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The connectivity at infinity of a manifold and $L^{q,p}$-Sobolev inequalities

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Abstract

The purpose of this paper is to give a self-contained proof that a complete manifold with more than one end never supports an $L^{q,p}$-Sobolev inequality ($2 \leq p, q \leq p^*$), provided the negative part of its Ricci tensor is small (in a suitable spectral sense). In the route, we discuss potential theoretic properties of the ends of a manifold enjoying an $L^{q,p}$-Sobolev inequality.

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0. Introduction

A classic subject in Riemannian geometry is the study of the interplay between curvature bounds and the topology of the underlying space. If, on the one hand, this interplay can take the form of a control of certain global topological invariants such as the homotopy or the homology groups of the manifold, on the other hand it can be visible in a control of the complexity at infinity of the space, e.g., the number of its ends. Recall that an

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end of a complete Riemannian manifold (M, g) with respect to a selected compact set K is any of the unbounded connected components of $M \setminus K$. Clearly, by enlarging K, the corresponding number of ends $n(K)$ increases and if $n(K)$ is constantly equal to 1 we say that M is connected at infinity. It is a well known consequence of the splitting theorem by J. Cheeger and D. Gromoll [7], that a complete manifold with non-negative Ricci curvature has at most two ends. Furthermore, if the Ricci curvature is positive at some point, then we have connectedness at infinity. According to works by H.-D. Cao, Y. Shen and S. Zhu [4], and P. Li and J. Wang [21] (see also [23]), the Ricci curvature assumption in Cheeger–Gromoll conclusion can be considerably relaxed provided a Sobolev inequality of the form

$$S_{2^*,2} \cdot \|\varphi\|_{L^{2^*}} \leq \|\nabla\varphi\|_{L^2}, \tag{1}$$

with $2^* = 2m/(m-2)$, for some constant $S_{2^*,2} > 0$ and for every $\varphi \in C_c^\infty(M)$ holds. More precisely, connectedness at infinity holds provided the Ricci curvature satisfies $\mathrm{Ric} \geq -q(x)$ where q is a non negative continuous function on M such that

$$\int_M q(x)\varphi(x)^2 d\ \mathrm{vol}_g \leq \int_M |\nabla\varphi|^2 d\ \mathrm{vol}_g \tag{2}$$

for all smooth functions φ with compact support in M. Note that this condition means that the function $q \geq 0$ is small in the following spectral sense:

$$\lambda_1^{-\Delta-q(x)}(M) = \inf_{\varphi\in C_c^1(M)\setminus\{0\}} \frac{\int_M \left(|\nabla\varphi|^2 - q\cdot\varphi^2\right)}{\int_M \varphi^2} \geq 0.$$

Observe that, by reversing the viewpoint, a complete manifold disconnected at infinity and with Ricci tensor subjected to the same conditions cannot support the $L^{2^*,2}$-Sobolev inequality (1). In general, it is an interesting and difficult problem to understand whether or not a complete manifold enjoys some $L^{q,p}$-Sobolev inequality

$$S_{q,p} \cdot \|\varphi\|_{L^q} \leq \|\nabla\varphi\|_{L^p}, \tag{3}$$

for some constant $S_{q,p} > 0$ and for every $\varphi \in C_c^\infty(M)$.

In this respect, we point out that the validity of the $L^{q,p}$-Sobolev inequality (3), when combined with a Ricci curvature assumption, implies some constraints on the fundamental group of the complete Riemannian manifold M. Indeed, a complete m-dimensional manifold with non-negative Ricci curvature enjoys (3) for some (hence every) $1 \leq p < m$ and $q = mp/(m-p)$ if and only if the volume growth is exactly Euclidean [29,8]. If this happens, then by a result due independently to M. Anderson [2], and P. Li [19], the fundamental group of the manifold is necessarily finite.

Further interesting connections between topology and $L^{q,p}$-Sobolev inequalities arise from a seminal work by Pansu [22], recently extended in [13]. Accordingly, the validity of (3) is related to a "global" cohomology theory which is sensitive only on the geometry at infinity of the underlying manifold, the so called $L^{q,p}$-cohomology, and gives information on the solvability of non-linear differential equations involving the p-Laplace operator (on differential forms). Very quickly, given $1 < p, q < +\infty$, the $L^{q,p}$-cohomology spaces of the complete Riemannian manifold M are defined as follows; we refer the reader to [13] for a detailed exposition. Let $L^q\left(M, \Lambda^k\right)$ denote the Banach space of L^q-integrable k-forms endowed with the obvious norm $\|\omega\|_q = \left(\int_M |\omega|^q\right)^{1/q}$. The usual exterior differential d on smooth, compactly supported k-forms extends weakly to $L^q\left(M, \Lambda^k\right)$ and gives rise to the Banach space

$$\Omega^k_{q,p}(M) = L^q(M, \Lambda^k) \cap d^{-1} L^p((M, \Lambda^{k+1}))$$

with norm $\|\omega\|_{q,p} = \|\omega\|_q + \|d\omega\|_p$. In this way, the weak exterior differential can be considered as a bounded linear operator $d^k_{q,p} : \Omega^k_{q,p}(M, \Lambda^k) \to L^p(M, \Lambda^{k+1})$. Since it satisfies the usual co-boundary rule $d \circ d = 0$ then, as in the classical de Rham cohomology, one is led to consider the corresponding subspaces of co-cycles $Z^k_{q,p}(M) = \ker d^k_{q,p}$ and co-boundaries $B^k_{q,p}(M) = d^{k-1}_{p,q}(\Omega^{k-1}_{q,p}(M)) \subset Z^k_{q,p}(M)$, and to define the kth space of the $L^{q,p}$-cohomology of M by setting

$$H^k_{q,p}(M) = Z^k_{q,p}(M)/B^k_{q,p}(M).$$

By continuity, $Z^k_{q,p}(M)$ is always closed, hence Banach, whereas $B^k_{q,p}(M)$ could be not. In case $B^k_{q,p}(M) = \overline{B^k_{q,p}(M)}$, the $L^{q,p}$-cohomology space $H^k_{q,p}(M)$ is said to be reduced. If M is compact and if $1 < p, q < \infty$ and $1/p - 1/q \leq 1/m$ then all the cohomology spaces $H^k_{q,p}(M)$ are reduced and coincide with the usual de Rham spaces $H^k_{dR}(M)$ [13, Theorem 12.10].

On the other hand, if M is bi-Lipschitz equivalent to M' then $H^k_{q,p}(M) \simeq H^k_{q,p}(M')$. In particular, if M is not compact, its $L^{q,p}$-cohomology is not affected by perturbing the Riemannian metric on a compact set. Now, in the language of $L^{q,p}$-cohomology, the validity of the Sobolev inequality (3) means precisely that the first $L^{q,p}$-cohomology space of M is reduced. Whence, if M is simply connected and $H^1_{p,r}(M) \neq 0$ for some $r > 1$, using a variational argument one can show that M supports a non-constant p-harmonic function with finite p-energy. Reversing the viewpoint, this circle of ideas shows that, under the validity of (3), if the first $L^{q,p}$-cohomology group vanishes, then the existence of a non-constant p-harmonic function $v : M \to \mathbb{R}$ with finite p-energy $|\nabla v| \in L^p(M)$ implies that M is not simply connected.

This brief and quite informal overview should have given some idea of the influence of $L^{q,p}$-Sobolev inequalities on the topology and the complexity at infinity of the space.

The goal of this paper is to give a complete and self contained proof of the following theorem that extends to every $p \geq 2$ the results by Cao–Shen–Zhu and Li–Wang alluded to at the beginning of paper.

Theorem 0.1. *Let (M, g) be a complete non compact Riemannian manifold of dimension m. Let $q > p \geq 2$ be such that*

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{m},$$

and assume that M supports an $L^{q,p}$-Sobolev inequality of the type

$$S_{q,p}\, \|\varphi\|_{L^q} \leq \|\nabla\varphi\|_{L^p}\,,$$

for some constant $S_{q,p} > 0$ and for every $\varphi \in C_c^\infty(M)$. Assume that the Ricci tensor of M is such that

$$^{M}\mathrm{Ric} \geq -q\,(x) \quad \text{on } M \tag{4}$$

for a suitable function $q \in C^0(M)$. If the Schrödinger operator $L = \Delta + Hq\,(x)$ satisfies

$$\lambda_1^{-L}(M) \geq 0, \tag{5}$$

for some constant $H > p^2/4\,(p-1)$, then, M is connected at infinity, i.e. for any compact set $F \subset M$, the complement $M \setminus F$ has exactly one unbounded connected component.

The paper is organized as follows. In Section 1 we give a rapid proof of the main theorem, which is based on three facts from the nonlinear potential theory of Riemannian manifolds. In Section 2 we provide the necessary background and in Sections 3–5, we give detailed proofs of the potential theoretical results that are used. Our proofs and viewpoints are independent of the existing literature (although we provide all the relevant references) and somewhat more direct. For instance, the potential theoretic properties of the ends are studied via a direct use of the doubling procedure and the equivalence between p-parabolicity in terms of p-capacity and p-subharmonic functions is obtained without the use of the non-linear Green kernel introduced by I. Holopainen. In the route, we also deduce a form of the Ahlfors maximum principle characterization of p-parabolicity using exterior domains and we extend a result by G. Carron concerning Sobolev inequalities outside a compact set.

1. Proof of the main theorem

We argue by contradiction. Assume that the complement $M \setminus F$ of a given compact subset $F \subset M$, contains at least two disjoint unbounded connected components E_1, E_2. From a theorem by S. Buckley and P. Koskela (see [3] and Theorem 3.4), we know that E_1, E_2 are *p-hyperbolic*. This means that for any compact subset $K_i \subset \overline{E}_i$ there exists $\alpha_i > 0$ such that

$$\int_M |\nabla u|^p d\,\mathrm{vol}_g \geq \alpha_i \tag{6}$$

for any $u \in C_c^1(M)$ such that $u \geq 1$ on K_i.

Using the p-hyperbolicity of E_1, E_2 together with a result of I. Holopainen (see [17] and Theorem 4.1), there exists a non-constant p-harmonic function w of finite p-energy, that is a function $w \in C^1(M)$ such that

$$\mathrm{div}\left(|\nabla w|^{p-2}\nabla w\right) = 0 \quad \text{and} \quad \int_M |\nabla w|^p d\,\mathrm{vol}_g < \infty.$$

The conclusion follows now from a Liouville type theorem recently proved by G. Veronelli and the first author (see [24], and Theorem 5.1). This theorem says that under the conditions (4) and (5) on the Ricci curvature, every p-harmonic function $u \in C^1(M)$ with finite p-energy $|\nabla u| \in L^p(M)$ must be constant if $p \geq 2$. Applying this result to the function w above gives us the required contradiction. $\square$

To sum up, the main theorem follows from (1) the fact that the Sobolev inequality implies that (M, g) has only p-hyperbolic ends, (2) the fact that a manifold with more than one hyperbolic end carries a non constant p-harmonic function with finite p-energy and (3) a Liouville type theorem saying that under our curvature assumption, every p-harmonic function with finite p-energy on M is constant. In the following sections, we give precise statements and independent complete proofs for these three facts.

2. Preliminary results from non-linear potential theory

A basic notion in geometric analysis is that of p-parabolicity and p-hyperbolicity of a Riemannian manifold ($1 \leq p < \infty$), see e.g. [16,27]. Recall first that the p-capacity of a compact set K in a Riemannian manifold (M, g) is defined as

$$\operatorname{cap}_p(K) = \inf \int_M |\nabla \varphi|^p \, d\operatorname{vol}_g,$$

where the infimum is taken with respect to all functions $\varphi \in C_c^1(M)$ such that $\varphi \geq 1$ on K.

Definition 2.1. A Riemannian manifold (M, g) is said to be *p-parabolic* if the p-capacity of every compact set $K \subset M$ vanishes. The manifold is *p-hyperbolic* if it contains a compact set of positive p-capacity.

Compact manifolds are obviously p-parabolic for any p. It is not hard to prove that on a connected p-hyperbolic manifold, every compact set of positive measure has positive p-capacity.

The next result gives further equivalent characterizations of p-parabolicity, we show in particular that p-parabolicity is equivalent to an exterior maximum principle. Note that the equivalence between (ii)–(iv) below is proved following arguments valid in the case $p = 2$ (see e.g. [23]) while the equivalence with condition (v) is a result in [11]. Furthermore, the equivalence (i)–(ii) was already observed in [16] using the non-linear Green function introduced by the author, and can be deduced from the results in [28]. However, we provide a new and direct argument. Finally, to the best of our knowledge, the explicit equivalence (iii)–(iv) has never been observed before.

Theorem 2.2. *Let (M, g) be a complete Riemannian manifold. The following conditions are equivalent.*

(i) *M is p-parabolic.*

(ii) *If $u \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ is a bounded above weak solution of $\Delta_p u \geq 0$ then u is constant.*

(iii) *There exists a relatively compact domain D in M such that, for every function $\varphi \in C(M \setminus D) \cap W^{1,p}_{\mathrm{loc}}(M \setminus \overline{D})$ which is bounded above and satisfies $\Delta_p \varphi \geq 0$ weakly in $M \setminus \overline{D}$, $\sup_{M \setminus D} \varphi = \max_{\partial D} \varphi$.*

(iv) *For every open set $\Omega \subset M$ with $\partial \Omega \neq \emptyset$, and for every $\psi \in C\left(\bar{\Omega}\right) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ which is bounded above and satisfies $\Delta_p \psi \geq 0$ weakly on Ω, $\sup_\Omega \psi = \sup_{\partial \Omega} \psi$.*

(v) *There exists a compact set* $K \subset M$ *with the following property. For every constant* $C > 0$*, there exists a compactly supported function* $v \in W^{1,p}(M) \cap C^0(M)$ *such that*

$$\|v\|_{L^p(K)} \geq C \|\nabla v\|_{L^p(M)}.$$

Proof. (i) $\Rightarrow$ (ii). Let M be p-parabolic, so that, for every compact set K, $\mathrm{cap}_p(K) = 0$, and assume by contradiction that there exists a positive, p-superharmonic function u. By translating and scaling, we may assume that $\sup u > 1$ and $\inf u = 0$. Note that, by the strong maximum principle (see, e.g., [15, Theorem 7.12]) u is strictly positive on M. Next let D be a relatively compact domain with smooth boundary contained in the superlevel set $\{u > 1\}$ and let D_i be an exhaustion of M consisting of relatively compact domains with smooth boundary such that $\overline{D} \subset\subset D_1$, and for every i let h_i be the solution of the Dirichlet problem

$$\begin{cases} \Delta_p h_i = 0, & \text{on } D_i \setminus D \\ h_i = 1, & \text{on } \partial D \\ h_i = 0, & \text{on } \partial D_i. \end{cases}$$

By a result of Tolksdorf [26], $h_i \in C^{1,\alpha}_{\mathrm{loc}}\left(D_i \setminus \overline{D}\right)$. Furthermore, since D and D_i have smooth boundaries, applying Theorem 6.27 in [15] with θ any smooth extension of the piecewise function

$$\theta_0 = \begin{cases} 1, & \text{on } \partial D \\ 0, & \text{on } \partial D_i, \end{cases}$$

we deduce that h_i is continuous on $\overline{D_i} \setminus D$. By the strong maximum principle, we have $0 < h_i < 1$ in $D_i \setminus \overline{D}$ and using the comparison principle, [15, Lemma 3.18], we deduce that $\{h_i\}$ is an increasing sequence. Hence, by the Harnack principle, $\{h_i\}$ converges locally uniformly on $M \setminus D$ a function h which is continuous on $M \setminus D$, p-harmonic on $M \setminus \overline{D}$ and satisfies $0 < h \leq 1$ on $M \setminus \overline{D}$ and $h = 1$ on ∂D. Again, $h \in C\left(M \setminus D\right) \cap C^{1,\alpha}_{\mathrm{loc}}\left(M \setminus \bar{D}\right)$.

Moreover, since h_i is the p-equilibrium potential of the condenser $(\overline{D}, D_i)$,

$$\mathrm{cap}_p\left(\overline{D}, D_i\right) = \int |\nabla h_i|^p = \inf \int |\nabla \varphi|^p,$$

where the infimum is taken with respect to $\varphi \in C^\infty_c\left(D_i\right)$ such that $\varphi = 1$ on ∂D. Think of each h_i extended to be zero off D_i. Therefore $\left\{\int_{M \setminus \overline{D}} |\nabla h_i|^p\right\}$ is decreasing and the sequence $\{h_i\} \subset W^{1,p}\left(\Omega\right)$ is bounded on every compact domain Ω of $M \setminus \overline{D}$. By the weak compactness theorem, see, e.g., Theorem 1.32 in [15], $h \in W^{1,p}\left(\Omega\right)$, and $\nabla h_i \to \nabla h$ weakly in $L^p\left(\Omega\right)$. In particular,

$$\int_\Omega |\nabla h|^p \leq \liminf_{i \to +\infty} \int_{D_i \setminus D} |\nabla h_i|^p.$$

On the other hand, it follows easily from the definition of capacity, that $\lim_i \mathrm{cap}_p(\overline{D}, D_i) = \mathrm{cap}_p(\overline{D}) = 0$. Thus, letting $\Omega \nearrow M \setminus \overline{D}$ we conclude that

$$\int_{M \setminus \overline{D}} |\nabla h|^p = 0,$$

so that h is constant, and since $h = 1$ on ∂D, $h \equiv 1$. Finally, since u is p-superharmonic and $u > h_i$ on $\partial D \cup \partial D_i$, by the comparison principle, $u \geq h_i$ on $D_i \setminus D$, and letting $i \to \infty$ we conclude that $u \geq 1$ on M, a contradiction.

(ii) $\Rightarrow$ (i). Given a relatively compact domain D, let h_i and h be the functions constructed above, and extend h to be 1 in D, so that h is continuous on M, bounded, and satisfies $\Delta_p h \leq 0$ weakly on M. Thus (ii) implies that h is identically equal to 1. On the other hand, since the functions h_i belong to $W_0^{1,p}(M)$, Lemma 1.33 in [15] shows that ∇h_i converges to ∇h weakly in $L^p(M)$. By Mazur's Lemma (see Lemma 1.29 in [15]) there exists a sequence v_k of convex combinations of the h_i's such that ∇v_k converges to ∇h strongly in L^p. Thus v_k is continuous, compactly supported, identically equal to 1 on $\overline{D}$ (because so are all the h_i's) and $\int_M |\nabla v_k|^p \to \int |\nabla h|^p = 0$, showing that $\mathrm{cap}_p(\overline{D}) = 0$, and M is p-parabolic.

(iii) $\Rightarrow$ (iv). Assume that (iii) holds, and suppose by contradiction that there exist a domain Ω and a function ψ as in (iv) for which $\sup_{\partial\Omega} \psi < \sup_\Omega \psi$. Note that, by the strong maximum principle, Ω is unbounded. Choose $0 < \varepsilon < \sup_\Omega \psi - \sup_{\partial\Omega} \psi$ sufficiently near to $\sup_\Omega \psi - \sup_{\partial\Omega} \psi$ so that $\overline{D} \cap \left\{\psi > \sup_{\partial\Omega} \psi + \varepsilon\right\} = \emptyset$. This is possible according to the strong maximum principle, because $\overline{D}$ is compact. Define $\tilde{\psi} \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ by setting

$$\tilde{\psi}(x) = \max\left\{\sup_{\partial\Omega} \psi + \varepsilon, \psi(x)\right\}$$

and note that $\Delta_p \tilde{\psi} \geq 0$ on M. According to property (iii),

$$\max_{\partial D} \tilde{\psi} = \sup_{M\setminus D} \tilde{\psi}.$$

However, since $\overline{D} \cap \left\{\psi > \sup_{\partial\Omega} \psi + \varepsilon\right\} = \emptyset$,

$$\max_{\partial D} \tilde{\psi} = \sup_{\partial\Omega} \psi + \varepsilon < \sup_{\Omega} \psi,$$

while

$$\sup_{M\setminus D} \tilde{\psi} = \sup_{\Omega} \psi.$$

The contradiction completes the proof.

(iv)$\Rightarrow$ (iii). Trivial.

(iv) $\Rightarrow$ (ii). Assume by contradiction that there exists $u \in C^0(M) \cap W^{1,p}_{\mathrm{loc}}(M)$ which is non-constant, bounded above and satisfies $\Delta_p u \geq 0$ weakly on M. Given $\gamma < \sup u$, the set $\Omega_\gamma = \{u > \gamma\}$ is open, and u is continuous and bounded above in $\overline{\Omega_\gamma}$, satisfies $\Delta_p u \geq 0$ weakly in Ω_γ and $\max_{\partial\Omega_\gamma} u < sup_{\Omega_\gamma} u$, contradicting (iv).

(ii) $\Rightarrow$ (iv). If there exists $\psi \in C\left(\overline{\Omega}\right) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying $\Delta_p \psi \geq 0$ and $\sup_\Omega \psi > \max_{\partial\Omega} \psi + 2\varepsilon$, for some $\varepsilon > 0$, then

$$\psi_\varepsilon = \begin{cases} \max\left\{\psi, \max\limits_{\partial\Omega}\psi + \varepsilon\right\} & \text{in } \Omega \\ \max\limits_{\partial\Omega} + \varepsilon & \text{in } M \setminus \Omega, \end{cases}$$

is a non-constant, bounded above, weak solution of $\Delta_p \psi_\varepsilon \geq 0$ on M. This contradicts (ii).

For the equivalence (i) $\Leftrightarrow$ (v), see [11, Theorem 3.1]. □

We now localize the concept of parabolicity on a given end. Recall that, by definition, an end E of M with respect to a compact domain F is any of the unbounded connected components of $M \setminus F$.

Definition 2.3. An end E of the Riemannian manifold (M, g) is said to be p-parabolic if, for every compact set $K \subset \bar{E}$,

$$\mathrm{cap}_p(K, E) = \inf \int_E |\nabla\varphi|^p = 0,$$

where the infimum is taken with respect to all $\varphi \in C_c^\infty\left(\bar{E}\right)$ such that $\varphi \geq 1$ on K.

We have the following characterizations of the parabolicity of ends.

Definition 2.4. The Riemannian double of a manifold E with smooth, compact boundary ∂E is defined to be a smooth Riemannian manifold (without boundary) $\mathcal{D}(E)$ such that (i) $\mathcal{D}(E)$ is complete (ii) $\mathcal{D}(E)$ is homeomorphic to the topological double of E and (iii) there is a compact set $K \subset \mathcal{D}(E)$ such that $\mathcal{D}(E) \setminus K$ has two connected components, both isometric to E.

Observe that this is not uniquely defined, but all such "doubles" are bi-Lipschitz equivalent.

Theorem 2.5. *An end E with smooth boundary ∂E is p-parabolic if and only if either one of the following equivalent conditions is satisfied:*

(i) *For every continuous $\phi : \bar{E} \to \mathbb{R}$ which is bounded above and p-subharmonic, $\sup_E \phi = \max_{\partial E} \phi$.*
(ii) *The (Riemannian) double $\mathcal{D}(E)$ of E is a p-parabolic manifold without boundary.*

Condition (i) in Theorem 2.5 yields easily the following necessary and sufficient condition for an end to be p-hyperbolic.

Corollary 2.6. *An end E is p-hyperbolic if and only if there exists a function $\psi \in C(\overline{E}) \cap W^{1,p}_{\mathrm{loc}}(E)$ which is p-superharmonic and such that $\inf_E \psi = 0$ and $\psi \geq 1$ on ∂E.*

Corollary 2.6 allows us to obtain the existence of special p-harmonic functions on p-hyperbolic ends (whose existence, in view of Theorem 2.2 in fact characterizes p-hyperbolic ends).

Lemma 2.7. *Let E be a p-hyperbolic end of (M, g) with smooth boundary. Then, there exists a non-constant p-harmonic function $h \in C\left(\bar{E}\right) \cap C^{1,\alpha}_{\mathrm{loc}}(E)$ such that:*

(1) $0 < h \leq 1$ *in* $\bar{E}$,
(2) $h = 1$ *on* ∂E,
(3) $\inf_{\bar{E}} h = 0$,
(4) $|\nabla h| \in L^p\left(\bar{E}\right)$.

Proof. Take a smooth exhaustion D_i of M with $\partial E \subset D_0$. Set $E_i = E \cap D_i$ and solve the Dirichlet problem

$$\begin{cases} \Delta_p h_i = 0, & \text{on } E_i \\ h_i = 1, & \text{on } \partial E \\ h_i = 0, & \text{on } \partial D_i \cap E. \end{cases}$$

By the arguments used in the proof of Theorem 2.2, $h_i \in C^{1,\alpha}_{\text{loc}}(E_i) \cap C(\overline{E}_i)$, $0 < h_i < 1$ in E_i, it is increasing and converges (locally uniformly) to a p-harmonic function h on $h \in C\left(\bar{E}\right) \cap C^{1,\alpha}_{\text{loc}}(E)$ satisfying $0 < h \leq 1$ and $h = 1$ on ∂E. Since E is p-hyperbolic, there exists a function ψ with the properties listed in Corollary 2.6. By the comparison principle, $h_i \leq \psi$ for every i, and passing to the limit, $h \leq \psi$, so that $\inf_E h = 0$ and in particular h is non-constant.

To prove that h has finite p-energy we argue as in the proof of Theorem 2.2, to show that $\left\{\int_E |\nabla h_i|^p\right\}$ (where h_i is extended to E by setting it equal to 0 in $E \setminus E_i$) is decreasing and, by Lemma 1.33 in [15], $\nabla h \in L^p(E)$ and ∇h_i converges to ∇h weakly in $L^p(E)$. □

Remark 2.8. Suppose that the end E is p-parabolic. Then, the same construction works but, in this case, by the boundary maximum principle characterization of parabolicity, we have $h \equiv 1$.

3. Sobolev inequalities, volume and hyperbolicity of ends

In this section we show that the validity of an $L^{q,p}$-Sobolev inequality implies that each of the ends of the underlying manifold is p-hyperbolic. Unlike previous investigations by Li–Wang [21] and Cao–Shen–Zhu [4] for $p = 2$, and Buckley–Koskela [3] for general p and general metric ambient spaces, our strategy is to use in a natural way the doubling construction on the given end, thus reducing the study to the case of a manifold without boundary. Technical difficulties arising from the validity of the Sobolev inequality only outside a compact set are overcome by extending a previous result by Carron.

We begin by describing the effect on volume growth of the validity of a Sobolev inequality. It is elementary to show that if an $L^{q,p}$-Sobolev inequality holds on a manifold then the manifold has infinite volume. Indeed, having fixed x_o in M we consider a family $\{\varphi_R\}_{R>0}$ of cut-off functions satisfying: (a) $0 \leq \varphi_R \leq 1$; (b) $\varphi_R = 1$ on $B_{R/2}(x_o)$; (c) $\operatorname{supp}(\varphi_R) \subset B_R(x_o)$; (d) $|\nabla \varphi_R| \leq 4/R$ on M. Using φ_R into the Sobolev inequality gives

$$S_{q,p}\mathrm{vol}\left(B_{R/2}(x_o)\right)^{1/q} \leq S_{q,p}\|\varphi_R\|_{L^q} \leq \|\nabla\varphi_R\|_{L^p} \leq \frac{4}{R}\mathrm{vol}\left(B_R(o)\right)^{1/p},$$

which, in turn, implies the non-uniform estimate

$$\mathrm{vol}\left(B_R(o)\right) \geq C R^p,$$

for every $R \geq 1$ and for some constant $C = \left(4^{-1}S_{q,p}\mathrm{vol}\left(B_1(o)\right)^{1/q}\right)^p > 0$. In particular $\mathrm{vol}(M) = +\infty$ and at least one of the ends of M has infinite volume.

In order to extend this conclusion to each individual end we can use a uniform volume estimate whose principle can be traced back to papers by G. Carron [5] and K. Akutagawa [1], see also [14, Lemma 2.2] and [25, Theorem 3.1.5]. We state this estimate in a form suitable for our purposes.

Proposition 3.1. *Let E be an end of the complete manifold M with respect to the compact set F and assume that the $L^{q,p}$-Sobolev inequality (3) holds on E, for some $q > p \geq 1$. Then there exists positive constant C_1 depending only on p, q and $S_{q,p}$ such that, for every geodesic ball $B_R(x_0) \subset E$*

$$\operatorname{vol}\left(B_R\left(x_0\right)\right) \geq C_1 R^{\frac{pq}{q-p}}. \tag{7}$$

In particular, if $F \subset B_{R_0}(o)$, then for every $x_0 \in E$ with $d(x_0, o) \geq R + R_0$ the ball $B_R(x_0)$ is contained in E, and E has infinite volume.

Proof. For every $\Omega \subset E$ let

$$\lambda\left(\Omega\right) = \inf \frac{\int_\Omega |\nabla\varphi|^p}{\int_\Omega |\varphi|^p},$$

the infimum being taken with respect to all $\varphi \in W_c^{1,p}\left(\Omega\right), \varphi \not\equiv 0$. By the Sobolev and Hölder inequalities, for every such φ we have

$$\int_\Omega \varphi^p \leq \operatorname{vol}(\Omega)^{\frac{q-p}{q}} \left(\int_\Omega \varphi^q\right)^{\frac{p}{q}} \leq \left(S_{q,p}\|\nabla\varphi\|_p\right)^p,$$

and therefore

$$\operatorname{vol}\left(\Omega\right)^{\frac{q-p}{q}} \lambda\left(\Omega\right) \geq S_{q,p}^p. \tag{8}$$

On the other hand, choosing $\Omega = B_R\left(x_0\right)$ and

$$\varphi\left(x\right) = R - d\left(x, x_0\right)$$

we deduce that

$$\begin{aligned} \lambda\left(B_R\left(x_0\right)\right) &\leq \frac{\operatorname{vol}\left(B_R\left(x_0\right)\right)}{\int_{B_R(x_0)}\left(R - d\left(x, x_0\right)\right)^p} \\ &\leq \frac{\operatorname{vol}\left(B_R\left(x_0\right)\right)}{\int_{B_{R/2}(x_0)}\left(R - d\left(x, x_0\right)\right)^p} \\ &\leq \frac{2^p \operatorname{vol}\left(B_R\left(x_0\right)\right)}{R^p \operatorname{vol}\left(B_{R/2}\left(x_0\right)\right)}. \end{aligned} \tag{9}$$

Combining (8) and (9) we obtain

$$\operatorname{vol}\left(B_R\left(x_0\right)\right)^{1+\frac{q-p}{q}} \geq 2^{-p} S_{q,p}^p R^p \operatorname{vol}\left(B_{R/2}\left(x_0\right)\right),$$

i.e.,

$$\operatorname{vol}\left(B_R\left(x_0\right)\right) \geq \left(2^{-p} S_{q,p}^p R^p\right)^\alpha \operatorname{vol}\left(B_{R/2}\left(x_0\right)\right)^\alpha,$$

with

$$0 < \alpha = \frac{1}{1+\frac{q-p}{q}} < 1.$$

Iterating this inequality k-times yields

$$\mathrm{vol}\,(B_R\,(x_0)) \geq 2^{-p\alpha\sum_{j=1}^{k} j\alpha^j}\left(2^{-p}S_{q,p}^{p}R^{p}\right)^{\sum_{j=1}^{k}\alpha^j}\mathrm{vol}\left(B_{R/2^k}\,(x_0)\right)^{\alpha^k}.$$

Since

$$\mathrm{vol}\,B_r\,(x_0) \sim \omega_0 r^m \quad \text{as } r \to 0\,(m = \dim M),$$

for k large enough

$$\mathrm{vol}\left(B_{R/2^k}\,(x_0)\right)^{\alpha^k} \geq \left(\frac{1}{2}\omega_0 R^m 2^{-km}\right)^{\alpha^k}$$

and letting $k \to +\infty$ finally gives

$$\mathrm{vol}\,(B_R\,(x_0)) \geq 2^{-p\bar{\alpha}}\left(2^{-p}S_{q,p}^{p}R^{p}\right)^{\frac{\alpha}{1-\alpha}},$$

where

$$\bar{\alpha} = \sum_{j=1}^{+\infty} j\alpha^j,$$

and estimate (7) holds since $\frac{p\alpha}{1-\alpha} = \frac{pq}{q-p}$.

To prove the second statement, assume that $x_0 \in E$ is such that $d(x_0, o) \geq R + R_0$, and consider the geodesic ball $B_R(x_0)$. If $x \in \overline{B_{R_0}(o)}$, then by the triangle inequality,

$$d\,(x_0, x) \geq d\,(x_0, o) - d\,(o, x) \geq R,$$

proving that $B_R\,(\bar{x}) \cap \overline{B_{R_0}\,(o)} = \emptyset$. On the other hand, if E' is a second connected component of $M \setminus K$ and $x'' \setminus B_{R_0}(o)$, let σ be a minimizing geodesic from x_0 to x'. By continuity, σ must intersect $\partial B_{R_0}\,(o)$ at some point x_1 and

$$d\left(x_0, x'\right) = \ell\,(\sigma) = d(x', x_1) + d(x_1, x_0) > d(x_1, \bar{x}) \geq R.$$

Therefore $B_R\,(x_0) \cap E' = \emptyset$ and we conclude that $B_R(x_0) \subset E$. Since $x_0 \in E$ can be chosen in such a way that $d\,(x_0, o)$ is arbitrarily large, letting $E \ni x_0 \to \infty$ gives that $\mathrm{vol}\,(E) = +\infty$. $\square$

We next prove that if an $L^{q,p}$-Sobolev inequality holds in the complement of a compact set of a complete Riemannian manifold, then each end is p-hyperbolic. The result is known for $p = 2$ [4,21,23]. The proof we give here is new and is based on the observation that if the $L^{q,p}$ Sobolev inequality (3) holds on M then M is necessarily p-hyperbolic. Indeed, if Ω is any compact domain then, for every $\varphi \in C_c^\infty(M)$ satisfying $\varphi \geq 1$ on Ω it holds

$$S_{q,p}\mathrm{vol}\,(\Omega)^{1/q} \leq S_{q,p}\,\|\varphi\|_{L^q} \leq \|\nabla\varphi\|_{L^p},$$

proving that

$$\mathrm{cap}_p\,(\Omega) \geq S_{q,p}^{p}\mathrm{vol}\,(\Omega)^{p/q} > 0.$$

This shows that M is p-hyperbolic, and therefore at least one of its ends is p-hyperbolic. To extend the conclusion to each end E of M, we are naturally led to applying the reasonings to the double $\mathcal{D}(E)$. By the very definition of the double of a manifold, it turns out that $\mathcal{D}(E)$ supports the Sobolev inequality (3) outside a compact neighborhood of the glued boundaries. Accordingly, to conclude that E is p-hyperbolic we can make a direct use of the following very general theorem that extends to any $L^{q,p}$-Sobolev inequality a previous result by Carron [6].

Theorem 3.2. *Let (M, g) be a possibly incomplete Riemannian manifold. Assume that M has infinite volume and that $M \setminus F$ supports the $L^{q,p}$-Sobolev inequality (3) for some compact $F \subset M$. Then, M is p-hyperbolic and the same Sobolev inequality, possibly with a different constant, holds on all of M.*

Remark 3.3. Clearly, if M is complete, according to Proposition 3.1 the assumption that M has infinite volume is automatically satisfied.

Proof. Let Ω be a precompact domain with smooth boundary such that $K \subset\subset \Omega$. Let also $W_\varepsilon \approx \partial\Omega \times (-\varepsilon, \varepsilon)$ be a bicollar neighborhood of $\partial\Omega$ such that $W_\varepsilon \subset M \setminus F$, and let $\Omega_\varepsilon = \Omega \cup W_\varepsilon$ and $M_\varepsilon = M \setminus \Omega_\varepsilon$. Note that, by assumption, the $L^{q,p}$-Sobolev inequality with Sobolev constant $S > 0$ holds on M_ε. Furthermore, the same $L^{q,p}$-Sobolev inequality, with some constant $S_\varepsilon > 0$ holds on the compact manifold with boundary $\overline{\Omega_\varepsilon}$ (start with the Euclidean L^1 Sobolev inequality and use Hölder's inequality a number of times). Now, let $\rho \in C_c^\infty(M)$ be a cut-off function satisfying $0 \le \rho \le 1$, $\rho = 1$ on $\Omega_{\varepsilon/2}$ and $\rho = 0$ on M_ε. Next, for any $v \in C_c^\infty(M)$, write $v = \rho v + (1-\rho)\, v$, and note that $\rho v \in C_c^\infty(\Omega_\varepsilon)$ whereas $(1-\rho)\, v \in C_c^\infty\left(M_{\varepsilon/2}\right)$ Therefore, we can apply the respective Sobolev inequalities and get

$$
\begin{aligned}
\|v\|_{L^q(M)} &\le \|v\rho\|_{L^q(\Omega_\varepsilon)} + \|v\,(1-\rho)\|_{L^q(M_{\varepsilon/2})} \\
&\le S_\varepsilon^{-1}\,\|\nabla(v\rho)\|_{L^p(\Omega_\varepsilon)} + S^{-1}\,\|\nabla(v\,(1-\rho))\|_{L^p(M_{\varepsilon/2})} \\
&\le \left(S_\varepsilon^{-1} + S^{-1}\right)\|\nabla v\|_{L^p(M)} + S_\varepsilon^{-1}\,\|v\nabla\rho\|_{L^p(\Omega_\varepsilon\setminus\Omega_{\varepsilon/2})} \\
&\quad + S^{-1}\,\|v\nabla\rho\|_{L^p(\Omega_\varepsilon)} \\
&\le \left(S_\varepsilon^{-1} + S^{-1}\right)\left\{\|\nabla v\|_{L^p(M)} + C\,\|v\|_{L^p(\Omega_\varepsilon)}\right\},
\end{aligned}
$$

where $C = \max_M |\nabla\rho|$. Summarizing, we have shown that, for every $v \in C_c^\infty(M)$,

$$
\|v\|_{L^q(M)} \le C_1\left\{\|\nabla v\|_{L^p(M)} + \|v\|_{L^p(\Omega_\varepsilon)}\right\}, \tag{10}
$$

for a suitable constant $C_1 > 0$.

With this preparation, we now prove that M is p-hyperbolic. To this end, using the fact that $\mathrm{vol}(M) = +\infty$, we choose a compact set $\Omega' \supset \Omega_\varepsilon$ satisfying

$$
\mathrm{vol}\left(\Omega'\right)^{1/p} \ge (2C_1)^q\,\mathrm{vol}\,(\Omega_\varepsilon)^{q/p}.
$$

Thus, applying (10) with a test function $v \in C_c^\infty(M)$ satisfying $v = 1$ on Ω', we deduce

$$
\mathrm{vol}\,(\Omega_\varepsilon) \le C_1^{-1}\mathrm{vol}\left(\Omega'\right)^{1/q} - \mathrm{vol}\,(\Omega_\varepsilon)^{1/p} \le \|\nabla v\|_{L^p(M)}.
$$

It follows that

$$\mathrm{cap}_p\left(\Omega'\right) \geq \mathrm{vol}\left(\Omega_\varepsilon\right) > 0,$$

proving that M is p-hyperbolic.

Finally, we show that the Sobolev inequalities on Ω_ε and on M_ε glue together. According to (10) it suffices to prove that there exists a suitable constant $E = E\left(\Omega_\varepsilon\right) > 0$ such that

$$\|v\|_{L^p(\Omega_\varepsilon)} \leq E \, \|\nabla v\|_{L^p(M)},$$

for every $v \in C_c^\infty(M)$. Since M is p-hyperbolic, this latter inequality follows from Theorem 2.2(ii). □

We can now prove the result announced at the beginning of this section.

Theorem 3.4. *Every end of a complete Riemannian manifold (M, g) supporting the $L^{q,p}$-Sobolev inequality (3) for some $q > p \geq 1$ is p-hyperbolic and, in particular, has infinite volume.*

Proof. Let E be an end with smooth boundary of the complete manifold M supporting the $L^{q,p}$-Sobolev inequality (3). We shall prove that the double $\mathcal{D}(E)$ of E is a p-hyperbolic manifold (without boundary). To this purpose, we note that $\mathcal{D}(E)$ has infinite volume because, by the first part of Theorem 3.4, E itself has infinite volume. Furthermore, E enjoys the Sobolev inequality (3) outside a compact neighborhood of the glued boundaries. Therefore, a direct application of Theorem 3.2 yields that $\mathcal{D}(E)$ is a p-hyperbolic manifold, as desired. □

Corollary 3.5. *Suppose that the complete manifold M has (at least) one p-parabolic end. Then the $L^{q,p}$-Sobolev inequality (3) fails.*

4. p-harmonic functions with finite p-energy

This section aims to giving a simple independent proof of a result of I. Holopainen [17], which extends to the nonlinear setting previous results of the Li–Tam theory [20]. Related results may be found in the paper by S.W. Kim, and Y.H. Lee [18].

Theorem 4.1. *Let (M, g) be a Riemannian manifold with at least two p-hyperbolic ends (with respect to some smooth, compact domain). Then, there exists a non-constant, bounded p-harmonic function $u \in C^0(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ satisfying $|\nabla u| \in L^p(M)$.*

Proof. Let E_i be the ends of M with respect to the smooth domain $\Omega \subset\subset M$. By assumption, we may suppose that E_1 and E_2 are p-hyperbolic. Let $\{D_t\}_{t\in\mathbb{N}}$ be a smooth exhaustion of M and set $E_{j,t} = E_j \cap D_t$.

For every $t \in \mathbb{N}$, let $u_t \in C^{1,\alpha}_{\mathrm{loc}}(D_t) \cap C\left(\overline{D_t}\right)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta_p u_t = 0 & \text{on } D_t \\ u_t = 1 & \text{on } E_1 \cap \partial D_t \\ u_t = 0 & \text{on } E_j \cap \partial D_t,\ j \neq 1. \end{cases}$$

Note that, by the strong maximum principle, $0 < u_t < 1$ in D_t. Moreover, as explained in Lemma 2.7, the sequence $\{u_t\}_{t\in\mathbb{N}}$ converges, locally uniformly, to a p-harmonic function $u \in C^0(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ satisfying $0 \leq u \leq 1$. Now, for every $j = 1, 2$, let h_j be the p-harmonic function associated to the ends E_j constructed in Lemma 2.7. Recall that h_j is the (locally uniform) limit of the p-harmonic function $h_{j,t}$ which satisfies $h_{j,t} = 1$ on ∂E_j and $h_{j,t} = 0$ on $E_j \cap \partial D_t$. Define $k_{1,t} = 1 - h_{1,t}$. Then, comparing u_t and $k_{1,t}$ on $E_{1,t}$ yields that $u_t \geq k_{1,t}$ on $E_{1,t}$. On the other hand, comparing u_t and $h_{2,t}$, gives $u_t \leq h_{2,t}$ on $E_{2,t}$. Therefore, taking limits as $t \to +\infty$, we deduce that $u \geq h_1$ on E_1 and $u \leq k_2$ on E_2. From this, using (3) in Lemma 2.7, we conclude that u is non-constant. We claim that $|\nabla u| \in L^p(M)$. Indeed, for every $j = 1, \ldots, n$, let $F_{j,t} = E_j \setminus E_{j,t}$. We think of u_t as extended to all of M by $u_t = 1$ on $F_{1,t}$ and $u_t = 0$ on $\cup_{i=2}^n F_{i,t}$. Then, by construction, u_t is the equilibrium potential of the condenser $\left(F_{1,t}, \cup_{i\geq 2} E_{i,t} \cup \Omega \cup E_1\right)$ and we have

$$\operatorname{cap}_p\left(F_{1,t}, \cup_{i\geq 2} E_{i,t} \cup \Omega \cup E_1\right) = \int_M |\nabla u_t|^p .$$

On the other hand, take $k_{1,t}$ and extend it to be one on $F_{1,t}$. Then, $k_{1,t}$ is the equilibrium potential of the condenser $\left(F_{1,t}, E_1\right)$ and we have

$$\operatorname{cap}_p\left(F_{1,t}, E_1\right) = \int_M \left|\nabla k_{1,t}\right|^p .$$

By the monotonicity properties of the p-capacity [15,12], and recalling that $\int_{E_{1,t}} \left|\nabla k_{1,t}\right|^p$ is decreasing in t, we deduce

$$\begin{aligned}\int_M |\nabla u_t|^p &= \operatorname{cap}_p\left(F_{1,t}, \cup_{i\geq 2} E_{i,t} \cup \Omega \cup E_1\right)\\ &\leq \operatorname{cap}_p\left(F_{1,t}, E_1\right) = \int_M \left|\nabla k_{1,t}\right|^p = \int_{E_{1,t}} \left|\nabla k_{1,t}\right|^p \leq C,\end{aligned}$$

for some constant $C > 0$ independent of t. Now observe that, for every domain $D \subset\subset M$, $\nabla u_t \to \nabla u$ weakly in $L^p(D)$ and therefore

$$\int_D |\nabla u|^p \leq \liminf_{t\to+\infty} \int_D |\nabla u_t|^p \leq C.$$

Letting $D \nearrow M$ completes the proof. □

5. A Liouville-type result for p-harmonic functions

The project of a self-contained proof of Theorem 0.1 will be completed once we have proved the following Liouville-type result for p-harmonic functions with finite p-energy.

Theorem 5.1. *Let (M, g) be a complete Riemannian manifold such that ${}^M\mathrm{Ric} \geq -q(x)$ for some continuous function $q(x) \geq 0$. Let $p \geq 2$ and assume that the Schrödinger operator $L_H = -\Delta - Hq(x)$ satisfies*

$$\lambda_1^{L_H}(M) \geq 0$$

for some $H > p^2/4\,(p-1)$. Then, every p-harmonic function $u : M \to \mathbb{R}$ of class C^1 and with finite p-energy $|\nabla u| \in L^p(M)$ must be constant.

Note that the spectral condition is equivalent to the strong positivity of the operator

$$-\Delta - \frac{s^2}{4\,(s-1)} q(x)$$

in the terminology of [9].

In the recent paper [24], the authors obtained a more general result for manifold-valued p-harmonic maps with low regularity. The proof in the real-valued case of Theorem 5.1 appears somewhat more direct.

Proof. Roughly speaking, the idea is to obtain a Caccioppoli-type inequality for the energy density $|\nabla u|$ of u and this is achieved by integrating the Bochner formula against suitable test functions.

Note that, by elliptic regularity, u is smooth on the open set

$$M_+ = \{x \in M : |\nabla u| \neq 0\}.$$

The standard Bochner formula, which is valid for a generic smooth function, states that

$$\frac{1}{2}\Delta |\nabla u|^2 = |\mathrm{Hess}\,(u)|^2 + \langle \nabla \Delta u, \nabla u\rangle + \mathrm{Ric}\,(\nabla u, \nabla u)\,, \quad \text{on } M_+.$$

Computing the Laplacian on the left hand side, using the Kato inequality

$$|\nabla\,|\nabla u|\;|^2 \le |\mathrm{Hess}(u)|^2$$

and recalling that $\mathrm{Ric} \ge -q\,(x)$, we deduce

$$|\nabla u|\,\Delta\,|\nabla u| \ge \langle \nabla \Delta u, \nabla u\rangle - q\,(x)\,|\nabla u|^2\,, \quad \text{on } M_+. \tag{11}$$

Let $0 \le \rho \in C_c^\infty(M_+)$ be a test function. We multiply both sides of (11) by $\rho^2\,|\nabla u|^{p-2}$ and we integrate by parts thus obtaining

$$-\int_{M_+} \left\langle \nabla \left(|\nabla u|^{p-1}\rho^2\right), \nabla\,|\nabla u|\right\rangle \ge -\int_{M_+} \Delta u \;\mathrm{div}\left(\rho^2\,|\nabla u|^{p-2}\,\nabla u\right) - \int_{M_+} q\,(x)\,\rho^2\,|\nabla u|^p\,. \tag{12}$$

We shall take care of each of the integrals in (12) separately.

(I) Direct computations and the Cauchy–Schwarz inequality show that

$$-\int_{M_+} \left\langle \nabla \left(|\nabla u|^{p-1}\rho^2\right), \nabla\,|\nabla u|\right\rangle \le 2\int_{M_+} \rho\,|\nabla \rho|\;\;|\nabla u|^{p-1}\,|\nabla\,|\nabla u|\;| - (p-1)\int_{M_+} \rho^2\,|\nabla u|^{p-2}\,|\nabla\,|\nabla u|\;|^2. \tag{13}$$

Let $\varepsilon > 0$ be any small number. Using the elementary inequality $2ab \le \varepsilon^2 a^2 + \varepsilon^{-2} b^2$ we obtain

$$\int_{M_+} 2\rho\,|\nabla\rho|\ \ |\nabla u|^{p-1}\,|\nabla\,|\nabla u|\ | \le \varepsilon^2 \int_{M_+} \rho^2\,|\nabla u|^{p-2}\,|\nabla\,|\nabla u|\ |^2 + \varepsilon^{-2}\int_{M_+} |\nabla\rho|^2\,|\nabla u|^p\,,$$

which, inserted into (13), yields

$$-\int_{M_+}\left\langle \nabla\left(|\nabla u|^{p-1}\,\rho^2\right),\nabla\,|\nabla u|\right\rangle \le \varepsilon^{-2}\int_{M_+}|\nabla\rho|^2\,|\nabla u|^p + \left(\varepsilon^2-(p-1)\right)\int_{M_+}\rho^2\,|\nabla u|^{p-2}\,|\nabla\,|\nabla u|\ |^2\,. \tag{14}$$

(II) Again, by direct computations,

$$-\int_{M_+}\Delta u\ \mathrm{div}\left(\rho^2\,|\nabla u|^{p-2}\,\nabla u\right) = -\int_{M_+}\rho^2\Delta u\,\Delta_p u - 2\int_{M_+}\rho\Delta u\,|\nabla u|^{p-2}\,\langle\nabla\rho,\nabla u\rangle\,. \tag{15}$$

Now, since u is p-harmonic, $\Delta_p u = 0$ and, therefore, the first summand on the right hand side vanishes. On the other hand, expanding the p-harmonicity condition we see that

$$\Delta u = -(p-2)\,|\nabla u|^{-1}\,\langle\nabla\,|\nabla u|\,,\nabla u\rangle\,,\quad \text{on } M_+.$$

Substituting this expression into (15) and manipulating as above, we conclude

$$\begin{aligned}\mathrm{LHS}(15) &= 2\,(p-2)\int_{M_+}\rho\,|\nabla u|^{-1}\,\langle\nabla\,|\nabla u|\,,\nabla u\rangle\,|\nabla u|^{p-2}\,\langle\nabla\rho,\nabla u\rangle\\ &\ge -2\,(p-2)\int_{M_+}\rho\,|\nabla\rho|\ \ |\nabla u|^{p-1}\,|\nabla\,|\nabla u|\ |\\ &\ge -\varepsilon^2\,(p-2)\int_{M_+}\rho^2\,|\nabla u|^{p-2}\,|\nabla\,|\nabla u||^2 - \varepsilon^{-2}\int_{M_+}|\nabla\rho|^2\,|\nabla u|^p\,. \end{aligned}\tag{16}$$

(III) Recall that, by the spectral assumption,

$$\int_M |\nabla\varphi|^2 - Hq\,(x)\,\varphi^2 \ge 0,$$

for every $\varphi\in C_c^\infty(M)$. Taking $\varphi = \rho\,|\nabla u|^{p/2}$ and performing the needed computations as above, we finally obtain

$$\begin{aligned}-\int_{M_+} q\,(x)\,\rho^2\,|\nabla u|^p \ge{}& -\left(H^{-1}+\varepsilon^{-2}H^{-1}p\right)\int_{M_+}|\nabla\rho|^2\,|\nabla u|^p\\ &-\left(\frac{p^2}{4}H^{-1}+\varepsilon^2H^{-1}p\right)\\ &\times\int_{M_+}\rho^2\,|\nabla u|^{p-2}\,|\nabla\,|\nabla u|\ |^2\,.\end{aligned}\tag{17}$$

Inserting (14), (16) and (17) into (12) we conclude that

$$A \int_{M_+} \rho^2 |\nabla u|^{p-2} |\nabla |\nabla u| \; |^2 \leq B \int_{M_+} |\nabla \rho|^2 |\nabla u|^p , \tag{18}$$

where we have set

$$A = p - 1 - \frac{p^2}{4} H^{-1} - \varepsilon^2 \left\{ p - 1 + H^{-1} p \right\}$$
$$B = H^{-1} + \varepsilon^{-2} \left\{ H^{-1} p + 2 \right\}.$$

Note that, by the assumption on H, $A > 0$ provided $0 < \varepsilon \ll 1$. Inequality (18) is almost the desired Caccioppoli-type inequality. The main problem to complete the argument, and to deduce the vanishing of $|\nabla u|$ by a standard choice of the cut-off functions, is that ρ must be supported in M_+. We need to extend the validity of (18) to any test function compactly supported in M. To this end, we use a trick introduced by F. Duzaar and M. Fucks in [10]. Namely, we define

$$\varphi_\delta = \min \left\{ \frac{|du|^{p/2}}{\delta}, 1 \right\}$$

for $\delta > 0$ and set $\xi = \varphi_\delta \eta$ for any $\eta \in C_c^\infty(M)$. Using the fact that $f(t) = t^{p/2}$ is a Lipschitz function for $p \geq 2$ and that, for a p-harmonic function, $|\nabla u|^{p/2-1} \nabla u \in W^{1,2}_{\mathrm{loc}}(M)$ (see e.g. [10]) it can be verified that $\xi \in W_0^{1,2}(M_+)$. Hence there exists a sequence $\{\rho_j\}_{j=1}^\infty \subset C_c^\infty(M_+)$ such that $\rho_j \to \xi$ in $W_0^{1,2}(M)$. Substituting $\rho = \rho_j$ into (18) and taking the liminf as $j \to \infty$, we get

$$A \int_{M_+} \eta^2 (\varphi_\delta)^2 |\nabla u|^{p-2} |\nabla |\nabla u| \; |^2 \leq 2B \int_{M_+} \eta^2 |\nabla \varphi_\delta|^2 |\nabla u|^p + 2B \int_{M_+} (\varphi_\delta)^2 |\nabla \eta|^2 |\nabla u|^p . \tag{19}$$

Finally, we let $\delta \to 0$. Note that $\varphi_\delta \to 1$ pointwise in M_+. Moreover

$$\begin{aligned} \int_{M_+} |\nabla u|^p |\nabla \varphi_\delta|^2 \eta^2 &= \int_{M_+} |\nabla u|^2 \frac{\left|\nabla |\nabla u|^{p/2}\right|^2}{\delta^2} \eta^2 \chi_{\{|\nabla u|^p < \delta^2\}} \\ &\leq \int_{M_+} \left|\nabla |\nabla u|^{p/2}\right|^2 \eta^2 \chi_{\{|\nabla u|^p < \delta^2\}} \end{aligned}$$

and the last term vanishes by dominated convergence as $\delta \to 0$. Therefore, letting $\delta \to 0$ in (19), we finally get the desired Caccioppoli inequality

$$\int_{M_+} \eta^2 |\nabla u|^{p-2} |\nabla |\nabla u| \; |^2 \leq C \int_{M_+} |\nabla u|^p |\nabla \eta|^2 , \quad \forall \eta \in C_c^\infty(M), \tag{20}$$

for a suitable constant $C > 0$.

As mentioned above, the argument can now be easily completed. By contradiction, suppose u is non-constant. For any fixed $R > 0$, we choose $\eta(x) = \eta_R(x)$ so as to satisfy

$$\begin{aligned}&\text{(a) } 0 \leq \eta(x) \leq 1, \qquad \text{(b) } \eta(x) = 1 \quad \text{on } B_R(o), \\ &\text{(c) } \eta(x) = 0 \quad \text{off } B_{2R}(o), \qquad \text{(d) } |\nabla \eta| \leq 2/R \quad \text{on } M.\end{aligned} \tag{21}$$

Whence, we deduce

$$\int_{B_R(o)\cap M_+} |\nabla u|^{p-2} |\nabla |\nabla u||^2 \leq \frac{4C}{R^2} \int_{B_{2R}(o)\cap M_+} |\nabla u|^p,$$

for some computable positive constant C, and letting $R \to +\infty$ we conclude

$$\int_{M_+} |\nabla u|^{p-2} |\nabla |\nabla u||^2 = 0,$$

proving that $|\nabla u| =$ const. on every connected component of M_+. Since u is non-constant, this implies that $M_+ = M$ and $|\nabla u| =$ const. $\neq 0$. Since, by assumption, $|\nabla u|^p \in L^1(M)$, we deduce that

$$\operatorname{vol} M < +\infty. \tag{22}$$

Using this information together with the spectral assumption and choosing $\eta = \eta_R$ to be the cut-off functions defined in (21), we get

$$\begin{aligned} 0 &\leq \lim_{R\to+\infty} \int_{B_{2R}(o)} \left\{ H^{-1} |\nabla \eta|^2 - q(x)\eta^2 \right\} \\ &\leq \lim_{R\to+\infty} \left\{ \frac{4 \operatorname{vol} B_{2R}(o)}{HR^2} - \int_{B_R(o)} q(x) \right\} \\ &= -\int_M q(x) \leq 0, \end{aligned}$$

proving that $q(x) = 0$, i.e., ${}^M\mathrm{Ric} \geq 0$. A well known result by S.T. Yau and E. Calabi now shows that M has at least a linear volume growth, contradicting (22). $\square$

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References

[1] K. Akutagawa, Yamabe metrics of positive scalar curvature and conformally flat manifolds, Differential Geom. Appl. 4 (3) (1994) 239–258.
[2] M. Anderson, On the topology of complete manifolds of nonnegative Ricci curvature, Topology 29 (1) (1990) 41–55.
[3] S.M. Bukley, P. Koskela, Ends of metric measure spaces and Sobolev inequalities, Math. Z. 252 (2006) 275–285.
[4] H.-D. Cao, Y. Shen, S. Zhu, The structure of stable minimal hypersurfaces in $\mathbb{R}^{n+1}$, Math. Res. Lett. 4 (5) (1997) 637–644.
[5] G. Carron, Inégalités isopérimétriques et inégalités de Faber–Krahn, in: Séminaire de Théorie Spectrale et Géométrie, No. 13, Année 1994–1995, pp. 63–66.

[6] G. Carron, Une suite exacte en L^2-cohomologie, Duke Math. J. 95 (2) (1998) 343–372.
[7] J. Cheeger, D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geom. 6 (1971) 119–128.
[8] T. Coulhon, L. Saloff-Coste, Variétés Riemanniennes isométriques à l'infini, Rev. Mat. Iberoam. 11 (1995) 687–726.
[9] T. Coulhon, Q. Zhang, Large time behavior of heat kernels on forms, J. Differential Geom. 77 (3) (2007) 353–384.
[10] F. Duzaar, M. Fuchs, On removable singularities of p-harmonic maps, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990) 385–405.
[11] V. Gol'dshtein, M. Troyanov, Axiomatic theory of Sobolev spaces, Expo. Math. 19 (2001) 289–336.
[12] V. Gol'dshtein, M. Troyanov, Capacities in metric spaces, Integral Equations Operator Theory 44 (2) (2002) 212–242.
[13] V. Gol'dshtein, M. Troyanov, Sobolev inequalities for differential forms and $L^{q,p}$ cohomology, J. Geom. Anal. 16 (2006) 597–631.
[14] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, in: CIMS Lecture Notes, Courant Institute of Mathematical Sciences, 1999.
[15] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, in: Oxford Mathematical Monographs. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993.
[16] I. Holopainen, Nonlinear potential theory and quasiregular mappings on Riemannian manifolds, Ann. Acad. Sci. Fenn. AI 74 (1990) 45.
[17] I. Holopainen, Rough isometries and p-harmonic functions with finite Dirichlet integral, Rev. Mat. Iberoam. 10 (1994) 143–176.
[18] S.W. Kim, Y.H. Lee, Energy finite solutions of elliptic equations on Riemannian manifolds, J. Korean Math. Soc. 45 (2008) 807–819.
[19] P. Li, Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature, Ann. of Math (2) 124 (1) (1986) 1–21.
[20] P. Li, L.F. Tam, Harmonic functions and the structure of complete manifolds, J. Differential Geom. 35 (1992) 359–383.
[21] P. Li, J. Wang, Minimal hypersurfaces with finite index, Math. Res. Lett. 9 (1) (2002) 95–103.
[22] P. Pansu, Cohomologie L^p des varietes a courbure negative, cas du degre 1, Rend. Semin. Mat. Univ. Politec. Torino (1989).
[23] S. Pigola, M. Rigoli, A.G. Setti, Vanishing and Finiteness Results in Geometric Analysis. A Generalization of the Bochner Technique, in: Progress in Mathematics, vol. 266, Birkhäuser Verlag, Basel, 2008.
[24] S. Pigola, G. Veronelli, On the homotopy class of maps with finite p-energy into non-positively corved manifolds, Geom. Dedicata 143 (2009) 109–116.
[25] L. Saloff-Coste, Aspects of Sobolev-Type Inequalities, in: London Mathematical Society Lecture Note Series, vol. 289, Cambridge University Press, Cambridge, 2002.
[26] P. Tolksdorf, Regularity of a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1984) 126–150.
[27] M. Troyanov, Parabolicity of manifolds, Siberian Adv. Math. 9 (4) (1999) 125–150.
[28] M. Troyanov, Solving the p-Laplacian on manifolds, Proc. Amer. Math. Soc. 128 (2) (2000) 541–545.
[29] N.Th. Varopoulos, Hardy–Littlewood theory for semigroups, J. Funct. Anal. 63 (1985) 240–260.