infty_0(U, \Lambda^{n-k})$ with compact support in U.

(b) A sequence $\{\alpha_j\} \subset L^1_{loc}(M, \Lambda^k)$ is said to *converge weakly* to $\alpha \in L^1_{loc}(M, \Lambda^k)$ if and only if

$$\lim_{j\to\infty} \int_U \alpha_j \wedge \omega \to \int_U \alpha \wedge \omega$$

for any oriented open subset $U \subset M$ and any $\omega \in C^\infty_0(U, \Lambda^{n-k})$. Convergence in L^1_{loc} implies weak convergence.

Proposition 2.1 *The weak exterior differential satisfies the following properties*:

(i) *If* $\theta \in L^1_{loc}(M, \Lambda^k)$ *satisfies* $\theta = d\phi$ *for some* $\phi \in L^1_{loc}(M, \Lambda^{k-1})$, *then* $d\theta = 0$.
(ii) *Let* $\alpha \in L^1_{loc}(M, \Lambda^k)$ *and* $\beta \in L^1_{loc}(M, \Lambda^{k+1})$. *If there exists a sequence* $\{\alpha_j\} \subset C^1(M, \Lambda^k)$ *such that* $\alpha_j \to \alpha$ *and* $d\alpha_j \to \beta$ *weakly, then* $d\alpha = \beta$.

Proof (i) The first assertion is clear, because for any smooth test form ω, we have $dd\omega = 0$.

(ii) For any smooth $(n-k-1)$-form ω with compact support in an oriented domain $U \subset M$, we have from Stokes' formula $\int_U d(\alpha \wedge \omega) = 0$. Hence

$$\int_U \alpha \wedge d\omega = \lim_{j\to\infty} \int_U \alpha_j \wedge d\omega = \lim_{j\to\infty} (-1)^{k+1} \int_U d\alpha_j \wedge \omega = (-1)^{k+1} \int_U \beta \wedge \omega.$$

By the definition of the weak exterior differential, this means that $d\alpha = \beta$. □

3 The Conformal de Rham Complex

Given a Riemannian manifold (M, g), we introduce the space

$$\Omega^k_{\text{conf}}(M, g) = \{\omega \in L^{n/k}(M, \Lambda^k) \mid d\omega \in L^{n/(k+1)}(M, \Lambda^{k+1})\}$$

of differential forms of degree k in $L^{n/k}(M)$ having a weak exterior differential in $L^{n/(k+1)}(M)$. It is a Banach space for the graph norm

$$\|\omega\|_{\text{conf}} = \left(\int_M |\omega|^{n/k} d\text{vol}_g\right)^{\frac{k}{n}} + \left(\int_M |d\omega|^{n/(k+1)} d\text{vol}_g\right)^{\frac{k+1}{n}}. \tag{3.1}$$

Definition The *conformal de Rham complex* of a Riemannian manifold (M, g) is the pair $(\Omega^\bullet_{\text{conf}}(M, g), d)$, where $\Omega^\bullet_{\text{conf}}(M, g) = \bigoplus_{k=1}^n \Omega^k_{\text{conf}}(M, g)$.

The name is justified by the following lemma:

Lemma 3.1 *$(\Omega^\bullet_{\text{conf}}(M, g), d)$ is a conformal invariant of (M, g). That is, if $g' = \lambda^2 g$ is a conformal deformation of the metric g (where λ is a smooth positive function on M), then $(\Omega^\bullet_{\text{conf}}(M, g'), d) = (\Omega^\bullet_{\text{conf}}(M, g), d)$, and the norm $\|\cdot\|_{\text{conf}}$ is the same for both metrics.*

Proof A direct calculation shows that the $L^{n/k}$-norm is conformally invariant on k-forms; see, e.g., [8]. □

Remark Observe that $\Omega^n_{\text{conf}}(M)$ is simply the space $L^1(M, \Lambda^n)$ of integrable n-forms on M and that $\Omega^0_{\text{conf}}(M)$ is the space of functions $u \in L^\infty(M)$ such that $du \in L^n(M)$. The spaces L^1 and L^∞ are not very well behaved Banach spaces; this causes some intricacies in our study of the conformal de Rham complex.

Lemma 3.2 *Smooth forms are dense in $\Omega^k_{\text{conf}}(M)$ for $k > 0$, that is,*

$$\overline{\Omega^k_{\text{conf}}(M) \cap C^\infty(M, \Lambda^k)} = \Omega^k_{\text{conf}}(M).$$

Smooth functions are dense in $\Omega^0_{\text{conf}}(M)$ for the weak convergence defined above.

Proof This is proved by regularization; see, e.g., [2, 4]; see also [8]. □

Definition A *Banach differential graded algebra* is a triple $(A,\cdot,d)$ where

(a) A is a Banach space;
(b) A is a direct sum $A=\bigoplus_{k\in\mathbb{N}} A^k$ where $A^k\subset A$ is a closed subspace;
(c) $d:A\to A$ is a bounded operator such that $d(A^k)\subset A^{k+1}$ and $d\circ d=0$;
(d) $(A,\cdot)$ is a real (or complex) algebra such that $x\in A^k$, $y\in A^l \Rightarrow x\cdot y\in A^{k+l}$;
(e) A is commutative in the graded sense, that is, $x\cdot y=(-1)^{kl}y\cdot x$ for any $y\in A^l$ and $x\in A^k$;
(f) $(A,\cdot)$ is a Banach algebra, that is, $\|x\cdot y\|\le\|x\|\|y\|$;
(g) The Leibniz rule holds:

$$d(x\cdot y)=d(x)\cdot y+(-1)^k x\cdot d(y),$$

for any $y\in A$ and $x\in A^k$.

Theorem 3.3 *The conformal de Rham complex* $(\Omega^\bullet_{\mathrm{conf}}(M),d)$ *is a unitary Banach differential graded algebra for the exterior product* $\wedge$.

Proof It is clear that each $\Omega^k_{\mathrm{conf}}(M)$ is a Banach space and $d:\Omega^k_{\mathrm{conf}}(M)\to\Omega^{k+1}_{\mathrm{conf}}(M)$ is a bounded operator. That $d\circ d=0$ follows from Proposition 2.1(i) above.

We need to prove that $\Omega^\bullet_{\mathrm{conf}}(M)$ is a Banach algebra, that is if $\alpha,\beta\in\Omega^\bullet_{\mathrm{conf}}(M)$, then $\alpha\wedge\beta\in\Omega^\bullet_{\mathrm{conf}}(M)$ and

$$\|\alpha\wedge\beta\|_{\mathrm{conf}}\le\|\alpha\|_{\mathrm{conf}}\cdot\|\beta\|_{\mathrm{conf}}. \tag{3.2}$$

Observe first that if $\alpha\in\Omega^k_{\mathrm{conf}}(M)$ and $\beta\in\Omega^\ell_{\mathrm{conf}}(M)$, then

$$\|\alpha\wedge\beta\|_{L^{n/(k+\ell)}}\le\|\alpha\|_{L^{n/k}}\|\beta\|_{L^{n/\ell}} \tag{3.3}$$

by Hölder's inequality, since $\frac{k}{n}+\frac{\ell}{n}=\frac{k+\ell}{n}$. Likewise,

$$\|(d\alpha)\wedge\beta\|_{L^{n/(k+\ell+1)}}\le\|d\alpha\|_{L^{n/(k+1)}}\|\beta\|_{L^{n/\ell}}$$

and

$$\|\alpha\wedge d(\beta)\|_{L^{n/(k+\ell+1)}}\le\|\alpha\|_{L^{n/k}}\|d\beta\|_{L^{n/(\ell+1)}}.$$

We now prove (3.2), assuming α,β are smooth. For such form the Leibniz rule is known:

$$d(\alpha\wedge\beta)=d(\alpha)\wedge\beta+(-1)^k\alpha\wedge(d\beta), \tag{3.4}$$

and we have, using the three inequalities above,

$$\begin{aligned}\|\alpha\wedge\beta\|_{\mathrm{conf}}&=\|\alpha\wedge\beta\|_{L^{n/(k+\ell)}}+\|d(\alpha\wedge\beta)\|_{L^{n/(k+\ell+1)}}\\&=\|\alpha\wedge\beta\|_{L^{n/(k+\ell)}}+\|d(\alpha)\wedge\beta\ \pm\ \alpha\wedge d(\beta)\|_{L^{n/(k+\ell+1)}}\\&\le\|\alpha\wedge\beta\|_{L^{n/(k+\ell)}}+\|d(\alpha)\wedge\beta\|_{L^{n/(k+\ell+1)}}+\|\alpha\wedge d(\beta)\|_{L^{n/(k+\ell+1)}}\\&\le\|\alpha\|_{L^{n/k}}\|\beta\|_{L^{n/\ell}}+\|d\alpha\|_{L^{n/(k+1)}}\|\beta\|_{L^{n/\ell}}+\|\alpha\|_{L^{n/k}}\|d\beta\|_{L^{n/(\ell+1)}}\end{aligned}$$

$$\leq \left(\|\alpha\|_{L^{n/k}} + \|d\alpha\|_{L^{n/(k+1)}}\right) \cdot \left(\|\beta\|_{L^{n/\ell}} + \|d\beta\|_{L^{n/(\ell+1)}}\right)$$
$$= \|\alpha\|_{\mathrm{conf}} \cdot \|\beta\|_{\mathrm{conf}}.$$

By the density of smooth forms for $k, \ell > 0$, the inequality (3.2) follows now for any forms in $\Omega^{\bullet}_{\mathrm{conf}}(M)$, as well as the Leibniz rule (3.3). If $k = 0$ or $\ell = 0$, we only have sequences of forms weakly converging to α and β; the argument also works in this case by the lower semi-continuity of the norm $\|\cdot\|_{\mathrm{conf}}$ for the weak convergence.

We have proved that $\Omega^{\bullet}_{\mathrm{conf}}(M)$ is a Banach differential graded algebra. It is unitary, since $1 \in \Omega^{0}_{\mathrm{conf}}(M) \subset L^{\infty}(M)$. □

Recall that $\Omega^{0}_{\mathrm{conf}}(M)$ is a subalgebra of $\Omega^{\bullet}_{\mathrm{conf}}(M)$. It is a closed subspace and thus a Banach algebra with norm

$$\|u\|_{\mathrm{conf}} = \|u\|_{L^{\infty}(M)} + \|du\|_{L^{n}(M)}.$$

Definition The *conformal Royden algebra* of M is the set

$$\mathcal{R}^{n}(M) = C(M) \cap \Omega^{0}_{\mathrm{conf}}(M)$$

of continuous functions in $\Omega^{0}_{\mathrm{conf}}(M)$. The closure of continuous functions with compact support in $\Omega^{0}_{\mathrm{conf}}(M)$ is denoted by $\mathcal{R}^{n}_{0}(M)$. These are closed subalgebras, and we have the following sequence of closed subalgebras:

$$\mathcal{R}^{n}_{0}(M) \subset \mathcal{R}^{n}(M) \subset \Omega^{0}_{\mathrm{conf}}(M) \subset \Omega^{\bullet}_{\mathrm{conf}}(M).$$

The Royden algebra is *not* dense in $\Omega^{0}_{\mathrm{conf}}(M)$; in particular, Lemma 3.2 does not hold for $k = 0$.

4 The Chain Rule for $W^{1,n}$-Sobolev Maps

The following result shows the usefulness of the conformal de Rham complex.

Proposition 4.1 *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be bounded domains and $f \in W^{1,n}(U, V)$. Then for any smooth differential form β defined on a neighborhood of V, we have $f^{*}\beta \in \Omega^{\bullet}_{\mathrm{conf}}(U)$ and*

$$f^{*}(d\beta) = d(f^{*}\beta).$$

Proof As in the case of smooth mappings, this result follows from the chain rule for 0-forms (i.e., functions) and the Leibniz rule. More precisely, let us write $f(x) = (f^{1}(x), f^{2}(x), \ldots, f^{m}(x))$. By hypothesis, we have f^{j} bounded and $df^{j} \in L^{n}(U, \Lambda^{1})$ for any $1 \leq j \leq m$; we thus have $f^{j} \in \Omega^{0}_{\mathrm{conf}}(U)$. In particular, we have $df^{j} \in \Omega^{1}_{\mathrm{conf}}(U)$, and since $\Omega^{\bullet}_{\mathrm{conf}}(U)$ is an algebra, we have

$$df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_k} \in \Omega^{k}_{\mathrm{conf}}(U) \tag{4.1}$$

for any $1 \le i_1, i_2, \dots, i_k \le n$ (this is of course simply the Hölder inequality).

We first prove the Proposition for 0-forms. So let us consider an arbitrary smooth function b defined on a neighborhood of V. The chain rule for functions and Sobolev maps (see, e.g., [4]) implies that $f^*(db) = d(b \circ f)$ in the weak sense. Observe that

$$d(b \circ f) = f^*(db) = \sum_{\nu=1}^{n} \left(\frac{\partial b}{\partial y^\nu} \circ f \right) df^\nu.$$

Our hypotheses imply that $b \circ f$ and $\frac{\partial b}{\partial y^\nu} \circ f$ are bounded, hence we have $d(b \circ f) = f^*(db) \in L^n(U, \Lambda^1)$ and thus $b \circ f \in \Omega^0_{\mathrm{conf}}(U)$.

We can now prove the Proposition for a general k-form. By linearity, it is sufficient to consider differential forms $\beta \in C^\infty(\mathbb{R}^n, \Lambda^k)$ of type $\beta = b(y)\, dy^{i_1} \wedge dy^{i_2} \wedge \cdots \wedge dy^{i_k}$. We just proved that $b \circ f$ and df^j belong to $f^*\beta \in \Omega^\bullet_{\mathrm{conf}}(U)$, thus

$$f^*(\beta) = (b \circ f)\, df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_k} \in \Omega^\bullet_{\mathrm{conf}}(U)$$

because $\Omega^\bullet_{\mathrm{conf}}(U)$ is an algebra. Now the Leibniz rule, together with $d(b \circ f) = f^*(db)$ and $d(df^j) = 0$, implies that

$$\begin{aligned} d(f^*(\beta)) &= d\left((b \circ f)\, df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_k}\right) \\ &= d(b \circ f) \wedge df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_k} \\ &= f^*(db) \wedge df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_k} = f^*(d\beta). \end{aligned}$$ □

A generalization of the previous result has been given in [9].

5 Conformal Capacity

Definition Let M be an n-dimensional Riemannian manifold and $F \subset U \subset M$ a pair of subsets with U open in M. The *conformal capacity* of the pair (F, U) is then defined as follows

$$\mathrm{Cap_n}(F, U) = \inf \left\{ \int_U |du|^n d\mathrm{vol}_g \;\middle|\; u \in \mathcal{A}(F, U) \right\},$$

where the set of admissible functions is given by

$$\mathcal{A}(F, U) = \{u \in \mathcal{R}^n_0(U) \mid u \ge 1 \text{ on a neighborhood of } F \text{ and } u \ge 0 \text{ a.e.}\}.$$

If $\mathcal{A}(F, U) = \emptyset$, then we set $\mathrm{Cap_n}(F, U) = \infty$. If $U = M$, we simply write $\mathrm{Cap_n}(F, M) = \mathrm{Cap_n}(F)$.

There are a number of important notions related to the concept of capacity, in particular the notion of parabolic manifolds and that of polar sets.

Definition A set $S \subset M$ is *conformally polar* if for any pair of open relatively compact sets $U_1 \subset U_2 \subset M$ such that $\operatorname{dist}(U_1, M \setminus U_2) > 0$, we have $\mathrm{Cap}_{\mathrm{n}}(S \cap U_1, U_2) = 0$.

Any finite set in M is conformally polar, while no set of Hausdorff dimension > 0 is conformally polar. In particular, a conformally polar set is always totally discontinuous; it even has Hausdorff dimension 0. See, e.g., [17] for these facts and more on polar sets.

Theorem 5.1 *Conformally polar sets are removable sets for the conformal de Rham complex, that is*

$$\Omega^{\bullet}_{\mathrm{conf}}(M \setminus S) = \Omega^{\bullet}_{\mathrm{conf}}(M)$$

for any conformally polar subset $S \setminus M$.

For instance $\Omega^{\bullet}_{\mathrm{conf}}(\mathbb{R}^n) = \Omega^{\bullet}_{\mathrm{conf}}(S^n)$, since the Euclidean space is conformally equivalent to a sphere with a point removed.

The proof will use the following proposition whose proof can be found in [22].

Proposition 5.2 *A set* $S \subset M$ *is conformally polar if and only if for every neighborhood* U *of* S *and every* $\epsilon > 0$, *there exists a function* $u \in C^1(M)$ *such that*

(i) *the support of* u *is contained in* $M \setminus S$;
(ii) $0 \leq u \leq 1$;
(iii) $u \equiv 1$ *on* $M \setminus U$;
(iv) $\int |du|^n \leq \epsilon$.

Proof of Theorem 5.1 Using Proposition 5.2, one can find a sequence of functions $u_j \in \Omega^0_{\mathrm{conf}}(M)$ such that $u_j = 0$ is a neighborhood of S and $u_j \to 1$ uniformly on any compact subset of $M \setminus S$ and $\int_M |du_j|^n \to 0$. Using the Lebesgue dominated convergence theorem, we see that

$$\lim_{j\to 0} \|(u_j\omega) - \omega\|_{L^{n/k}} = 0 \quad \text{and} \quad \lim_{j\to 0} \|(u_j d\omega) - d\omega\|_{L^{n/(k+1)}} = 0$$

for any k-form $\omega \in \Omega^{\bullet}_{\mathrm{conf}}(M)$. We also have from Hölder's inequality

$$\lim_{j\to 0} \|du_j \wedge \omega\|_{L^{n/(k+1)}} \leq \lim_{j\to\infty} \|du_j\|_{L^n} \cdot \|\omega\|_{L^{n/k}} = 0,$$

thus, by the Leibniz rule, we have

$$\lim_{j\to 0} \|(u_j\omega) - \omega\|_{\mathrm{conf}} = 0.$$

It follows that the sequence of bounded operators

$$T_j : \Omega^{\bullet}_{\mathrm{conf}}(M \setminus S) \to \Omega^{\bullet}_{\mathrm{conf}}(M)$$

defined by $T_j(\omega) = u_j\omega$ converges to an isometry between these Banach algebras. □

Definitions The Riemannian manifold M is *conformally parabolic* if $\mathrm{Cap_n}(F, M) = 0$ for any compact set $F \subset M$. It is *conformally hyperbolic* otherwise.

Lemma 5.3 *An n-dimensional Riemannian manifold M is conformally parabolic if and only if there exists a sequence $\{\eta_j\} \subset \mathcal{R}_0^n(M)$ such that $\eta_j \to 1$ uniformly on each compact set and* $\lim_{j\to\infty} \int_M |d\eta_j|^n = 0$.

Proof Assume that M is conformally parabolic and choose a sequence $D_1 \subset D_2 \subset \cdots \subset M$ of compact sets whose union is M. By hypothesis, we have $\mathrm{Cap_n}(D_j, M) = 0$, hence one can find for each j a function $u_j \in \mathcal{R}_0^n(M)$ such that $u_j \geq 1$ on D_j and $\int_M |du_j|^n \leq 1/j$. The desired sequence is obtained by truncation: $\eta_j = \min\{u_j, 1\}$.

Conversely, if such a sequence $\{\eta_j\}$ exists, then $\mathrm{Cap_n}(F, M) = 0$ for all compact sets F in M by definition, and M is thus conformally parabolic. □

Remark The notions of conformal capacity, parabolic manifolds and polar sets have been intensively studied; see, e.g., [5, 12, 13, 17, 22, 27] among other works. The conformal capacity is essentially equivalent to the notion of a modulus of families of paths.

6 Quasiconformal Maps

Quasiconformal maps have been intensively studied since 1960; see, e.g., [11, 23, 25, 26] for some background on this subject. Let us recall that a map $f : (M, g) \to (N, h)$ between two n-dimensional Riemannian manifolds is said to be *quasiconformal* if it satisfies the following properties:

(i) f is a homeomorphism;
(ii) $f \in W^{1,n}_{loc}(M, N)$, where $n = \dim(M) = \dim(N)$;
(iii) there exists a constant K such that

$$|df_x|^n \leq K |J_f(x)| \tag{6.1}$$

almost everywhere, where $|df_x|$ is the norm of the weak differential of f at $x \in M$ and $J_f(x)$ is its Jacobian.

This is the analytic definition; there is also a well known geometric definition:

Theorem 6.1 *The homeomorphism $f : (M, g) \to (N, h)$ is quasiconformal if and only if for all x in M we have*

$$\limsup_{r\to 0} \frac{\sup\{d(f(x), f(y)); d(x, y) = r\}}{\inf\{d(f(x), f(y)); d(x, y) = r\}} \leq H$$

for some $H < \infty$.

See, e.g., [23, 25].

Remarks A continuous map $f : M \to N$ satisfying the conditions (ii) and (iii) is called a *quasiregular map* or a *mapping with bounded distortion*. These maps are an important generalization of the class of quasiconformal maps; basic references on them are the books [17, 18].

Theorem 6.2 *Any quasiconformal* $f : (M, g) \to (N, h)$ *satisfies the following properties*:

(a) f *is differentiable almost everywhere*;
(b) f *maps sets of measure zero in* M *to sets of measure zero in* N (*this is called the Luzin property*);
(c) *the inverse map* $f^{-1} : N \to M$ *is also quasiconformal*;
(d) *the composition of two quasiconformal maps is again a quasiconformal map*;
(e) *the "change of variables formula" for integrals holds*: *for any measurable function* $v : N \to \mathbb{R}_+$ *on* N, *the pull-back* $v \circ f$ *is measurable on* M *and*

$$\int_M (v \circ f)(x)|J_f(x)|\, d\operatorname{vol}_g(x) = \int_N v(y)\, d\operatorname{vol}_h(y).$$

See [17, 23] for proofs of these facts.

We quote a few additional classical results on quasiconformal maps:

Theorem 6.3 *The homeomorphism* $f : (M, g) \to (N, h)$ *is quasiconformal if and only if for any pair of sets* $F \subset U \subset M$ *with* U *open and* F *compact we have*

$$\frac{1}{K}\operatorname{Cap}_{\mathrm{n}}(F, U) \le \operatorname{Cap}_{\mathrm{n}}(f(F), f(U)) \le K\operatorname{Cap}_{\mathrm{n}}(F, U).$$

(See [23] for a proof using the modulus of path families instead of capacities.)

Corollary 6.4 *Let* $f : (M, g) \to (N, h)$ *be a quasiconformal homeomorphism*, *then*

(i) *a subset* $S \subset M$ *is conformally polar if and only if* $f(S) \subset N$ *is conformally polar*;
(ii) M *is conformally parabolic if and only if* N *is conformally parabolic*.

The proof is obvious after the previous theorem.

Theorem 6.5 *Let* $f : (M, g) \to (N, h)$ *be a homeomorphism between two Riemannian manifolds*. *Then* f *is a quasiconformal map if and only if the pull-back of differential forms defines an isomorphism of Banach differential algebras*

$$f^* : \Omega^\bullet_{\mathrm{conf}}(N) \longrightarrow \Omega^\bullet_{\mathrm{conf}}(M).$$

Proof Assume that f is a K-quasiconformal map. Then for any differential form θ on N of degree k, we have a.e.

$$|f^*\theta_x| \le |df(x)|^k |\theta_{f(x)}| \le K^{k/n} |J_f(x)|^{k/n} |\theta_{f(x)}|$$

where K is the constant in inequality (6.1). By the change of variables formula, we then have

$$\int_M |f^*\theta_{f(x)}|^{n/k}\, d\operatorname{vol}_g \le K \int_M |\theta_{f(x)}|^{n/k} |J_f(x)|\, d\operatorname{vol}_g \le K \int_N |\theta_{f(x)}|^q\, d\operatorname{vol}_h.$$

This shows that $\|f^*\theta\|_{L^{n/k}(M,\Lambda^k)} \le K^{k/n} \|\theta\|_{L^{n/k}(N,\Lambda^k)}$. Likewise, we also have

$$\|f^*(d\theta)\|_{L^{n/(k+1)}(M,\Lambda^{k+1})} \le K^{(k+1)/n} \|d\theta\|_{L^{n/(k+1)}(N,\Lambda^{k+1})}.$$

Thus, we have proved that f^* acts as a bounded operator on the de Rham complexes and

$$\|f^*\theta\|_{\text{conf}} \le K^{(k+1)/n} \|\theta\|_{\text{conf}}.$$

It remains to prove the chain rule $d(f^*\theta) = f^*(d\theta)$ for any form θ in the conformal de Rham complex. But for smooth forms, the chain rule follows from Proposition 4.1. Since smooth forms are dense in the conformal de Rham complex and $f^* : \Omega^\bullet_{\text{conf}}(N) \to \Omega^\bullet_{\text{conf}}(M)$ is continuous, the chain rule holds for any $\theta \in \Omega^\bullet_{\text{conf}}(N)$.

Conversely, $f^* : C_0(N) \to C_0(M)$ is an isomorphism because f is a homeomorphism. If $f^* : \Omega^\bullet_{\text{conf}}(N) \xrightarrow{\simeq} \Omega^\bullet_{\text{conf}}(M)$ is also an isomorphism, then it defines an isomorphism at the level of the Royden algebras: $f^* : \mathcal{R}^n_0(N) \xrightarrow{\simeq} \mathcal{R}^n_0(M)$. By definition, the conformal capacities in M and N are then comparable, and thus f is quasiconformal by Theorem 6.3. □

We conclude by stating the following deep result of J. Lelong-Ferrand; see [14].

Theorem 6.6 *Two manifolds M and N are quasiconformally equivalent if and only if the Royden algebras $\mathcal{R}^n_0(M)$ and $\mathcal{R}^n_0(N)$ are isomorphic as abstract Banach algebras.*

7 Conformal Cohomology

7.1 Definitions

Given an arbitrary Banach complex, one can define a *reduced cohomology*, a *non-reduced cohomology* and a *torsion* (see [8, 10, 15]). Let us recall these definitions in the present situation. First set $Z^k_{\text{conf}}(M) = L^{n/k}(M, \Lambda^k) \cap \ker d$ (it is the set of closed forms in $L^{n/k}(M, \Lambda^k)$) and

$$B^k_{\text{conf}}(M) = d\left(L^{n/k-1}(M, \Lambda^{k-1})\right) \cap L^{n/k}(M, \Lambda^k).$$

Observe that $Z^k_{\text{conf}}(M) \subset \Omega^k_{\text{conf}}(U)$ is a closed linear subspace and that

$$B^k_{\text{conf}}(M) \subset \overline{B}^k_{\text{conf}}(M) \subset Z^k_{\text{conf}}(M).$$

where $\overline{B}^k_{\text{conf}}(M)$ is the closure of $B^k_{\text{conf}}(M)$.

Definition The *conformal cohomology* of M is the quotient

$$H^k_{\mathrm{conf}}(M) = Z^k_{\mathrm{conf}}(M)/B^k_{\mathrm{conf}}(M),$$

the *reduced conformal cohomology* of M is

$$\overline{H}^k_{\mathrm{conf}}(M) = Z^k_{\mathrm{conf}}(M)/\overline{B}^k_{\mathrm{conf}}(M),$$

and the *torsion* is

$$T^k_{\mathrm{conf}}(M) = \overline{B}^k_{\mathrm{conf}}(M)/B^k_{\mathrm{conf}}(M).$$

Remarks These are vector spaces. The reduced cohomology inherits a norm from that of $Z^k_{\mathrm{conf}}(M)$ and it is a Banach space. The unreduced cohomology is a Banach space if and only if the torsion vanishes. If the torsion $T^k_{\mathrm{conf}}(M) \neq 0$, then it is infinite dimensional and we always have the exact sequence

$$0 \to T^k_{\mathrm{conf}}(M) \to H^k_{\mathrm{conf}}(M) \to \overline{H}^k_{\mathrm{conf}}(M) \to 0. \tag{7.1}$$

It follows from Lemma 3.1 that the conformal cohomology is a conformal invariant; it only depend on the conformal class of the metric. The same holds for the reduced conformal cohomology and the torsion.

Lemma 7.1 *$Z^\bullet_{\mathrm{conf}}(M)$ is a subalgebra of $\Omega^\bullet_{\mathrm{conf}}(M)$, while $B^\bullet_{\mathrm{conf}}(M)$ and $\overline{B}^\bullet_{\mathrm{conf}}(M)$ are ideals in $Z^\bullet_{\mathrm{conf}}(M)$.*

Proof The Leibniz rule implies that if $\alpha, \beta \in \Omega^\bullet_{\mathrm{conf}}(M)$ are closed, then $d(\alpha \wedge \beta) = 0$, thus $Z^\bullet_{\mathrm{conf}}(M)$ is a subalgebra. Now if $\alpha = d\gamma \in B^\bullet_{\mathrm{conf}}(M)$ and $\beta \in Z^\bullet_{\mathrm{conf}}(M)$, then

$$d(\gamma \wedge \beta) = \alpha \wedge \beta,$$

hence $\alpha \wedge \beta \in B^\bullet_{\mathrm{conf}}(M)$, and that set is thus an ideal in $Z^\bullet_{\mathrm{conf}}(M)$. Now the closure of an ideal in a Banach algebra is clearly also an ideal. This concludes the proof. □

This lemma immediately implies the following

Corollary 7.2 *The wedge product defines structures of algebra on $H^\bullet_{\mathrm{conf}}(\mathbb{B}^n)$, $\overline{H}^\bullet_{\mathrm{conf}}(\mathbb{B}^n)$ and $T^\bullet_{\mathrm{conf}}(\mathbb{B}^n)$.*

Observe also that the exact sequence (7.1) holds in the category of graded algebras.

7.2 Relation with Sobolev Inequalities

The $L_{q,p}$-cohomology of a Riemannian manifold (M, g) has an interpretation in terms of Sobolev inequalities for differential forms [8]. In the conformal case, the results can be stated as follow:

Theorem 7.3 (i) $H^k_{\text{conf}}(M, g) = 0$ *if and only if there exists a constant* $C < \infty$ *such that for any closed form* $\omega \in L^{n/k}(M, \Lambda^k)$ *there exists a differential form* $\theta \in L^{n/(k-1)}(M, \Lambda^{k-1})$, *such that* $d\theta = \omega$ *and*

$$\|\theta\|_{L^{k/(k-1)}} \leq C\, \|\omega\|_{L^{k/n}} .$$

(ii) *If* $T^k_{\text{conf}}(M) = 0$, *then there exists a constant* C *such that for any differential form* $\theta \in L^{n/(n-k)}(M, \Lambda^{k-1})$ *there exists a form* $\zeta \in L^{n/(n-k)}(M, \Lambda^{k-1})$ *such that* $d\zeta = 0$ *and*

$$\|\theta - \zeta\|_{L^{k/(k-1)}} \leq C\, \|d\theta\|_{L^{n/k}} . \tag{7.2}$$

(iii) *Conversely, if* $1 < k < n$ *and if a Sobolev inequality* (7.2) *holds, then* $T^k_{\text{conf}}(M) = 0$.

Corollary 7.4 *For every manifold, we have* $T^1_{\text{conf}}(M) \neq 0$.

Proof Let $x \in M$ be an arbitrary point and $U \subset M$ be a neighborhood of x. Then we can find an essentially unbounded function $u \in L^1(M)$, with support in U and such that $du \in L^n(U)$ (see [19, p. 119]). In particular, the inequality (7.2) never holds. □

7.3 The Poincaré Lemma

We have the following Poincaré Lemma:

Theorem 7.5 (The Poincaré Lemma) $H^k_{\text{conf}}(\mathbb{B}^n) = 0$ *if* $1 < k < n$. *We also have* $H^0_{\text{conf}}(\mathbb{B}^n) = \overline{H}^0_{\text{conf}}(\mathbb{B}^n) = \mathbb{R}$ *and* $H^1_{\text{conf}}(\mathbb{B}^n) \neq 0$.

Proof This is Theorem 11.5 in [8] (in that paper, one should assume $q < \infty$, and not $q \leq \infty$). The fact that $H^1_{\text{conf}}(\mathbb{B}^n) \neq 0$ follows from Corollary 7.4, and the equality $H^0_{\text{conf}}(\mathbb{B}^n) = \mathbb{R}$ is obvious. □

Whether $H^n_{\text{conf}}(\mathbb{B}^n)$ and $\overline{H}^n_{\text{conf}}(\mathbb{B}^n)$ vanish is still an open question for us.

Corollary 7.6 *The conformal cohomology of the hyperbolic space* $\mathbb{H}^n$ *vanishes for any* $1 < k < n$.

Proof The hyperbolic space is conformally equivalent to the unit ball via the Poincaré ball model. Thus $H^k_{\text{conf}}(\mathbb{H}^n) = H^k_{\text{conf}}(\mathbb{B}^n)$. □

7.4 Other Results

For conformally parabolic n-manifolds, we have:

Theorem 7.7 *Let* M *be a connected oriented conformally parabolic* n*-manifolds. Then* $\overline{H}^n_{\text{conf}}(M) \cong \mathbb{R}$.

This result is proved in Theorem 9.2 below.

Theorem 7.8 *Any cohomology class (or reduced cohomology class) of degree* $1 < k < n$ *in* M *can be represented by a smooth form.*

This is proved by regularization; see Theorem 12.8 in [8]. □

Theorem 7.9 *Any quasiconformal map* $f : (M, g) \to (N, h)$ *defines an isomorphism at the level of the conformal cohomology*

$$f^* : H^{\bullet}_{\mathrm{conf}}(N) \longrightarrow H^{\bullet}_{\mathrm{conf}}(M),$$

and likewise for reduced cohomology and torsion.

Proof This follows immediately from Theorem 6.5. □

Theorem 7.10 *Let* M *be an arbitrary Riemannian manifold and* $S \subset M$ *a conformally polar set. Then* $M \setminus S$ *and* M *have the same conformal cohomology and torsion.*

Proof It is immediate from Theorem 5.1. □

Corollary 7.11 *If* M *is compact and* N *is quasiconformally equivalent to* $M \setminus S$ *where* $S \subset M$ *is a conformally polar set, then*

$$H^k_{\mathrm{conf}}(N) = H^k_{\mathrm{de\ Rham}}(M)$$

and $T^k_{\mathrm{conf}}(N) = 0$ *for any* $1 < k < n$.

In particular, since $\mathbb{R}^n$ is conformally a sphere with a point removed, we have $H^k_{\mathrm{conf}}(\mathbb{R}^n) = H^k(S^n) = 0$ for any $1 < k < n$. We also have $H^n_{\mathrm{conf}}(\mathbb{R}^n) = H^n(S^n) = \mathbb{R}$ by Theorem 7.7 and $H^0_{\mathrm{conf}}(\mathbb{R}^n) = H^0(S^n) = \mathbb{R}$ for obvious reasons. On the other hand, $T^1_{\mathrm{conf}}(\mathbb{R}^n) \neq 0$ by Corollary 7.4.

8 Interior Conformal de Rham Complex

Let us denote by $\Omega^k_{\mathrm{conf};0}(M)$ the closure of smooth forms with compact support in $\Omega^k_{\mathrm{conf}}(M)$ for the graph norm (3.1). Concretely, a k-form θ belongs to $\Omega^k_{\mathrm{conf};0}(M)$ if and only if $\theta \in \Omega^k_{\mathrm{conf}}(M)$ and there exists a sequence $\{\theta_j\}$ of smooth k-forms with compact support such that $d\theta_j \in L^{n/(k+1)}$ for all j and

$$\lim_{j\to\infty} \|\theta - \theta_j\|_{\mathrm{conf}} = 0.$$

We also define $Z^k_{\mathrm{conf};0}(M) = \ker(d) \cap \Omega^k_{\mathrm{conf};0}(M)$ and $B^k_{\mathrm{conf};0}(M) = d(\Omega^{k-1}_{\mathrm{conf};0}(M))$. Observe that the closure $\overline{B}^k_{\mathrm{conf};0}(M)$ is simply the closure of $d(C^\infty_0(M, \Lambda^k))$ in the space $L^{n/k}(M, \Lambda^k)$. The corresponding cohomologies are defined in the usual way:

$$H^k_{\text{conf};0}(M) = Z^k_{\text{conf};0}(M)/B^k_{\text{conf};0}(M) \quad \text{and}$$
$$\overline{H}^k_{\text{conf};0}(M) = Z^k_{\text{conf};0}(M)/\overline{B}^k_{\text{conf};0}(M).$$

Proposition 8.1 *The interior conformal de Rham complex satisfies the following properties*:

(i) $\Omega^\bullet_{\text{conf};0}(M) \subset \Omega^\bullet_{\text{conf}}(M)$ *is a closed ideal*;
(ii) $\Omega^0_{\text{conf};0}(M) = \mathcal{R}^n_0(M)$.

The proof is elementary.

Theorem 8.2 *If M is conformally parabolic then $\Omega^k_{\text{conf};0}(M) = \Omega^k_{\text{conf}}(M)$ for any $k \geq 1$.*

Proof The proof is similar to that of Theorem 5.1: Choose a sequence $\{\eta_j\} \in \mathcal{R}^n_0(M)$ as in Lemma 5.3. It is easy to check that for any $\omega \in \Omega^k_{\text{conf}}(M)$, we have $\eta_j\omega \in \Omega^k_{\text{conf};0}(M)$, and the Lebesgue dominated convergence theorem implies that

$$\lim_{j\to 0} \|(\eta_j\omega) - \omega\|_{L^{n/k}} = 0 \quad \text{and} \quad \lim_{j\to 0} \|(\eta_j d\omega) - d\omega\|_{L^{n/(k+1)}} = 0.$$

We also have from Hölder's inequality

$$\lim_{j\to 0} \|d\eta_j \wedge \omega\|_{L^{n/(k+1)}} \leq \lim_{j\to\infty} \|d\eta_j\|_{L^n} \cdot \|\omega\|_{L^{n/k}} = 0,$$

thus, by the Leibniz rule, we have

$$\lim_{j\to 0} \|(\eta_j\omega) - \omega\|_{\text{conf}} = 0.$$

It follows that $\omega = \lim_{j\to\infty}(\eta_j\omega) \in \Omega^k_{\text{conf};0}(M)$, and thus $\Omega^k_{\text{conf};0}(M) = \Omega^k_{\text{conf}}(M)$. □

Remark The proof clearly does not work for $k = 0$. In fact, we have $1 \in \Omega^0_{\text{conf}}(M) \setminus \Omega^0_{\text{conf};0}(M)$ for any non-compact manifold (because for any $u \in C^\infty_0(M)$, we have $\|1 - u\|_{\text{conf}} \geq \|1 - u\|_{L^\infty} \geq 1$).

9 The Integration Operator

In this section, M is an oriented n-dimensional Riemannian manifold.

Since $\Omega^n_{\text{conf}}(M) = L^1(M, \Lambda^n)$, we have a well defined bounded linear form

$$I : \Omega^n_{\text{conf}}(M) \to \mathbb{R} \tag{9.1}$$

given by integration: $I(\omega) = \int_M \omega$.

Proposition 9.1 *Let M be an oriented Riemannian manifold of dimension n. Then the integration operator given by (9.1) vanishes on $\overline{B}^n_{\mathrm{conf};0}(M)$. Hence there is is a well defined bounded linear form*

$$\overline{H}^n_{\mathrm{conf};0}(M) \to \mathbb{R}$$

given by $[\omega] \mapsto \int_M \omega$. This linear form is surjective; in particular, $\overline{H}^n_{\mathrm{conf};0}(M) \neq 0$.

Proof By Stokes' theorem, the integration operator vanishes on smooth exact forms with compact support and thus on $B^n_{\mathrm{conf};0}(M)$ and $\overline{B}^n_{\mathrm{conf};0}(M)$ by density and continuity. It is thus well defined on the quotient $\overline{H}^n_{\mathrm{conf}}(M) = \Omega^n_{\mathrm{conf}}(M)/\overline{B}^n_{\mathrm{conf}}(M)$. Since we can always find a smooth n-form with compact support and non-vanishing integral, this linear form is surjective. □

For conformally parabolic n-manifolds, we have a better result:

Theorem 9.2 *Let M be a connected oriented conformally parabolic n-manifolds. Then the integration operator*

$$I : \overline{H}^n_{\mathrm{conf}}(M) \xrightarrow{\simeq} \mathbb{R} \tag{9.2}$$

defines an isomorphism.

Proof Since M is conformally parabolic, we know from Theorem 8.2 that $\overline{H}^n_{\mathrm{conf}}(M) = \overline{H}^n_{\mathrm{conf},0}(M)$ and thus the linear form I is well defined. As observed above, this linear form is surjective.

By Theorem 3 in [7], we know that a smooth n-form ω with compact support has a primitive $\theta \in L^{n/(n-1)}$ if and only if $I(\omega) = 0$. In other words, if M is conformally parabolic, then

$$\ker(I) \cap C_0^\infty(M, \Lambda^n) \subset B^n_{\mathrm{conf}}(M).$$

But $C_0^\infty(M, \Lambda^n)$ is dense in $\Omega^n_{\mathrm{conf};0}(M) = \Omega^n_{\mathrm{conf}}(M)$; we thus have the exact sequence

$$0 \to \overline{B}^n_{\mathrm{conf}}(M) \to \Omega^n_{\mathrm{conf}}(M) \xrightarrow{I} \mathbb{R} \to 0.$$

Therefore, the operator $I : \overline{H}^n_{\mathrm{conf}}(M) = \Omega^n_{\mathrm{conf}}(M)/\overline{B}^n_{\mathrm{conf}}(M) \to \mathbb{R}$ is an isomorphism. □

In contrast to the previous theorem, we have

Theorem 9.3 (The Kelvin-Nevanlinna-Royden criterion) *Let M be an oriented Riemannian manifold of dimension n. Then*

$$I : B^n_{\mathrm{conf}}(M) \to \mathbb{R}$$

is non-trivial if and only if M is conformally hyperbolic.

This result implies in particular that an integration operator $I : \overline{H}^n_{\mathrm{conf}}(M) \to \mathbb{R}$ is well defined only if M is conformally parabolic.

Proof If M is conformally parabolic, then $I(B^n_{\mathrm{conf}}(M)) = I(B^n_{\mathrm{conf};0}(M)) = 0$. The converse direction is proved in [7, Proposition 1], which says that if M is conformally hyperbolic, then there exists $\beta \in \Omega^{n-1}_{\mathrm{conf}}(M)$ such that $\int_M d\beta \neq 0$. □

10 Complements

10.1 An Application to SOL Geometry

Let us denote by $\mathbb{H}^3$ the three-dimensional hyperbolic space and by SOL the group of 3×3 real matrices of the form

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

This is a solvable and unimodular three-dimensional Lie group. It is diffeomorphic to $\mathbb{R}^3$ (with coordinates x, y, z), and a left invariant Riemannian metric is $ds^2 = e^{-2z}dx^2 + e^{2z}dy^2 + dz^2$. Its volume measure is given by $dxdydz$ and is bi-invariant. See [6, 21] for more information on the geometry of this group.

Theorem 10.1 $\mathbb{H}^3$ *and SOL are not quasiconformally equivalent.*

This result is not easy to prove directly, because *SOL* and $\mathbb{H}^3$ are both conformally hyperbolic, and they both have exponential volume growth.

Proof $\mathbb{H}^3$ is conformally equivalent to the three ball $\mathbb{B}^3$, thus $H^2_{\mathrm{conf}}(\mathbb{H}^3) = H^2_{\mathrm{conf}}(\mathbb{B}^3) = 0$. On the other hand, we have $H^2_{\mathrm{conf}}(SOL) \neq 0$, a fact proven in [6]. The result follows then from Theorem 6.5. □

10.2 Relative Conformal Cohomology

Let us now set

$$\hat{\Omega}^k_{\mathrm{conf}}(M) = \Omega^k_{\mathrm{conf}}(M)/\Omega^k_{\mathrm{conf};0}(M).$$

The cohomology of this complex is the *relative conformal cohomology*, or the *conformal cohomology at infinity* of M, denoted by $H^k_{\mathrm{conf}}(M)$. From the short exact sequence of complexes

$$0 \to \Omega^k_{\mathrm{conf};0}(M) \to \Omega^k_{\mathrm{conf}}(M) \to \hat{\Omega}^k_{\mathrm{conf}}(M) \to 0,$$

we deduce a long exact sequence in cohomology

$$\cdots \to H^k_{\mathrm{conf};0}(M) \to H^k_{\mathrm{conf}}(M) \to \hat{H}^k_{\mathrm{conf}}(M) \to H^{k+1}_{\mathrm{conf};0}(M) \to \cdots.$$

Of course, this is useful only for conformally hyperbolic manifolds, since we know from Theorem 8.2 that for a conformally parabolic manifold we have $\hat{\Omega}^{\bullet}_k(M) = 0$ for all $k \geq 1$.

10.3 The Local Conformal de Rham Complex

It also useful to consider a *local conformal de Rham complex*: we denote it by $\Omega^n_{\text{lconf}}(M)$. By definition, a locally integrable differential form α belongs to $\Omega^k_{\text{lconf}}(M)$ if and only if $h \cdot \alpha \in \Omega^k_{\text{conf}}(M)$ for any smooth function $h : M \to \mathbb{R}$ with compact support. This local complex is a differential algebra, but not a Banach space. It has been introduced in the work of Sullivan and Donaldson; see [3]. For any $k \neq 1, n$, the cohomology of this complex coincides with the de Rham cohomology (use the Poincaré lemma (Theorem 7.5) and the sheaf theoretical proof of de Rham Theorem).

A large supply of forms in $\Omega^k_{\text{lconf}}(M)$ are provided by the following

Proposition 10.2 *Given a locally bounded*[1] *map $f : M \to N$ in the class $W^{1,n}_{loc}$ between two Riemannian manifolds and a smooth differential form $\omega \in \Omega^{\bullet}_{\text{deRham}}(N)$, we have $f^*(\omega) \in \Omega^{\bullet}_{\text{lconf}}(M)$. Furthermore, $df^*(\omega) = f^*(d\omega)$. In other words, we have a well defined morphism of chain complexes*

$$f^* : \Omega^{\bullet}_{\text{de Rham}}(N) \to \Omega^{\bullet}_{\text{lconf}}(M).$$

This follows from Proposition 4.1. □

The local conformal de Rham complex can be defined without using any smooth structure. Indeed, by Theorem 6.5, it is enough if the manifold has a quasiconformal structure, i.e., an $\mathbb{R}^n$ valued atlas whose transition between local coordinates are quasiconformal maps of domains in $\mathbb{R}^n$. Sullivan has proved in [20] that any topological n-manifold admits a unique quasiconformal structure if $n \neq 4$. It follows that on any topological manifold of dimension $n \neq 4$, we may represent cohomology classes by differential forms in the local conformal de Rham complex.

On the other hand, Sullivan and Donaldson have proved that there are topological 4-manifolds which do not carry any quasiconformal structure, and also that there are pairs of quasiconformal 4-manifolds which are homeomorphic but not quasiconformally equivalent. Such exotic quasiconformal structures even exist on $\mathbf{R}^4$; see [3].

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[1] The map is locally bounded if it maps compact sets to bounded sets.

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