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Weak Finsler structures and the Funk weak metric

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Abstract

We discuss general notions of metrics and of Finsler structures which we call *weak metrics* and *weak Finsler structures*. Any convex domain carries a canonical weak Finsler structure, which we call its *tautological weak Finsler structure*. We compute distances in the tautological weak Finsler structure of a domain and we show that these are given by the so-called *Funk weak metric*. We conclude the paper with a discussion of geodesics, of metric balls, of convexity, and of rigidity properties of the Funk weak metric.

1. *Introduction*

A *weak metric* on a set is a function defined on pairs of points in that set which is nonnegative, which can take the value ∞, which vanishes when the two points coincide and which satisfies the triangle inequality. Compared to an ordinary metric, a weak metric can thus take the values 0 or ∞, and it need not be symmetric. This general notion turns out to be useful in various situations. The terminology "weak metric" is due to Ribeiro [**19**], but the notion can at least be traced back to the work of Hausdorff (see [**13**]). In the paper [**16**], a number of natural weak metrics are discussed. In this paper, we are mostly interested in a class of weak metrics that is related to convex geometry and to a general notion of Finsler structure on manifolds.

A basic construction in convex geometry is the notion of *Minkowski norm*, which associates to any convex set containing the origin in a vector space V a translation-invariant homogenous weak metric on V. Finsler geometry is an extension of this construction to an arbitrary manifold. We define a *weak Finsler strucure* on a differentiable manifold to be a field of convex sets on that manifold. More precisely, a weak Finsler structure is a subset of the tangent space of the manifold whose intersection with each fiber is a convex set containing the origin. The Minkowski norm in each tangent space of a manifold endowed with a weak Finsler structure gives rise to a function defined on the total space of the tangent bundle. We call this function the *Lagrangian* of the weak Finsler structure. Integrating this

Lagrangian on piecewise smooth curves in the manifold defines a length structure and thus a notion of distance on the manifold. This distance is generally a weak metric.

A case of special interest is when the manifold is a convex domain Ω in $\mathbb{R}^n$ and when the weak Finsler structure is obtained by replicating at each point of Ω the domain Ω itself. We call this the *tautological* weak Finsler structure, and we study some of its basic properties in the present paper. More precisely, we first give a formula for the distance between two points. It turns out that this distance coincides with the weak metric introduced by P. Funk in [**12**]. We then study the geometry of balls and the geodesics in the Funk weak metric.

Modern references on Finsler geometry include [**1**, **3**, **4**, **10**]. One of Herbert Busemann's major ideas, expressed in [**6–9**] is that Finsler geometry should be developed without local coordinates and without the use of differential calculus. This paper is a contribution to this program.

2. *Preliminaries on convex geometry*

In this section, we recall a few notions in convex geometry that will be used in the sequel.

Given a convex subset Ω of $\mathbb{R}^n$, we shall denote by $\overline{\Omega}$ its closure, $\overset{o}{\Omega}$ its interior, and $\partial\Omega = \overline{\Omega} \setminus \overset{o}{\Omega}$ its boundary.

Let $\Omega \subset \mathbb{R}^n$ be a (not necessarily open) convex set and let x be a point in Ω.

Definition 2·1. *The* radial function of Ω with respect to x *is the function* $r_{\Omega,x} : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ *defined as*

$$r_{\Omega,x}(\xi) = \sup\{t \in \mathbb{R} \mid (x + t\xi) \in \Omega\}.$$

Definition 2·2. *The* Minkowski function of Ω with respect to x *is the function* $p_{\Omega,x} : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ *defined by*

$$p_{\Omega,x}(\xi) = \frac{1}{r_{\Omega,x}(\xi)}.$$

Classically, the Minkowski function is associated to an open convex subset Ω of $\mathbb{R}^n$ containing the origin 0, and taking $x = 0$. This function is sometimes called the *Minkowski weak norm* of the convex set (see e.g. [**11**, **15**, **20**, **21**]).

We also recall that for any convex set Ω in $\mathbb{R}^n$, there exists a unique smallest affine subspace of $\mathbb{R}^n$ containing Ω, which is called the *affine span* of Ω, and which we denote by $\mathrm{Aff}(\Omega)$. The intersection of Ω with $\mathrm{Aff}(\Omega)$ has nonempty interior in $\mathrm{Aff}(\Omega)$; we call this interior the *relative interior of* Ω, and we denote it by $\mathrm{RelInt}(\Omega)$.

The following proposition collects a few basic properties of the Minkowski function. In particular, Property (8) tells us that we can reconstruct the relative interior of Ω from the Minkowski function of Ω at any point. The proofs are easy.

PROPOSITION 2·3. *Let Ω be a convex subset of $\mathbb{R}^n$. For every x in Ω and for every ξ and η in $\mathbb{R}^n$, we have*:

(1) $p_{\Omega,x}(\xi) = \inf\{t \geqslant 0 \mid \xi \in t(\Omega - x)\}$. (*Here, $\Omega - x$ denotes the Minkowski sum of Ω and the singleton $\{-x\}$.*);

(2) *if the ray $\{x + t\xi \mid t \geqslant 0\}$ is contained in Ω, then* $p_{\Omega,x}(\xi) = 0$;

(3) $p_{\Omega,x}(\lambda\xi) = \lambda p_{\Omega,x}(\xi)$ *for* $\lambda \geqslant 0$;

(4) $p_{\Omega,x}(\xi + \eta) \leqslant p_{\Omega,x}(\xi) + p_{\Omega,x}(\eta)$;

(5) *the Minkowski function $p_{\Omega,x}$ is convex*;

(6) *if* x *is in* $\overset{o}{\Omega}$, *then* $p_{\Omega,x}$ *is continuous*;
(7) *if* Ω *is closed, then* $\Omega = \{y \in \mathbb{R}^n \mid y = x + \xi,\ p_{\Omega,x}(\xi) \leqslant 1\}$;
(8) $\mathrm{RelInt}(\Omega) = \{y = x + \xi \mid p_{\Omega,x}(\xi) < 1\}$;
(9) *if* $\Omega_1 = \mathrm{RelInt}(\Omega)$, *then* $p_{\Omega_1,x} = p_{\Omega,x}$.

In some cases, we can give explicit formulas for the Minkowski function $p_{\Omega,x}$. For instance, the Minkowski function of the closed ball $B = B(0, R)$ in $\mathbb{R}^n$ of radius R and center 0 with respect to any point x in B is given by

$$p_{B,x}(\xi) = \frac{\sqrt{\langle \xi, x\rangle^2 + (R^2 - |x|^2)|\xi|^2} + \langle \xi, x\rangle}{(R^2 - |x|^2)}.$$

The Minkowski function of a half-space $H = \{x \in \mathbb{R}^n | \langle \nu, x\rangle \leqslant s\}$, where ν is a vector in $\mathbb{R}^n$ (which is orthogonal to the hyperplane bounding H) and where s is a real number, with respect to a point x in H, is given by

$$p_{H,x}(\xi) = \max\left(\frac{\langle \nu, \xi\rangle}{s - \langle \nu, x\rangle}, 0\right).$$

We shall use this formula later on in this paper. We also recall the following:

Definition 2·4 (Support hyperplane). *Let* Ω *be a nonempty subset of* $\mathbb{R}^n$. *An affine hyperplane* A *in* $\mathbb{R}^n$ *is called a* support hyperplane *for* Ω *if* Ω *is contained in one of the two closed half-spaces bounded by* A *and if* $\overline{\Omega} \cap A \neq \emptyset$.

If A *is a support hyperplane for* Ω *and if* x *is a point in* $\overline{\Omega} \cap A$, *then* A *is called a* support hyperplane for Ω at x. *When* $\Omega \subset \mathbb{R}^2$, *then* A *is called a* support line.

The case where Ω is contained in some hyperplane A is an uninteresting example of a support hyperplane, which is the hyperplane A itself.

Suppose now that Ω is a convex subset of $\mathbb{R}^n$. It is known that any point on the boundary of Ω is contained in at least one of its support hyperplanes (see e.g. [**11**, p. 20]). The intersection of Ω with any of its support hyperplanes is a convex set which is nonempty if Ω is closed. This intersection is not always reduced to a point.

We recall the notion of a strictly convex subset in $\mathbb{R}^n$, and before that we note the following well-known proposition (see [**11**]):

PROPOSITION 2·5. *Let* Ω *be an open convex subset of* $\mathbb{R}^n$. *Then, the following are equivalent*:
(1) $\partial\Omega$ *does not contain any nonempty open affine segment*;
(2) *each support hyperplane of* Ω *intersects* $\partial\Omega$ *in exactly one point*;
(3) *support hyperplanes at distinct points of* $\partial\Omega$ *are distinct*;
(4) *any linear function on* $\mathbb{R}^n$ *has exactly one maximum on* $\partial\Omega$.

Definition 2·6 (Strictly convex subset). *Let* Ω *be an open convex subset of* $\mathbb{R}^n$. *Then,* Ω *is said to be* strictly convex *if one (or, equivalently, all) the properties of Proposition* 2·5 *are satisfied.*

3. *The notion of weak metric*

Definition 3·1 (Weak metric). *A* weak metric *on a set* X *is a function* $\delta : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ *satisfying*:
(1) $\delta(x, x) = 0$ *for all* x *in* X;
(2) $\delta(x, z) \leqslant \delta(x, y) + \delta(y, z)$ *for all* x, y *and* z *in* X.

We say that such a weak metric δ is *symmetric* if $\delta(x, y) = \delta(y, x)$ for all x and y in X; that it is *finite* if $\delta(x, y) < \infty$ for every x and y in X; that δ is *strongly separating* if we have the equivalence

$$\min(\delta(x, y), \delta(y, x)) = 0 \iff x = y;$$

and that δ is *weakly separating* if we have the equivalence

$$\max(\delta(x, y), \delta(y, x)) = 0 \iff x = y.$$

We recall that the notion of weak metric already appears in the work of Hausdorff (cf. [**13**], in which Hausdorff defines asymmetric distances on various sets of subsets of a metric space).

Definition 3·2 (Geodesic). *Let (X, δ) be a weak metric space and let $I \subset \mathbb{R}$ be an interval. We shall say that a map $\gamma : I \to X$ is* geodesic *if for every t_1, t_2 and t_3 in I satisfying $t_1 \leqslant t_2 \leqslant t_3$ we have*

$$\delta(\gamma(t_1), \gamma(t_2)) + \delta(\gamma(t_2), \gamma(t_3)) = \delta(\gamma(t_1), \gamma(t_3)). \tag{3·1}$$

In the classical terminology, Equation 3·1 says that *the point $\delta(t_2)$ is between the points $\delta(t_1)$ and $\delta(t_3)$* (cf. [**8**]).

Weak metrics were extensively studied by Busemann, cf. [**6–9**]. A basic example of a weak metric defined on a convex set in $\mathbb{R}^n$ is the following:

Example 3·3. *Let $\Omega \subset \mathbb{R}^n$ be a convex set such that $0 \in \overline{\Omega}$ and let*

$$p(\xi) = p_{\Omega,0}(\xi) = \inf\{t > 0 \mid \xi \in t\,\Omega\}$$

be the Minkowski weak norm with respect to 0 *of* Ω*. Then, the function $\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \bigcup \{\infty\}$ defined by*

$$\delta(x, y) = p(y - x)$$

is a weak metric on $\mathbb{R}^n$. For this weak metric, we have the following equivalences:

(1) *δ is finite $\Leftrightarrow 0 \in \overset{o}{\Omega}$*;
(2) *δ is symmetric $\Leftrightarrow \Omega = -\Omega$*;
(3) *δ is strongly separating $\Leftrightarrow \Omega$ is bounded*;
(4) *δ is weakly separating $\Leftrightarrow \Omega$ does not contain any Euclidean line.*

The weak metric on $\mathbb{R}^n$ defined in Example 3·3 is called the *Minkowski weak metric associated to* Ω. The associated weak metric space $(\mathbb{R}^n, \delta)$ is called a *weak Minkowski space*.

4. *Weak length spaces*

Let X be a topological space. We shall say that a collection Γ of continuous paths $\gamma : [a, b] \to X$, where $[a, b]$ can be any compact interval of $\mathbb{R}$, is a *semigroupoid of paths* on X if the following properties hold:

(1) if $\gamma_1 : [a, b] \to X$ and $\gamma_2 : [c, d] \to X$ satisfy $\gamma_1(b) = \gamma_2(c)$, then the concatenation $\gamma_1 * \gamma_2$ is in Γ,
(2) any constant path belongs to Γ.

A typical example of a semigroupoid of paths is given by the set of all piecewise smooth paths in a smooth manifold.

Remark. Regarding to the abstract notion of semigroupoid, it would not be necessary to assume that all constant paths belong to Γ, but this hypothesis is convenient and does not reduce the generality of our concepts.

We shall use the following notion:

Definition 4·1 (weak length structure). *Let X be a topological space and let Γ be a semigroupoid of paths on X. A* weak length structure *on (X,Γ) is a function $\ell:\Gamma\to[0,\infty]$ such that the following three properties are satisfied:*

(1) *(additivity.) For every γ_1 and γ_2 in Γ, we have $\ell(\gamma_1 * \gamma_2) = \ell(\gamma_1) + \ell(\gamma_2)$;*
(2) *for any constant path c, we have $\ell(c) = 0$;*
(3) *(invariance under reparametrization.) If $[a,b]$ and $[c,d]$ are intervals of $\mathbb{R}$, if $\gamma\colon[a,b]\to X$ is a path in X which is in Γ and if $f\colon[c,d]\to[a,b]$ is a continuous surjective nondecreasing map such that $\gamma\circ f$ is in Γ, then $\ell(\gamma) = \ell(\gamma\circ f)$.*

Definition 4·2. *A* weak length space *is a triple (X,Γ,ℓ) where X is a topological space, Γ is a semigroupoid of paths on X and ℓ is a weak length structure on (X,Γ).*

Let us give a few additional definitions that use the above notation:

(i) the weak length structure Γ is said to be *separating* if $\ell(\gamma) > 0$ for any non constant path γ in Γ;
(ii) the weak length structure Γ is *reversible* if for every γ in Γ we have $\gamma^{-1}\in\Gamma$ and $\ell(\gamma^{-1}) = \ell(\gamma)$, where γ^{-1} is the reverse path of γ (with the obvious definition);
(iii) let (X,Γ,ℓ) be a weak length space such that $\gamma^{-1}\in\Gamma$ for every γ in Γ. Then the *arithmetic symmetrization* of the weak length structure ℓ is the weak length structure ${}^s\ell$ on (X,Γ) given by

$$ {}^s\ell(\gamma) = \frac{1}{2}\left(\ell(\gamma^{-1}) + \ell(\gamma)\right). $$

We study symmetrizations of weak metrics, in relation with the Funk weak metric and the Hilbert metric, in [**17**].

Given a groupoid of paths Γ on a topological space X, for x and y in X, we let

$$\Gamma_{x,y} = \{\gamma\in\Gamma \mid \gamma \text{ joins } x \text{ to } y\}.$$

LEMMA 4·3. *Let (X,Γ,ℓ) be a topological space equipped with a semigroupoid of paths and with a weak length structure. Then the function $\delta_\ell\colon X\times X\to\mathbb{R}$ defined by*

$$\delta_\ell(x,y) = \inf_{\gamma\in\Gamma_{x,y}} \ell(\gamma), \tag{4·1}$$

is a weak metric on X. This weak metric is symmetric if ℓ is reversible. If δ_ℓ is separating, then ℓ is separating.

The proof is immediate from the definitions.

Definition 4·4 (Weak length metric space). *Let (X,Γ,ℓ) be a topological space equipped with a semigroupoid of paths and with a weak length structure. The weak metric δ_ℓ defined in Definition* 4·1 *is called the* weak metric associated to the weak length structure ℓ. *A* weak length metric space *is a weak metric space obtained from the triple (X,Γ,ℓ) by equipping X with the associated weak metric δ_ℓ.*

5. *Weak Finsler structures*

We introduce a general notion of Finsler structure, which we call *weak Finsler structure*, and which can be considered as an infinitesimal notion of weak length structure.

Definition 5·1 (Weak Finsler structure). *Let M be a C^1 manifold and let TM be its tangent bundle. A* weak Finsler structure *on M is a subset $\widetilde{\Omega} \subset TM$ such that for each x in M, the subset $\Omega_x = \widetilde{\Omega} \cap T_xM$ of the tangent space T_xM of M at x is convex and contains the origin.*

We provide the set of all weak Finsler structures on M with the order relation $\preceq$ defined as follows:

$$\widetilde{\Omega_1} \preceq \widetilde{\Omega_2} \;\Leftrightarrow\; \widetilde{\Omega_1} \supset \widetilde{\Omega_2}.$$

Examples 5·2. In the following examples, M is a C^1 manifold.

(1) $\widetilde{\Omega} = TM$ is a weak Finsler structure, which we call the *minimal* weak Finsler structure.
(2) $\widetilde{\Omega} = M \subset TM$, embedded as the zero section, is a weak Finsler structure which we call the *maximal* weak Finsler structure.
(3) If $\widetilde{\Omega}$ and $\widetilde{\Omega}'$ are two weak Finsler structures on M, then $\widetilde{\Omega} \cap \widetilde{\Omega}' \subset TM$ is also a weak Finsler structure.
(4) If $\widetilde{\Omega}$ and $\widetilde{\Omega}'$ are two Finsler structures on M, then, taking the union of the Minkowski sums $\Omega_x + \Omega'_x$ of the convex sets in each tangent space T_xM, we obtain the *Minkowski sum weak Finsler structure* $\widetilde{\Omega} + \widetilde{\Omega}' \subset TM$.
(5) If ω is a differential 1-form on M, then

$$\widetilde{\Omega}_\omega = \{(x, \xi) \in TM \mid \omega_x(\xi) \leqslant 1\}$$

and

$$\widetilde{\Omega}_{|\omega|} = \{(x, \xi) \in TM \mid |\omega_x|(\xi) \leqslant 1\}$$

are weak Finsler structures on M.
(6) If ω and ω' are two 1-forms on M, then $\max(\omega, \omega')$ defines a weak Finsler structure on M.
(7) If $\widetilde{\Omega}$ is a weak Finsler structure on M and if $N \subset M$ is a C^1 submanifold, then $\widetilde{\Omega}_N = \widetilde{\Omega} \cap TN$ is a weak Finsler structure on N, called the weak Finsler structure *induced* by the embedding $N \subset M$.
(8) If $\widetilde{\Omega}$ is a weak Finsler structures on M, if N is a C^1 manifold and if $f : N \to M$ is a C^1 map, then $(Tf)^{-1}(\widetilde{\Omega}) \subset TN$ is a weak Finsler structure on N. We denote it by $f^*(\widetilde{\Omega})$ and we call it the *pull-back* of $\widetilde{\Omega}$ by the map f.

Definition 5·3 (Lagrangian). *The* Lagrangian *of a weak Finsler structure $\widetilde{\Omega}$ on a C^1 manifold M is the function on the tangent bundle TM whose restriction to each tangent space T_xM is the Minkowski function of Ω_x with respect to the origin of $T_xM \simeq \mathbb{R}^n$. It is thus defined by*

$$p(x, \xi) = p_{\widetilde{\Omega}}(x, \xi) = \inf\{t \mid t^{-1}\xi \in \Omega_x\}.$$

The quantity $p(x, \xi)$ is also called the *Finsler norm* of the vector (x, ξ) relative to the given weak Finsler structure.

Example 5·4. Let g be a Riemannian metric on M, let ω be a differential 1-form and let μ be a smooth function on M satisfying $|\mu\omega_x| < 1$ at every point x in M. Then, $p = \sqrt{g} + \mu\omega$ is the Lagrangian of a weak Finsler structure on M. Such a weak Finsler structure is usually called a *Randers metric* on M, and it has applications in physics (cf. e.g. [**4**, Section 11·3], and see also [**5**] for the relation of this metric with the Zermelo navigation problem.)

LEMMA 5·5. *Let $\widetilde{\Omega}$ be a weak Finsler structure on M. Assume that M (considered as embedded in TM as the zero section) is contained in the interior of $\widetilde{\Omega} \subset TM$. Then the associated Lagrangian $p: TM \to \mathbb{R}$ is upper semi-continuous.*

Proof. The hypothesis implies that for every x in M, the interior of each convex set $\Omega_x = \widetilde{\Omega} \cap T_xM \subset T_xM$ is nonempty. Therefore, the usual interior and the relative interior of Ω_x coincide. Property (9) of Proposition 2·3 implies then that the Lagrangian of $\widetilde{\Omega}$ coincides with the Lagrangian of its interior $\mathrm{Int}\,(\widetilde{\Omega})$.

One may therefore assume without loss of generality that $\widetilde{\Omega} \subset TM$ is an open set, and in particular, that

$$\widetilde{\Omega} = \{(x, \xi) \in TM |\ p(x, \xi) < 1\}$$

(see Proposition 2·3 (8)). Now for any $t \in \mathbb{R}$, the sublevel set $\{(x, \xi) \in TM \mid p(x, \xi) < t\}$ is either empty (when $t \leqslant 0$) or it is homothetic to the open set $\widetilde{\Omega} \subset TM$ (when $t > 0$). In any case, it is an open subset of TM, and $p : TM \to \mathbb{R}$ is therefore upper semi-continuous.

PROPOSITION 5·6. *Let $\widetilde{\Omega}$ be a Finsler structure on a C^1 manifold M and let $p_{\widetilde{\Omega}} : TM \to \mathbb{R}$ be the associated Lagrangian. Then:*

(1) *for every x in M, the function $\xi \mapsto p(x, \xi)$ is a weak norm on T_xM;*

(2) *if $\widetilde{\Omega}' \subset TM$ is another Finsler structure on M, with associated Lagrangian $p_{\widetilde{\Omega}'}$, then we have the equivalence*

$$\widetilde{\Omega} \preceq \widetilde{\Omega}' \iff p_{\widetilde{\Omega}} \leqslant p_{\widetilde{\Omega}'};$$

(3) *$p_{\widetilde{\Omega}} : TM \to \mathbb{R}$ is Borel-measurable.*

Proof. The first two assertions are easy to check and we only prove the last one. If M (identified to the zero section in TM) is contained in the interior of $\widetilde{\Omega} \subset TM$, then, by Lemma 5·5, the Lagrangian p is upper semi-continuous and therefore Borel measurable. In the general case, M is contained in $\widetilde{\Omega}$ but not necessarily in its interior. We consider a sequence

$$TM \preceq \widetilde{\Omega}_1 \preceq \widetilde{\Omega}_2 \preceq \cdots \preceq \widetilde{\Omega}$$

of weak Finsler structures such that M is contained in the interior of $\widetilde{\Omega}_j \subset TM$ for every $j \in \mathbb{N}$ and

$$\widetilde{\Omega} = \bigcap_{j=1}^{\infty} \widetilde{\Omega}_j.$$

We then have $p_{\widetilde{\Omega}_1} \leqslant p_{\widetilde{\Omega}_2} \leqslant \cdots \leqslant p_{\widetilde{\Omega}}$ and

$$p_{\widetilde{\Omega}} = \sup_j p_{\widetilde{\Omega}_j} = \lim_{j\to\infty} p_{\widetilde{\Omega}_j}.$$

Each function $p_{\widetilde{\Omega}_i}$ is upper semi-continuous by the previous lemma and in particular it is Borel measurable. Therefore $p_{\widetilde{\Omega}}$ is the limit of a sequence of Borel measurable functions and is thus Borel measurable.

We shall say that the weak Finsler structure $\widetilde{\Omega}$ is *smooth* if p is smooth in the complement of the zero section.

Definition 5·7 (The weak length structure associated to a weak Finsler structure). *Let M be a C^1 manifold equipped with a weak Finsler structure $\widetilde{\Omega}$ with Lagrangian p. There is an associated weak length structure on M, defined by taking Γ to be the semigroupoid of piecewise C^1 paths, and defining, for each $\gamma : [a, b] \to M$ in Γ,*

$$\ell(\gamma) = \int_a^b p(\gamma(t), \dot{\gamma}(t))\, dt. \tag{5·1}$$

Remark 5·8. In Equation 5·1, γ and $\dot{\gamma}$ are continuous, and since p is Borel-measurable, the map $t \mapsto p(\gamma(t), \dot{\gamma}(t))$ is nonnegative and measurable. Therefore, the Lebesgue integral in Definition 5·1 is well defined.

Examples 5·9.

(1) Weak Minkowski spaces constitute a class of examples of weak length spaces associated to weak Finsler structures. A weak Minkowski metric space $(\mathbb{R}^n, \delta)$ is associated to a weak Finsler structure $\widetilde{\Omega} = \bigcup_{x \in M} \Omega_x$ obtained by taking, for each point x in $\mathbb{R}^n$, $\Omega_x = \widetilde{\Omega} \cap T_x\mathbb{R}^n$ to be the (open or closed) unit ball of the weak Minkowski norm, using the natural identification between the unit tangent space $T_x\mathbb{R}^n$ and the ambient space $\mathbb{R}^n$.

(2) A more general class of examples of weak Finsler spaces is the class of submanifolds of weak Minkowski spaces, obtained as follows. Let $M \subset \mathbb{R}^n$ be a C^1 submanifold, where $\mathbb{R}^n$ is equipped with a weak Minkowski length metric δ. The weak length structure on M induced by the weak length structure $(\mathbb{R}^n, \delta)$ is associated to a weak Finsler structure on M, where the convex set Ω_x in the tangent space $T_x M$ at any point x of M is the intersection of the unit ball of the weak Minkowski structure with $T_x M$ (using the natural identification of that tangent space with a vector subspace of $\mathbb{R}^n$).

The class of examples in (2) is, with respect to the class of examples in (1), a class of induced weak Finsler structures, in the sense defined in Example 5·2 (7) above. We note that Finsler structures on submanifolds of weak Minkowski structures is considered in [**2**].

6. *The tautological weak Finsler structure*

In this section, Ω is an open convex subset of $\mathbb{R}^n$. We shall use the natural identification $T\Omega \simeq \Omega \times \mathbb{R}^n$.

Definition 6·1 (The tautological weak Finsler structure). *The* tautological weak Finsler structure *on Ω is the weak Finsler structure $\widetilde{\Omega} \subset T\Omega$ defined by*

$$\widetilde{\Omega} = \{(x, \xi) \in \Omega \times \mathbb{R}^n \mid x \in \Omega \text{ and } x + \xi \in \Omega\}.$$

This structure is called "tautological" because the fiber over each point x of Ω is the set Ω itself, with the point x as origin.

The proof of the next proposition follows easily from the definitions.

PROPOSITION 6·2. *Let Ω be an open convex subset of $\mathbb{R}^n$ equipped with its tautological weak Finsler structure $\widetilde{\Omega}$. Then, for every x in Ω, the Finsler norm of any tangent vector ξ at x is given by $p_{\Omega,x}(\xi)$, where $p_{\Omega,x}$ is the Minkowski function of Ω with respect to x.*

Given an open convex subset Ω of $\mathbb{R}^n$, we denote by d_Ω the weak length metric associated to the tautological weak Finsler structure on Ω. Thus, this weak metric is given by

$$d_\Omega(x, y) = \inf_{\gamma \in \Gamma_{x,y}} \int_\gamma p_{\Omega,\gamma(t)}(\dot{\gamma}(t))dt. \tag{6·1}$$

where $\Gamma_{x,y}$ is the set of piecewise C^1 paths joining x to y.

The following lemma is easy to check.

LEMMA 6·3. *Let Ω and Ω' be two convex open subsets of $\mathbb{R}^n$ satisfying $\Omega \subset \Omega'$, then $d_{\Omega'} \leqslant d_\Omega$.*

In the rest of this paper, we shall use the following notation: for x and y in $\mathbb{R}^n$, we denote by $|x - y|$ their Euclidean distance. Given two distinct points x and y in Ω, $R(x, y)$ denotes the Euclidean ray starting at x and passing through y. In the case where $R(x, y) \not\subset \Omega$ we set $a^+ = a^+(x, y) = R(x, y) \bigcap \partial\Omega$.

We now state and prove the main Theorem of the present paper.

THEOREM 6·1. *Let Ω be an open convex subset of $\mathbb{R}^n$ equipped with its tautological weak Finsler structure. Then, for every x and y in Ω, the Euclidean segment connecting x and y is of minimal length, and the associated weak metric on Ω is given by*

$$d_\Omega(x, y) = \begin{cases} \log \dfrac{|x - a^+|}{|y - a^+|} & \textit{if } x \neq y \textit{ and } R(x, y) \not\subset \Omega \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. As before, we let d_Ω denote the weak metric defined by the tautological weak Finsler structure on Ω. We also denote by $\ell(\gamma)$ the length of a path γ for the tautological weak Finsler length structure.

The proof of the theorem is done in four steps.

Step 1. Suppose that $R(x, y) \subset \Omega$. Consider the linear path γ: $[0, |x - y|] \to \Omega$ defined by

$$\gamma(t) = x + t\frac{y - x}{|y - x|}. \tag{6·2}$$

The derivative of the path γ is the constant vector

$$\dot{\gamma}(t) = \frac{y - x}{|y - x|}.$$

Therefore, $p_{\widetilde{\Omega}}(\gamma(t), \dot{\gamma}(t)) = \frac{1}{|y-x|} p_{\widetilde{\Omega}}(\gamma(t), y - x)$, which is equal to 0 since $R(x, y) \subset \Omega$.

Now the path γ has length zero and satisfies $\gamma(0) = x$ and $\gamma(|y - x|) = y$. Therefore $d_\Omega(x, y) = 0$.

In the rest of this proof, we suppose that $R(x, y) \not\subset \Omega$.

Step 2. We show that for every distinct points x and y in Ω and for every Euclidean segment γ joining x to y, we have

$$d_\Omega(x, y) \leqslant \ell(\gamma) = \log \frac{|x - a^+|}{|y - a^+|}. \tag{6·3}$$

Using the radial function $r_{\Omega,x}$ introduced in Section 2, we can write

$$a^+ = a^+(x,y) = x + r_{\Omega,x}(y-x)\cdot(y-x).$$

To compute the Finsler length of the Euclidean segment $[x, y]$, we parametrize it as the path γ defined in (6·2).

For $0 \leqslant t \leqslant |x-y|$, let $r(t) = |x-\gamma(t)|$. Then, $r(t) = r_{\Omega,x}(\gamma(t), \dot{\gamma}(t))$, and it is easy to see that

$$r(t) = |x-a^+| - t.$$

Then, we have $r'(t) = -1$ and therefore

$$\ell(\gamma) = \int_0^{|y-x|}\frac{dt}{r(t)} = -\int_0^{|y-x|}\frac{r'(t)dt}{r(t)} = -\log\big(r(t)\big)\Big|_{t=0}^{t=|y-x|} = \log\frac{|x-a^+|}{|y-a^+|}.$$

This gives the desired inequality (6·3).

Step 3. We complete the proof of the theorem in the particular case where Ω is a half-space. By the invariance of the tautological weak Finsler structure under the group of affine transformations, it suffices to consider the case where Ω is a half-space $H \subset \mathbb{R}^n$, which we can assume to be defined by an equation

$$H = \{x \in \mathbb{R}^n | \langle \nu, x\rangle \leqslant s\},$$

for some vector ν in $\mathbb{R}^n$ (which is orthogonal to the hyperplane bounding H) and for some s in $\mathbb{R}$. Recall that the Minkowski function associated to H is given by the formula

$$p_H(x,\xi) = \max\left\{\frac{\langle \nu, \xi\rangle}{s-\langle \nu, x\rangle}, 0\right\}.$$

Consider now an arbitrary piecewise C^1 path $\alpha\colon [0,1] \to H$ such that $x = \alpha(0)$ and $y = \alpha(1)$. Then,

$$\ell(\alpha) = \int_0^1 \max\left\{\frac{\langle \nu, \dot{\alpha}(t)\rangle}{s-\langle \nu, \alpha(t)\rangle}, 0\right\}dt \geqslant \int_0^1 \frac{\langle \nu, \dot{\alpha}(t)\rangle}{s-\langle \nu, \alpha(t)\rangle}dt.$$

We have

$$\frac{\langle \nu, \dot{\alpha}(t)\rangle}{s-\langle \nu, \alpha(t)\rangle} = -\frac{d}{dt}(\log(s-\langle \nu, \alpha(t)\rangle)).$$

Therefore,

$$\ell(\alpha) \geqslant -\log(s-\langle \nu, \alpha(1)\rangle) + \log(s-\langle \nu, \alpha(0)\rangle) = \log\frac{s-\langle \nu, x\rangle}{s-\langle \nu, y\rangle}.$$

Now we note that

$$s-\langle \nu, x\rangle = s-\langle \nu, x-a^+\rangle - \langle \nu, a^+\rangle = \langle \nu, a^+-x\rangle.$$

Likewise,

$$s-\langle \nu, y\rangle = \langle \nu, a^+-y\rangle.$$

Thus, we obtain

$$\ell(\alpha) \geqslant \log\frac{\langle \nu, a^+-x\rangle}{\langle \nu, a^+-y\rangle}.$$

Now using the fact that the three points x, y, a^+ are aligned in that order and that ν is not parallel to the vector $x - y$, we obtain

$$\frac{\langle \nu, a^+ - x\rangle}{\langle \nu, a^+ - y\rangle} = \frac{|x - a^+|}{|y - a^+|},$$

which gives

$$\ell(\alpha) \geqslant \log \frac{|x - a^+|}{|y - a^+|}.$$

Since α is arbitrary, we have

$$d_H(x, y) \geqslant \log \frac{|x - a^+|}{|y - a^+|}.$$

Combining this inequality and the inequality (6·3), we obtain, in the case where $\Omega = H$ is a half-space,

$$d_H(x, y) = \log \frac{|x - a^+|}{|y - a^+|}.$$

In particular any Euclidean segment is length minimizing.

Step 4. Now we prove the proposition for a general open convex set Ω.

Let x and y be two elements in Ω and consider the Euclidean ray $R(x, y)$.

By hypothesis, we have $R(x, y) \not\subset \Omega$, and as before, we set $a^+ = R(x, y) \bigcap \partial\Omega$. We let A denote a support hyperplane to Ω through a^+, and we let H be the open half-space containing Ω and whose boundary is equal to A. Using Lemma 6·3 and Step 3, we have

$$d_\Omega(x, y) \geqslant d_H(x, y) = \log \frac{|x - a^+|}{|y - a^+|}.$$

Combining this with the inequality (6·3) we obtain $d_\Omega(x, y) = \log \frac{|x-a^+|}{|y-a^+|}$. The argument also proves that any Euclidean segment γ is length minimizing. This completes the proof of Theorem 6·1.

7. *The Funk weak metric*

In this and the following section, we give a quick overview of the Funk weak metric, of its geodesics, of its balls and of its topology.

The Funk weak metric is a nice example of a weak metric, and a geometric study of this weak metric is something which seems missing in the literature. We study this weak metric in more detail in [**18**].

In this section, Ω is a nonempty open convex subset of $\mathbb{R}^n$. We use the notation a^+, $R(x, y)$, etc. established in the preceding section.

Definition 7·1 (The Funk weak metric). *The* Funk weak metric *on Ω, denoted by F_Ω, is defined, for x and y in Ω, by the formula*

$$F_\Omega(x, y) = \begin{cases} \log \dfrac{|x - a^+|}{|y - a^+|} & \text{if } x \neq y \text{ and } R(x, y) \not\subset \Omega \\ 0 & \text{otherwise.} \end{cases}$$

As a first example, let us consider:

Example 7·2 (The upper half-plane). *Let $\Omega = H \subset \mathbb{R}^2$ be the upper half-plane, that is,*

$$H = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}.$$

Then, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in H, we have

$$F_H(x, y) = \max\left\{\log \frac{x_2}{y_2}, 0\right\}.$$

Observe that Theorem 6·1 says that the Funk weak metric of any convex set is the weak metric associated to the tautological weak Finsler structure in Ω. In particular the triangle inequality is satisfied. Another proof of the triangle inequality is given in [**22**, p. 85]. This proof is not trivial and uses arguments similar to those of the classical proof of the triangle inequality for the Hilbert metric, as given by D. Hilbert in [**14**].

If $\Omega = \mathbb{R}^n$, then $F \equiv 0$. We shall henceforth assume that $\Omega \neq \mathbb{R}^n$ whenever we shall deal with the Funk weak metric of a nonempty open convex subset Ω of $\mathbb{R}^n$.

The Funk weak metric is always unbounded. Indeed, if x is any point in Ω and if x_n is any sequence of points in that space converging to a point on $\partial\Omega$ (convergence here is with respect to the Euclidean metric), then $F_\Omega(x, x_n) \to \infty$. Notice that on the other hand the sequence $F_\Omega(x_n, x)$ is bounded.

The following three propositions are easy consequences of the definitions and they will be used below. We take Ω to be again a nonempty open subset of $\mathbb{R}^n$.

PROPOSITION 7·3. *Let $\Omega' \subset \Omega$ be the intersection of Ω with an affine subspace of $\mathbb{R}^n$, and suppose that $\Omega' \neq \emptyset$. Then, $F_{\Omega'}$ is the weak metric induced by F_Ω on Ω'.*

PROPOSITION 7·4. *In the case where Ω is bounded, the Funk weak metric F_Ω is strongly separating, and we have the following equivalences:*

$$F_\Omega(x, x_n) \longrightarrow 0 \iff F_\Omega(x_n, x) \longrightarrow 0 \iff |x - x_n| \longrightarrow 0. \tag{7·1}$$

PROPOSITION 7·5. *Let Ω_1 and Ω_2 be two open convex subsets of $\mathbb{R}^n$. Then,*

$$F_{\Omega_1 \cap \Omega_2} = \max\{F_{\Omega_1}, F_{\Omega_1}\}.$$

8. *On the geometry of the Funk weak metric*

In this section, Ω is again an open convex subset in $\mathbb{R}^n$. We study the geodesics, and then, the geometric balls of the Funk weak metric $F = F_\Omega$.

PROPOSITION 8·1. *Let x, y and z be three points in Ω lying in that order on a Euclidean line. Then, we have $F(x, y) + F(y, z) = F(x, z)$.*

This results follows from Theorem 6·1, but it is also quite simple to prove it directly.

Proof. We can assume that the three points are distinct, otherwise the proof is trivial. We have $R(x, y) \subset \Omega \Leftrightarrow R(x, z) \subset \Omega \Leftrightarrow R(y, z) \subset \Omega$, and this holds if and only if the three quantities $F(x, y)$, $F(y, z)$ and $F(x, z)$ are equal to 0. Thus, the conclusion also holds trivially in this case. Therefore, we can assume that $R(x, y) \not\subset \Omega$. In this case, we have $a^+(x, y) = a^+(x, z) = a^+(y, z)$. Denoting this common point by a^+, we have

$$\frac{|x - a^+|}{|y - a^+|} \frac{|y - a^+|}{|z - a^+|} = \frac{|x - a^+|}{|z - a^+|},$$

which implies

$$\log \frac{|x - a^+|}{|y - a^+|} + \log \frac{|y - a^+|}{|z - a^+|} = \log \frac{|x - a^+|}{|z - a^+|},$$

which completes the proof.

COROLLARY 8·2. *The Euclidean segments in Ω are geodesic segments for the Funk weak metric on Ω.*

Since the open set Ω is convex, Corollary 8·2 implies that (Ω, F_Ω) is a geodesic weak metric space (any two points can be joined by a geodesic segment). Metrics on subsets of Euclidean space for which the Euclidean segments are geodesic segments are important, in particular because they are related to Hilbert's fourth Problem which precisely asks for a characterization of such metrics. H. Busemann calls a metric on a space embedded in $\mathbb{R}^n$ a "Desarguesian space" if Euclidean segments are geodesics, and if it satisfies some further conditions that has to do with the uniqueness of geodesics (see [**8**, chapter II]). The Funk weak metric is not always a (weak version of a) Desarguesian space in the sense of Busemann, because in general a geodesic between two points is not unique. We shall discuss this fact below, and we shall indeed see that in general, the Euclidean segments are not the only geodesic segments for a Funk weak metric. Desarguesian metrics are also often termed *pojective*.

The following proposition implies that there exist other types of geodesic segments in Ω, provided there exists a Euclidean segment of nonempty interior contained in the boundary of Ω.

PROPOSITION 8·3. *Let Ω be an open convex subset of $\mathbb{R}^n$ such that $\partial\Omega$ contains some non degenerate Euclidean segment $[p, q]$ and let x and z be two distinct points in Ω such that $R(x, z) \bigcap [p, q] \neq \emptyset$. Let Ω' be the intersection of Ω with the affine subspace of $\mathbb{R}^n$ spanned by $\{x\} \bigcup [p, q]$. Then, for any point y in Ω' satisfying $R(x, y) \bigcap [p, q] \neq \emptyset$ and $R(y, z) \bigcap [p, q] \neq \emptyset$, we have $F(x, y) + F(y, z) = F(x, z)$.*

Proof. It suffices to work in the space Ω'. Let x', y' and z' denote the feet of the perpendiculars from x, y and z respectively on the Euclidean line joining the points p and q (see Figure 1). Let $b = R(x, z) \bigcap [p, q]$. Since the triangles bxx' and bzz' are similar, we have

$$F(x, z) = \log \frac{|x - b|}{|z - b|} = \log \frac{|x - x'|}{|z - z'|}.$$

Similar formulas hold for $F(x, y)$ and $F(y, z)$. Therefore,

$$\begin{aligned} F(x, z) &= \log \frac{|x - x'|}{|z - z'|} \\ &= \log \left(\frac{|x - x'|}{|y - y'|} \frac{|y - y'|}{|z - z'|} \right) \\ &= \log \left(\frac{|x - x'|}{|y - y'|} \right) + \log \left(\frac{|y - y'|}{|z - z'|} \right) \\ &= F(x, y) + F(y, z). \end{aligned}$$

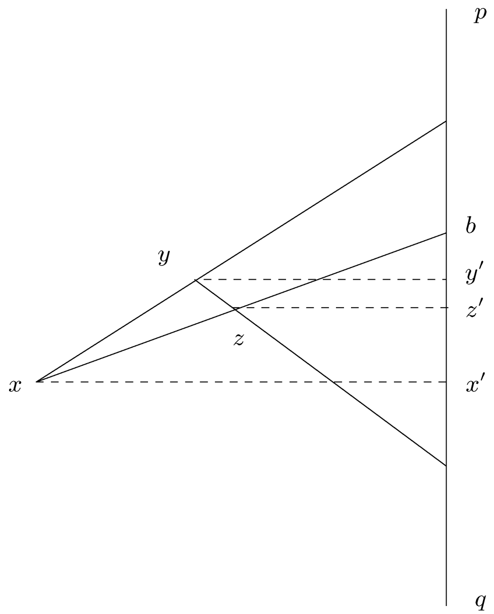

Fig. 1.

COROLLARY 8·4. *For any triple of points x, y, z as in Proposition* 8·3*, the union of the two segments* $[x, y] \bigcup [y, z]$ *is a geodesic segment.*

Proof. Let x', y', z' be three points on the topological segment $[x, y] \bigcup [y, z]$ such that the points x, x', y', z', z are in that order. If the three points x', y', z' are in that order on a Euclidean segment and then, by Proposition 8·1, they satisfy $F(x', y') + F(y', z') = F(x', z')$. In the other case, it is easy to see by elementary Euclidean geometry that the triple of points x', y', z' satisfy the properties of the triple x, y, z of Proposition 8·3, and in that case, we also have $F(x', y') + F(y', z') = F(x', z')$.

Remark 8·5. Corollary 8·4 allows us to construct polygonal paths that are not Euclidean segments but that are geodesics for the Funk weak metric of open sets containing a nonempty open segment in their boundary. By taking limits of such polygonal paths, we can easily construct, from Proposition 8·3, smooth paths which are not Euclidean paths and which are geodesic for the Funk weak metric.

PROPOSITION 8·6. *Let Ω be an open convex subset of $\mathbb{R}^n$. Let x and z be two distinct points in Ω such that $R(x, z) \bigcap \partial\Omega \neq \emptyset$ and such that at the point $b = R(x, z) \bigcap \partial\Omega$, there is a support hyperplane whose intersection with $\partial\Omega$ is reduced to b. Let y be a point in Ω such that the three points x, y, z in Ω do not lie on the same affine line. Then, $F(x, z) < F(x, y) + F(y, z)$.*

Proof. To prove the proposition, we work in the affine plane spanned by x, y and z and therefore we can assume without loss of generality that $n = 2$.

We assume that the intersection points of $R(x, y)$ and $R(y, z)$ with $\partial\Omega$ are not empty, and we let a and c be respectively these points. From the hypothesis, there is a support line of Ω (which we call D) at b whose intersection with $\partial\Omega$ is reduced to the point b.

For the proof, we distinguish three cases.

Case 1. The two rays $R(x, y)$ and $R(y, z)$ intersect the line D (see Figure 2).

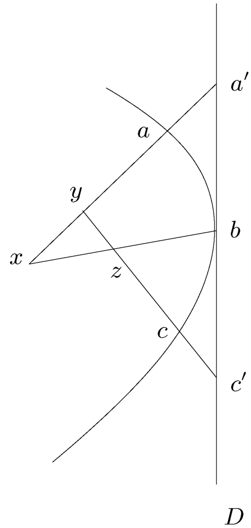

Fig. 2.

Let a' and c' be respectively these intersection points. Note that the three points a', b and c' are in that order on D. By reasoning with the projections on the line D and arguing as we did in the proof of Proposition 8·3, we have

$$\frac{|x-b|}{|z-b|} = \frac{|x-a'|}{|y-a'|}\frac{|y-c'|}{|z-c'|}.$$

Since we have

$$\frac{|x-a'|}{|y-a'|} < \frac{|x-a|}{|y-a|}$$

and

$$\frac{|y-c'|}{|z-c'|} < \frac{|y-c|}{|z-c|},$$

we obtain

$$\frac{|x-b|}{|z-b|} < \frac{|x-a|}{|y-a|}\frac{|y-c|}{|z-c|}$$

which gives, by taking logarithms, $F(x, z) < F(x, y) + F(y, z)$.

Case 2. The ray $R(x, y)$ intersects D and the ray $R(y, z)$ does not intersect D (Figure 3). We let as before a' denote the point $R(x, y) \bigcap D$.

Let D' be the Euclidean line passing through z and parallel to D. The hypotheses in the case considered imply that the line D' intersects the segment $[x, y]$. Let y' be this intersection point. The point y' is contained in Ω.

We have, as in Case 1,

$$F(x, z) = \log\frac{|x-b|}{|z-b|}$$

and

$$F(x, y) = \log\frac{|x-a|}{|y-a|} > \log\frac{|x-a'|}{|y-a'|}.$$

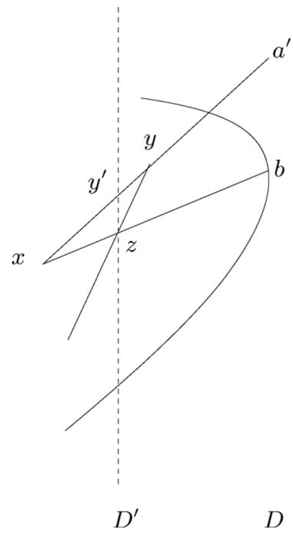

Fig. 3.

Now we have

$$\frac{|x-b|}{|z-b|}=\frac{|x-a'|}{|y'-a'|}<\frac{|x-a'|}{|y-a'|},$$

that is, $F(x,z)<F(x,y)$, which implies the desired result.
Case 3. The ray $R(x,y)$ does not intersect the line D. This case can be treated as Case 2, and we have in this case $F(x,y)<F(y,z)$, which implies the desired result.

The following is a direct consequence of Proposition 8·6.

COROLLARY 8·7. *Let Ω be an open bounded strictly convex subset of $\mathbb{R}^n$ and let x, y and z be three points in Ω that are not contained in an affine segment. Then, $F(x,z)<F(x,y)+F(y,z)$.*

COROLLARY 8·8. *Let Ω be an open bounded strictly convex subset of $\mathbb{R}^n$. Then, the affine segments in Ω are the only geodesic segments for the Funk weak metric of Ω.*

Proof. This follows from the previous corollary and Corollary 8·2, which says that the affine segments are geodesic segments for the Funk weak metric.

We recall that a subset Y in a (weak) metric space X is said to be *geodesically convex* if for any two points x and y in Y, any geodesic segment in X joining x and y is contained in Y.

COROLLARY 8·9. *Let Ω be an open bounded strictly convex subset of $\mathbb{R}^n$ and let Ω' be a subset of Ω. Then, Ω' is convex with respect to the affine structure of $\mathbb{R}^n$ if and only if Ω' is a geodesically convex subset of Ω with respect to the Funk weak metric F_Ω.*

Remark 8·10. Note the formal analogy between Corollary 8·8 and the following well known result on the geodesic segments of a Minkowski metric on $\mathbb{R}^n$: if the unit ball of a Minkowski metric is strictly convex, then the only geodesic segments of this metric are the affine segments.

We now consider spheres and balls in a Funk weak metric space (Ω, F). As this weak metric is non-symmetric, we have to distinguish between right and left spheres and balls, and we use the following notation. For any point x in Ω and for any nonnegative real number δ, we set

(i) $B(x,\delta) = \{y \in \Omega \mid F_\Omega(x,y) < \delta\}$ (the *right open ball of center x and radius* δ);
(ii) $B'(x,\delta) = \{y \in \Omega \mid F_\Omega(y,x) < \delta\}$ (the *left open ball of center x and radius* δ);
(iii) $S(x,\delta) = \{y \in \Omega \mid F_\Omega(x,y) = \delta\}$ (the *right sphere of center x and radius* δ);
(iv) $S'(x,\delta) = \{y \in \Omega \mid F_\Omega(y,x) = \delta\}$ (the *left sphere of center x and radius* δ).

In [**7**, p. 20], H. Busemann discusses topologies for general weak metric spaces. In the case of a genuine metric space, the open balls define the topology of that space. In general, the collections of left and of right open balls in a weak metric space generate two different topologies.

We shall see below that if Ω is a bounded convex open subset of $\mathbb{R}^n$ equipped with its Funk weak metric, then, the collections of left and of right open balls are sub-bases of the same topology on Ω, and this topology coincides with the topology induced from the inclusion of Ω in $\mathbb{R}^n$.

In the case where the convex open set Ω is unbounded, the left and the right open balls of the Funk weak metric are always noncompact. In the next proposition, we study these balls in the case where Ω is bounded. Note that a convex subset of $\mathbb{R}^n$ is unbounded if and only if it contains a Euclidean ray.

PROPOSITION 8·11. *Let Ω be a bounded convex open subset of $\mathbb{R}^n$, let x be a point in Ω and let δ be a nonnegative real number. Then,*

(1) *The right sphere $S(x,\delta)$ is the image of $\partial\Omega$ by the Euclidean homothety of center x and factor $(1-e^{-\delta})$. The right ball $B(x,\delta)$ is convex as a subset of $\mathbb{R}^n$, and it is relatively compact. In the case where the set Ω is strictly convex, then the right ball $B(x,\delta)$ is metrically convex (with respect to the Funk weak metric).*
(2) *The left ball $B'(x,\delta)$ is convex as a subset of $\mathbb{R}^n$, and it is equal to the intersection with Ω of the image of Ω by the Euclidean homothety of center x and of factor $(e^\delta - 1)$, followed by the Euclidean central symmetry of center x. The ball $B'(x,\delta)$ is not necessarily relatively compact.*

Proof. Let y be a point in Ω and let us set, as before, $a^+ = R(x,y) \cap \partial\Omega$. We have the following equivalences:

$$y \in S(x,\delta) \iff \log\frac{|x-a^+|}{|y-a^+|} = \delta \iff \frac{|x-a^+|}{|y-a^+|} = e^\delta,$$

which is easily seen to be equivalent to $|y-x| = |x-a^+|(1-e^{-\delta})$. From this fact the first statement in Property (1) follows easily. The rest of Property (1) follows using Corollary 8·9.

To prove Property (2), let $a^- = R(y,x) \cap \partial\Omega$. We have the following equivalences:

$$\log\frac{|y-a^-|}{|x-a^-|} = \delta \iff |y-a^-| = e^\delta|x-a^-|,$$

which is also equivalent to

$$|y-x| = (e^\delta - 1)|x-a^-|.$$

Thus, $y \in S'(x, \delta)$ if and only if y is in the intersection of Ω with the image $h(\partial\Omega)$ of $\partial\Omega$ by the Euclidean homothety h with center x and of factor $(e^\delta - 1)$, followed by the Euclidean central symmetry of center x. The interior of this intersection is convex as a subset of $\mathbb{R}^n$ but it is not necessarily a compact subset of (Ω, F). The ball $B'(x, \delta)$ is relatively compact if and only if $h(\partial\Omega)$ is contained in Ω.

We note the following "local-implies-global" property of Funk weak metrics. The meaning of the statement is clear, and it follows directly from Proposition 8·11 (1).

COROLLARY 8·12. *We can reconstruct the boundary $\partial\Omega$ of Ω from the local geometry at any point of Ω.*

We also highlight the following *strong rigidity* result for Funk weak metrics.

COROLLARY 8·13. *Let Ω and Ω' be two convex sets in $\mathbb{R}^n$, and denote by F and F' their corresponding Funk metrics. If there exist open subsets $U \subset \Omega$ and $U' \subset \Omega'$ such that (U, F) and (U', F') are isometric, then (Ω, F) and (Ω', F') are globally isometric.*

COROLLARY 8·14. *Let Ω be a bounded open strictly convex subset of $\mathbb{R}^n$. Then, the left and right open balls of Ω are geodesically convex with respect to the Funk weak metric F_Ω.*

Proof. This follows from Proposition 8·11 and from Corollary 8·9.

We also deduce from Proposition 8·11 that for any x and x' in Ω and for any two positive real numbers δ and δ', the right spheres $S(x, \delta)$ and $S(x', \delta')$ are homothetic.

Thus, for instance, if Ω is the interior of a Euclidean sphere (respectively, of a Euclidean ellipsoid) in $\mathbb{R}^n$, then any right sphere $S(x, \delta)$ is a Euclidean sphere (respectively, an ellipsoid).

Note that the proof of Proposition 8·11 shows that for a fixed x, any two right spheres $S(x, \delta)$ and $S(x, \delta')$ are homothetic by a Euclidean homothety of center x, but that in general, a homothety which sends a sphere $S(x, \delta)$ to a sphere $S(x', \delta')$ does not necessarily send the center x of $S(x, \delta)$ to the center x' of $S(x', \delta')$. One can see this fact on the following example: Let Ω be an open Euclidean disk in $\mathbb{R}^n$, and let us take x to be the Euclidean center of that disk. Then, by symmetry, for any $\delta > 0$, the right sphere $S(x, \delta)$ is a Euclidean sphere whose Euclidean and whose metric centers are both at x. Now let x' be a point which is close to the boundary of Ω. Obviously, the Euclidean homothety that sends $\partial\Omega$ to $S(x', \delta)$ does not send the center of $\partial\Omega$ to the (Funk-)geometric center of the sphere $S(x', \delta)$ (recall that the center of this homothety is the point x). Now taking a composition of two homotheties, we obtain a Euclidean homothety that sends the geometric sphere $S(x, \delta)$ to the geometric sphere $S(x', \delta)$, and that does not preserve the geometric centers of these spheres.

Remark 8·15. The property for a Funk weak metric on a subset Ω of $\mathbb{R}^n$ that all the right spheres are homothetic is also shared by the metrics induced by Minkowski weak metrics on $\mathbb{R}^n$.

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