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# FINSLER CONFORMAL LICHNEROWICZ-OBATA CONJECTURE

by V. S. MATVEEV, H.-B. RADEMACHER, M.TROYANOV & A. ZEGHIB

Abstract. — We prove the Finsler analog of the conformal Lichnerowicz-Obata conjecture showing that a complete and essential conformal vector field on a non-Riemannian Finsler manifold is a homothetic vector field of a Minkowski metric.

Résumé. — Nous démontrons une variante de la conjecture de Lichnerowicz-Obata sur les transformations conformes des variétés finslériennes. Plus précisément, un champ de vecteurs conforme complet et essentiel sur une variété finslérienne non-riemannienne, est un champ homothétique sur un espace vectoriel normé.

## 1. Definitions and results

In this paper a *Finsler metric* on a smooth manifold $M$ is a function $F : TM \to \mathbb{R}_{\geqslant 0}$ satisfying the following properties:

(1) It is smooth on $TM \setminus TM_0$, where $TM_0$ denotes the zero section of $TM$,

(2) For every $x \in M$, the restriction $F_{|T_xM}$ is a norm on $T_xM$, i.e., for every $\xi$, $\eta \in T_xM$ and for every nonnegative $\lambda \in \mathbb{R}_{\geqslant 0}$ we have
  (a) $F(\lambda \cdot \xi) = \lambda \cdot F(\xi)$,
  (b) $F(\xi + \eta) \leqslant F(\xi) + F(\eta)$,
  (c) $F(\xi) = 0 \Longrightarrow \xi = 0$.

We do not require that (the restriction of) the function $F$ is strictly convex. In this point our definition is more general than the usual definition. In addition we do not assume the metric to be reversible, i.e., we do not

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assume that $F(-\xi) = F(\xi)$. Geometrically speaking a Finsler metric is characterized by a smooth family $x \in M \mapsto \{\xi \in T_xM \mid F(\xi) = 1\} \subset T_xM$ of convex hypersurfaces (sometimes called *indicatrices*, cf. [6]) containing the zero section in the tangent bundle.

Recent references for Finsler geometry include [4, 20, 7, 3]. Particular classes of Finsler metrics which occur in our results are the following:

*Example 1.1 (Riemannian metric).* — For every Riemannian metric $g$ on $M$ the function $F(x,\xi) := \sqrt{g_{(x)}(\xi,\xi)}$ is a Finsler metric. Geometrically the Finsler metric is a smooth family of ellipsoids.

*Example 1.2 (Minkowski metric).* — Consider a norm on $\mathbb{R}^n$, i.e., a function $p : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ satisfying 2a, 2b, 2c. We canonically identify $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates $\Big(\underbrace{(x_1, ..., x_n)}_{x\in\mathbb{R}^n}, \underbrace{(\xi_1, ..., \xi_n)}_{\xi\in T_x\mathbb{R}^n}\Big)$. Then, $F(x,\xi) := p(\xi)$ is a Finsler metric. The metric is translation invariant, on the other hand every translation invariant Finsler metric is a Minkowski metric. Due to the translation invariance the Finsler metric is uniquely characterized by a convex hypersurface in a single tangent space $T_xM$.

Two Finsler metrics $F$ and $F_1$ on an open subset $U \subseteq M$ are called *conformally equivalent*, if $F_1 = \lambda \cdot F$ for a nowhere vanishing function $\lambda$ on $U$. We say that a differentiable mapping $f : (M_1, F_1) \to (M_2, F_2)$ is *conformal*, if the pullback of the metric $F_2$ is conformally equivalent to $F_1$, i.e., if for every $\xi \in T_xM$ we have $F_2\left(df_x(\xi)\right) = \lambda(x)F_1\left(\xi\right)$. If the conformal factor $\lambda$ is constant the map is called *homothetic*, for $\lambda = 1$ it is *isometric*. A vector field is called *conformal* (resp. *homothetic* or *isometric*) if its local flow acts by *conformal* (resp. homothetic or isometric) local diffeomorphisms. If the conformal vector field $v$ is *complete* then the flow $\phi^t : M \to M, t \in \mathbb{R}$ of $v$ is a one-parameter group of conformal diffeomorphisms of the manifold $M$. Obviously, if a metric $F_1$ is conformally equivalent to $F$, then every conformal vector field for $F$ is also a conformal vector field for $F_1$.

*Example 1.3.* — For the Finsler metric $F := \sqrt{g(\xi,\xi)}$ from Example 1.1, conformal vector fields for the Riemannian metric $g$ are conformal vector fields for the Finsler metric $F$, and vice versa. For Euclidean space $\mathbb{R}^n$ the description of conformal mappings for $n = 3$ is due to Liouville [15] and for $n \geqslant 3$ to Lie [14], for recent expositions cf. for example [5, Thm. A.3.7], [12] and [11].

*Example 1.4.* — For the Minkowski Finsler metric $F(x,\xi) := p(\xi)$ from Example 1.2, the mappings $x \in \mathbb{R}^n \mapsto H_t(x) = t \cdot x \in \mathbb{R}^n$ are homotheties for all $t > 0$. Then, the vector field $v(x) = \frac{d}{dt}\big|_{t=0} H_t(x) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is the corresponding homothetic vector field.

Now we can state our main result:

THEOREM 1.5. — *Suppose $v$ is a conformal and complete vector field on a connected Finsler manifold $(M, F)$ of dimension $n \geqslant 2$. Then, at least one of the following statement holds.*

1. (1) *There exists a Finsler metric $F_1$ conformally equivalent to $F$ such that the flow of $v$ preserves the Finsler metric $F_1$.*
2. (2) *The manifold $M$ is conformally equivalent to the sphere $S^n$ with its standard Riemannian metric.*
3. (3) *The manifold $M$ is diffeomorphic to Euclidean space $\mathbb{R}^n$, and the Finsler metric $F$ is conformally equivalent to a Minkowski metric, cf. Example 1.2. The vector field $v$ with respect to the Minkowski metric is homothetic.*

For Riemannian metrics, the statement above is called the conformal *Lichnerowicz-Obata* conjecture, and was proved independently by D.Alekseevksii [2], J.Ferrand [9], M.Obata [18] and R.Schoen [19], see also [13, 10]. Of course, in the Riemannian case, Example 1.2 corresponds to the Euclidean metric on $\mathbb{R}^n$. A conformal vector field satisfying the assumptions of the first case is also called *inessential,* otherwise it is called *essential.*

*Remark 1.6.* — Theorem 1.5 also implies the following result: If the conformal group is essential, i.e. if there is no conformally equivalent metric such that the conformal group becomes the isometry group, then the metric is conformally equivalent to the round sphere, or to a Minkowski space.

Theorem 1.5 was announced in [24] under the following additional assumptions: $M$ is closed, and the Finsler metric $F$ is *strictly convex*, i.e., the second derivative of $F^2_{|T_pM}$ has rank $n-1$ at every point on $T_pM - TM_0$. The proof is sketched in [24]. It is long and actually is a repeating of the proof from [9] (which is technically very nontrivial) in the Finsler case.

Our proof of Theorem 1.5 is much shorter. It is based on the following observation: for every Finsler metric $F$ we can canonically construct a Riemannian metric $g$ such that if $v$ is a conformal vector field for $F$, then it is also a conformal vector field for $g$. Then, by the Riemannian version of Theorem 1.5, the following two cases are possible:

- The flow of $v$ acts by isometries of a certain Riemannian metric $g_1$ conformally equivalent to the Riemannian metrics $g$. This case will be called "trivial case" in the proof of Theorem 1.5. In this case, it immediately follows that the flow of $v$ acts by the isometries of a particular metric $F_1$ conformally equivalent to $F$.

- The manifold is $S^n$ or $\mathbb{R}^n$, and the metric $g$ is conformally equivalent to the standard metric. In this case, all possible essential conformal vector fields $v$ can be explicitly described, cf. Example 1.3. A direct analysis of the flow of such vector field shows that the only Finsler metrics for which $v$ is a conformal vector field are as in Theorem 1.5.

*Remark 1.7.* — In conformal geometry the case of surfaces $n = 2$ is special due to the existence of holomorphic functions. Any holomorphic function defined on an open subset in the complex plane $\mathbb{C}$ with everywhere non-vanishing derivative is conformal. This shows that the part of Liouville's theorem on conformal transformations of Euclidean spaces stating that a conformal diffeomorphism between open subsets of Euclidean space is the restriction of a conformal diffeomorphism of the standard sphere only holds for dimensions $n \geqslant 3$. On the other hand the description of the conformal diffeomorphism of the $n$-dimensional sphere $S^n$ as compositions of homotheties and inversions in the Euclidean space $\mathbb{R}^n \cong S^n - \{p\}$ also holds for $n = 2$, as one concludes from the standard classification of (anti)holomorphic functions on $\mathbb{C}$ resp. $\mathbb{C}P^1$. It is shown by Alekseevskii [2, Thm.8] that an essential and complete conformal vector field on a surface only exists on the 2-sphere with the standard metric or on Euclidean 2-space. Therefore for our main result the case $n = 2$ is not exceptional.

## 2. Averaged Riemannian metric

For a given smooth norm $p$ on $\mathbb{R}^n$ we construct canonically a positive definite symmetric bilinear form $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

For a Finsler metric $F$, the role of $p$ will play the restriction of $F$ to $T_xM$. We will see that the constructed $g$ will smoothly depend on $x$, i.e., $g_{(x)}$ is a Riemannian metric.

Consider the unit sphere $S_1 = \{\xi \in \mathbb{R}^n \mid p(\xi) = 1\}$ of the norm $p$. Consider the (unique) volume form $\Omega$ on $\mathbb{R}^n$ such that the volume of the 1-ball $B_1 = \{\xi \in \mathbb{R}^n \mid p(\xi) \leqslant 1\}$ equals 1.

Denote by $\omega$ the volume form on $S_1$, whose values on the vectors $\eta_1, ..., \eta_{n-1}$ tangent to $S_1$ at the point $\xi \in S_1$ are given by $\omega(\eta_1, ..., \eta_{n-1}) := \Omega(\xi, \eta_1, \eta_2, ..., \eta_{n-1})$.

Now, for every point $\xi \in S_1$, consider the symmetric bilinear form $b_{(\xi)} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $b_{(\xi)}(\eta, \nu) = D^2_{(\xi)} p^2(\eta, \nu)$. In this formula, $D^2_{(\xi)} p^2$ is the second differential at the point $\xi$ of the function $p^2$ on $\mathbb{R}^n$. The analytic expression for $b_{(\xi)}$ in the coordinates $(\xi_1, ..., \xi_n)$ is

$$b_{(\xi)}(\eta,\nu) = \sum_{i,j} \frac{\partial^2 p^2(\xi)}{\partial \xi_j \partial \xi_j} \eta_i \nu_j. \tag{2.1}$$

Since the norm $p$ is convex, the bilinear form (2.1) is nonnegative definite: for all $\eta$ we have

$$b_{(\xi)}(\eta,\eta) \geqslant 0. \tag{2.2}$$

Clearly, for every $\xi \in S_1$, we have

$$b_{(\xi)}(\xi,\xi) > 0 \tag{2.3}$$

Now consider the following bilinear symmetric $2-$form $g$ on $\mathbb{R}^n$: for $\eta, \nu \in \mathbb{R}^n$, we put

$$g(\eta,\nu) = \int_{S_1} b_{(\xi)}(\eta,\nu)\omega.$$

We assume that the orientation of $S_1$ is chosen in such a way that $\int_{S_1} \omega \geqslant 0$. Because of (2.2) and (2.3), $g$ is positive definite.

*Remark 2.1.* — If the norm $p$ comes from a scalar product, i.e., if $p(\xi) = \sqrt{b_1(\xi,\xi)}$ for a certain positive definite symmetric 2-form $b_1$, then $b$ is equal to $b_1$ multiplied by a constant only depending on the dimension.

Starting with a Finsler metric $F$, we can use this construction for every tangent space $T_xM$ of the manifold, the role of $p$ is played by the restriction $F_{|T_xM}$ of the Finsler metric to the tangent space $T_xM$. Since this construction depends smoothly on the point $x \in M$, we obtain a Riemannian metric $g = g(F)$ on $M$. We call this metric the *averaged Riemannian metric* of the Finsler metric $F$.

*Remark 2.2.* — It is easy to check that for the metric $F_1 := \lambda(x) \cdot F$ the constructed metric $g_1$ is conformally equivalent to the metric $g$ constructed for $F$. More precisely, $g_1 = \lambda(x)^2 \cdot g$. Then, a conformal diffeomorphism (conformal vector field, respectively) for $F$ is also a conformal diffeomorphism (conformal vector field, respectively) for $g$. Moreover, if $v$ is conformal for $F$ and is an isometry (homothety, respectively) for $g$, then it is an isometry (homothety, respectively) for $F$ as well.

*Remark 2.3.* — This averaging construction is quite natural and it is very possible that other researchers in Finsler geometry already thought about it, but we could not find any reference about it in the literature, nor any significant result in Finsler geometry whose proof is based on the averaged metric. It would certainly be worthwile to further investigate its properties. Recently, Szabo [21] uses a similar averaging construction to explicitely construct all Finlser Berwald metrics. There are other canonical constructions of a bilinear form starting from a norm. R. Schneider told us, that a for a convex geometer the natural bilinear form corresponding to a convex body is one corresponding to the John ellipsoid of this convex body.

These constructions have the nice properties listed in Remarks 2.1, 2.2. We still prefer our averaged Riemannian metric, since the method of Szabo assumes that the norm is strictly convex, and since it is not clear whether the John ellipsoid depends smoothly on the norm. In [22] Torrome suggests another averaging construction for Finsler metrics.

## 3. Proof of Theorem 1.5

Let $v$ be a complete conformal vector field on a connected Finsler manifold $(M, F)$. Then, it is also a conformal vector field for the averaged Riemannian metric $g$. Then, by the Riemannian version of our Theorem, which, as we explained in the introduction, was proved in [18, 2, 23, 9, 19], we have the following possibilities:

> (**Trivial case**) $v$ is a Killing vector field of a conformally equivalent metric $\lambda(x)^2 g$.
>
> (**Interesting case**) For a certain function $\lambda$, the Riemannian manifold $(M^n, \lambda(x)^2 g)$ is $(\mathbb{R}^n, g_0)$, or $(S^n, g_1)$, where $g_0$ resp. $g_1$ is the Euclidean metric on $\mathbb{R}^n$ resp. the standard metric of sectional curvature 1 on $S^n$.

In the **trivial case**, as we explained in Remark 2.2, for a certain function $\lambda$, $v$ is a Killing vector field for the metric $F_1 := \lambda \cdot F$, which was one of the possibilities in Theorem 1.5.

Now we treat the **interesting case**. Without loss of generality, we can assume that $(M, g)$ is $(\mathbb{R}^n, g_0)$, or $(S^n, g_1)$.

### 3.1. Case 1: $(M, g) = (\mathbb{R}^n, g_0)$.

Since the vector field is complete, it generates a one parameter group $\phi^t : \mathbb{R}^n \to \mathbb{R}^n$ of conformal transformations with respect to the Finsler metric $F$ and the averaged Riemannian metric $g_0$. It follows from Liouville's theorem that for any $t$ the mapping $\phi^t$ is a homothety of the Riemannian metric $g_0$. In other words, in an appropriate cartesian coordinate system $(x_1, ..., x_n)$, the conformal diffeomorphism $\phi = \phi^1$ has the form

$$\phi(x_1, ..., x_n) = \mu \cdot (x_1, ..., x_n)A, \tag{3.1}$$

where $A$ is an orthogonal $(n \times n)$-matrix. Without loss of generality we can assume that $0 < \mu < 1$. We will show that in this case the metric $F$ is as in Example 1.2.

We identify $T_x\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$ with the help of the cartesian coordinates $x = (x_1, ..., x_n)$. We assume that the first component of the product $\mathbb{R}^n \times \mathbb{R}^n$ corresponds to our manifold $\mathbb{R}^n$, and that the second component of the product $\mathbb{R}^n \times \mathbb{R}^n$ corresponds to the tangent space. The coordinates on the tangent spaces will be denoted by $\xi$, so $\Big(\underbrace{(x_1, ..., x_n)}_{x\in\mathbb{R}^n}, \underbrace{(\xi_1, ..., \xi_n)}_{\xi\in T_x\mathbb{R}^n}\Big)$ is a coordinate system on $T_x\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

Clearly, the differential of the mapping $\phi$ given by (3.1) is given by

$$d\phi_x(\xi) = (\mu \cdot (x_1, ..., x_n)A, \mu \cdot (\xi_1, ..., \xi_n)A)\,.$$

Then, for every $\xi, \eta \in T_x\mathbb{R}^n$, we have $g_{\phi(x)}(d\phi_x(\xi), d\phi_x(\eta)) = \mu^2 \cdot g_{(x)}(\xi, \eta)$. Hence, by Remark 2.2, $F\left(\phi(x), d\phi_x(\xi)\right) = \mu \cdot F\left(x, \xi\right)$. Consider the mapping

$$h : T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,$$
$$h(x_1, ...x_n, \xi_1, ..., \xi_n) = (\mu \cdot (x_1, ..., x_n)A, (\xi_1, ..., \xi_n)A)\,.$$

By construction, this mapping satisfies $F\left(h(x, \xi)\right) = F(x, \xi)$. Since the orthogonal group $\mathbb{O}(n)$ is compact, we can choose a sequence $m_j \to \infty$ such that $A^{m_j} \to 1 \in \mathbb{O}(n)$ for $j \to \infty$. Then, $(0, \xi) = \lim_{j\to\infty} h^{m_j}(x, \xi)$. Hence,

$$F(0, \xi) = F\left(\lim_{j\to\infty} h^{m_j}(x, \xi)\right) = \lim_{j\to\infty} F\left(h^{m_j}(x, \xi)\right) = F(x, \xi).$$

Thus, $F$ is translation invariant and therefore a Minkowski metric, cf. Example 1.2. Hence in this case, up to conformal equivalence, the Finsler metric is a Minkowski metric, and the conformal vector field is homothetic.

### 3.2. Case 2: $(M, g) = (S^n, g_1)$

Then, by [13, Thm. 12] any essential conformal vector field $v$ vanishes at exactly one (Case 2a) or exactly two (Case 2b) points. We denote by $v^{-1}(0) = \{x \in M \mid v(x) = 0\}$ the set of zeroes. If we assume $v(x) = 0$ we use the stereographic projection $s_x : S^n - \{x\} \to \mathbb{R}^n$ and obtain with the push forward of the vector field $v$ a complete and conformal vector field on $\mathbb{R}^n$.

#### 3.2.1. Case 2a: Suppose $v^{-1}(0) = \{x, y\}, x \neq y$

Suppose the conformal vector field $v$ vanishes precisely at two points $x$ and $y$ of the sphere. We will show that the Finsler metric $F$ is in fact Riemannian.

Denote by $s_+ : (S^n - \{x\}, g_1) \to (\mathbb{R}^n, g_0)$ the stereographic projection from $x$ which is conformal with respect to the standard Riemannian metrics $g_0, g_1$ with conformal factor $\sigma_+$. Here, $\mathbb{R}^n$ should be identified with the hyperplane through the origin parallel to the tangent spaces $T_x S^n$. Then we define a Finsler metric $F_+$ by $s_+^* F_+ = \sigma_+ F$. Then the averaged Riemannian metric of $F_+$ coincides with the Euclidean metric $g_0$. The push foward vector field $v_+ := s_+^* v$ is a conformal and complete vector on $\mathbb{R}^n$ with respect to the Finsler metric $F_+$ as well with respect to the standard metric $g_1$. This vector field has exactly one zero on $\mathbb{R}^n$. Therefore, by section 3.1, the Finsler metric $F_+$ is a Minkowski metric, i.e., translation invariant. In particular we can assume without loss of generality that the zero point of $v_+$ is the origin of $\mathbb{R}^n$. Hence we can assume that the zero points of $v$ on $S^n$ are antipodal points, i.e., $v^{-1}(0) = \{\pm x\}$.

The stereographic projection $s_- : (S^n - \{-x\}, g_1) \to (\mathbb{R}^n, g_0); s_-(q) = s_+(-q)$ from $-x$ is a conformal mapping with conformal factor $\sigma_-$ with $\sigma_+(-q) = \sigma_-(q), q \in S^n$ i.e., $s_\pm^* g_0 = \sigma_\pm^2 g_1$. Then we define also the Finsler metric $F_-$ on $\mathbb{R}^n$ by $s_-^* F_- = \sigma_- F$. The averaged Riemannian metric of $F_-$ equals the Euclidean metric $g_0$. The push-forward $v_- := (s_-)_* v$ of the vector field $v$ is a conformal vector field on $\mathbb{R}^n$ with respect to the Finsler metric $F_-$ and, hence, with respect to the standard metric $g_0$. Both vector fields $v_\pm$ are evidently complete and have precisely one zero at the origin. Therefore, by section 3.1, the Finsler metrics $F_\pm$ are Minkowski metrics, i.e., translation invariant.

It is well known that the composition $s_- \circ s_+^{-1} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n$ equals the inversion $I(q) = q/g_0(q, q)$ at the unit sphere. Therefore, the inversion defines a conformal transformation $I : (\mathbb{R}^n - \{0\}, F_+) \to (\mathbb{R}^n - \{0\}, F_-)$ between the two Minkowski metrics. The differential $dI_q$ of the inversion at a point $q \in S^{n-1} := \{u \in \mathbb{R}^n \mid g_0(u, u) = 1\}$ equals the reflection $R_q$ at the hyperplane normal to $q$. This implies that $dI_q^* F_+ = R_q^* F_+ = F_-$ for any $q \in S^{n-1}$. Since the reflections generate the orthogonal group and since the Finsler metrics $F_\pm$ are translation invariant, it follows that the norms $F_\pm|T_0M$ at the origin are invariant under the full orthogonal group and hence Euclidean.

### 3.2.2. Case 2b: $v^{-1}(0) = \{x\}$

We assume that the vector field $v$ on $S^n$ vanishes precisely at one point $x \in S^n$. We will again show that the metric $F$ is Riemannian.

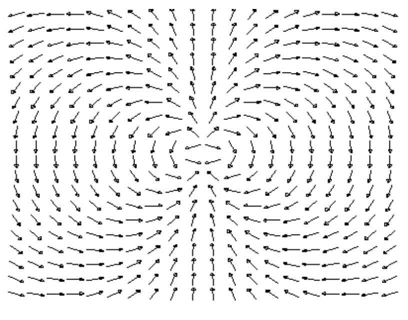
We again consider the stereographic projections $s_\pm : S^n - \{\pm x\} \to \mathbb{R}^n$ from the points $x, -x$ as introduced in Section 3.2.1, and denote by $F_\pm := (s_\pm)_* F$ the induced Finsler metrics on $\mathbb{R}^n$. The push-forward $v_+$ of $v$ with respect to $s_+$ vanishes nowhere on $\mathbb{R}^n$ and is complete, let $\psi^t$ be its flow on $\mathbb{R}^n$. Then Liouville's theorem implies that for an arbitrary $t$ the conformal diffeomorphism $f = \psi^t$ has the form $f(x) = \mu A x + b$ with an orthogonal matrix $A$ and $\mu > 0, b \in \mathbb{R}^n$. Since the mapping $f$ has no fixed point it follows that $b \neq 0\,;\, \mu = 1$ and $Ab = b$. We introduce the following notation: $f_{A,b}(q) = Aq + b$ for an orthogonal matrix $A$ and $b \in \mathbb{R}^n$ with $Ab = b$.

If we use the stereographic projection $s_-$, then the push-forward of $v$ has a zero in the origin 0 and the mapping $f$ transforms to $\overline{f_{A,b}} = I \circ f_{A,b} \circ I$ where $I = \sigma_- \circ \sigma_+^{-1}$ is the inversion at the unit sphere. Hence $\overline{f}_{A,b}(q) = \frac{Aq + b\|q\|^2}{1+2\langle Aq,b\rangle + \|b\|^2\|q\|^2}$ where $< .,. >= g_0(.,.)$ with related norm $\|.\|$. The conformal factor is given by $\psi(q) = \frac{1}{1+2\langle Aq,b\rangle+\|b\|^2\|q\|^2}$. In particular the conformal mapping $\overline{f}_{A,b}$ induces at the fixed point 0 the map

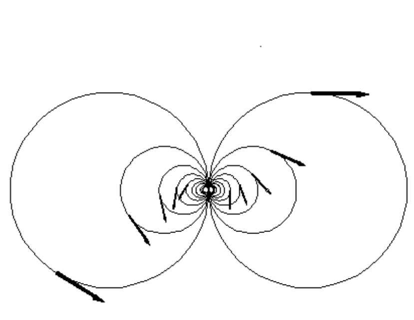
$$\xi \in T_0\mathbb{R}^n \mapsto d\,\left(\overline{f}_{A,b}\right)_0(\xi) = A\xi \in T_0\mathbb{R}^n \tag{3.2}$$

which is an isometry also with respect to the restriction of the Finsler metric $F_-$ to 0 since $\psi(0) = 1$.

For an orthogonal mapping $A$ we introduce the map $h_A : z \in \mathbb{R}^n \to Az \in \mathbb{R}^n$ with induced mapping $(z,\xi) \in T_z\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \mapsto dh_A(z,\xi) = (Az, A\xi) \in T_z\mathbb{R}^n$ . We want to show that the map $h_A$ is an isometry for the Finsler metric $F_-$. Let $v_1$ be the vector field on $S^n$ which corresponds to the parallel vector field $b$ on $\mathbb{R}^n$ with respect to the stereographic projection $s_+$, i.e., $ds_+(v_1)(q) = b$ for all $q \in S^n$. The vector field $v_1$ is a conformal vector field with respect to the standard Riemannian metric $g_1$ with exactly one zero in $x$. The flow lines of $v_1$ consist of the circles passing through $x$ with a common tangent vector, see the picture. Hence we obtain the following properties of the flow $\phi^t : S^n \to S^n$ of the conformal vector field $v_1$ on $S^n$ :



**The vector field $v_1/|v_1|$ in dimension 2**



**The integral curves of $v_1$ in dimension 2**

*Remark 3.1.* — The flow $\phi^t$ of the conformal vector field $v_1$ defined above satisfies the following properties:

(a) For any point $q \in S^n : x = \lim_{t\to\pm\infty} \phi^t(q)$.

(b) For any tangent vector $\xi \in T_x S^n, F(x,\xi) = 1$ there is a sequence $q_i \in S^n - \{x\}$ with $\lim_{i\to\infty} q_i = x$ and $\xi = \lim_{i\to\infty} \frac{v_1(q_i)}{F(q_i, v(q_i))}$

Now we show that the mapping $h_A$ is an isometry also for the Finsler metric $F_-$ : We can choose a sequence $m_i \to \infty$ with $A^{m_i} \to 1$. In particular for a given $(z,\xi) \in T_z\mathbb{R}^n; z \neq 0$ there is a unique $(0,\xi_0) \in T_0\mathbb{R}^n; F((0,\xi_1)) = 1$ such that

$$(0,\xi_1) = \lim_{i\to\infty} \frac{d\overline{f}^{m_i}_{A,b}(z,\xi)}{F_-\left(d\overline{f}^{m_i}_{A,b}(z,\xi)\right)}. \tag{3.3}$$

Since the mapping $\overline{f}_{A,b}$ is conformal for $F_-$ and since $h_A$ and $f_{A,b}$ commute it follows that

$$\begin{aligned}\frac{F_-\left(dh_A(z,\xi)\right)}{F_-\left((z,\xi)\right)} &= \lim_{i\to\infty} \frac{F_-\left(d\overline{f}^{m_i}_{A,b}\, dh_A\,(z,\xi)\right)}{F_-\left(d\overline{f}^{m_i}_{A,b}(z,\xi)\right)} \\ &= \lim_{i\to\infty} \frac{F_-\left(dh_A\left(d\overline{f}^{m_i}_{A,b}(z,\xi)\right)\right)}{F_-\left(d\overline{f}^{m_i}_{A,b}(z,\xi)\right)} = \frac{F_-\left(d\,h_A\,(0,\xi_0)\right)}{F_-\left((0,\xi_0)\right)} \\ &= \frac{F_-\left(d\,\left(\overline{f}_{A,b}\right)_0(0,\xi_0)\right)}{F_-\left((0,\xi_0)\right)} = 1\end{aligned}$$

as shown above, cf. Equation 3.2. Therefore the mapping $h_A$ is an isometry of the Finsler metric $F_-$. This implies that also the flow generated by $\overline{f}_{1,b} = \overline{f}_{A,b} \circ h_A^{-1}$ is conformal for the Finsler metric $F_-$. Therefore the vector field $v_1$ on $S^n$ is also a conformal vector field for the Finsler metric $F$ on $S^n$.

Let us now consider the following functions $m, M : S^n \to \mathbb{R}_{\geqslant 0}$:

$$m(q) := \frac{F^2(q, v_1(q))}{g_{(q)}(v_1(q), v_1(q))},$$
$$M(q) := \max_{\eta \in T_q S^n,\ \eta \neq 0} \frac{F^2(q,\eta)}{g_{(q)}(\eta,\eta)} - \min_{\eta \in T_q S^n,\ \eta \neq 0} \frac{F^2(q,\eta)}{g_{(q)}(\eta,\eta)}.$$

Both are continuous functions invariant with respect to the flow $\phi^t$ of $v_1$. It follows from Remark 3.1(a) that the function $m$ is a constant, i.e., there exists $\mu > 0$ such that $F^2(q, v_1(q)) = \mu\, g_{(q)}(v_1(q), v_1(q))$. Part (b) of Remark 3.1 implies that for every $0 \neq \eta \in T_x S^n$ we have $\frac{F^2(x,\eta)}{g_{(x)}(\eta,\eta)} = \mu$. Hence,

$$M(q) = \max_{\eta \in T_q S^n,\ \eta \neq 0} \frac{F^2(q,\eta)}{g_{(q)}(\eta,\eta)} - \min_{\eta \in T_q S^n,\ \eta \neq 0} \frac{F^2(q,\eta)}{g_{(q)}(\eta,\eta)} = \mu - \mu = 0.$$

But since $M$ is also flow invariant by Remark 3.1(a), we have $M(q) = 0$ for all $q \in S^n$, i.e., $F$ is up to a constant the norm of the standard metric $g$. Theorem 1.5 is proved. □

As a consequence of the Proof of Theorem 1.5 the inversion of the averaged Riemannian metric is not a conformal map for a non-Euclidean Minkowski metric, cf. Section 3.2.1. Therefore one obtains from Liouville's theorem on the conformal transformations of an Euclidean vector space the following description of the conformal transformations of a Minkowski space:

*Remark 3.2.* — Let $V$ be an $n$-dimensional vector space with a Minkowski norm $F$ which is not Euclidean. Denote by $g$ the corresponding averaged Euclidean metric. If $f : (U, F) \to (V, F)$ is a conformal mapping from an open subset $U$ and $n \geqslant 3$ then $f$ is a a similarity with respect to the Minkowski metric $F$ and with respect to the Euclidean metric $g$. Hence it is of the form $x \in V \mapsto \mu A x + b \in V$ for some $\mu > 0; b \in V$ and an orthogonal mapping $A$ of $(V, g)$.

## 4. Conclusion

Theorem 1.5 describes complete conformal vector fields of Finsler metrics; it appears that no new phenomena (with respect to the Riemannian case) appear. Our proof is based on the construction of averaged metric in

Section 2, and on the description of conformal vector fields for Riemannian metrics due to [14, 15, 18, 2, 23, 13, 10, 9, 19].

As an interesting and much more involved problem in Finsler geometry related to transformation groups we would like to suggest to generalize the projective Lichnerowicz-Obata conjecture for Finsler metrics, see [16, 17] for the proof of the Riemannian version, see also [20].

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