Chapter 3

Funk and Hilbert geometries from the Finslerian viewpoint

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1 Introduction

The hyperbolic space $\mathbb{H}^n$ is a complete, simply connected Riemannian manifold with constant sectional curvature $K = -1$. It is unique up to isometry and several concrete models are available and well known. In particular, the *Beltrami–Klein* model is a realization in the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ in which the hyperbolic lines are represented by the affine segments joining pairs of points on the boundary sphere $\partial\mathbb{B}^n$ and the distance between two points p and q in $\mathbb{B}^n$ is half of the logarithm of the cross ratio of a, p with b, q, where a and b are the intersections of the line through p and q with $\partial\mathbb{B}^n$. We refer to [1] for a historical discussion of this model. In 1895, David Hilbert observed that the Beltrami–Klein construction defines a distance in any convex set $\mathcal{U} \subset \mathbb{R}^n$, and this metric still has the properties that affine segments are the shortest curves connecting points. More precisely, if the point x belongs to the segment $[p,q]$, then

$$d(p,q) = d(p,x) + d(x,q). \tag{1.1}$$

Metrics satisfying this property are said to be *projective* and Hilbert's IV$^{\text{th}}$ problem asks to classify and describe all projective metrics in a given domain $\mathcal{U} \subset \mathbb{R}^n$, see Chapter 15 of this volume [42].

The Hilbert generalization of Klein's model to an arbitrary convex domain $\mathcal{U} \subset \mathbb{R}^n$ is no longer a Riemannian metric, but it is a *Finsler metric*. This means that the distance between two points p and q in $\mathcal{U}$ is the infimum of the length of all smooth curves $\beta\colon [0,1] \to \mathcal{U}$ joining these two points, where the length is given by an integral of the type

$$\ell(\beta) = \int_0^1 F(\beta(t), \dot{\beta}(t))dt. \tag{1.2}$$

Here, $F\colon T\mathcal{U} \to \mathbb{R}$ is a sufficiently regular function called the *Lagrangian*. Geometrical considerations lead us to assume that F defines a norm on the tangent space at any point of $\mathcal{U}$; in fact it is also useful to consider non-symmetric and possibly degenerate norms (which we call *weak Minkowski norms*, see [46]).

General integrals of the type (1.2) are the very subject of the classical calculus of variations and Finsler geometry is really a daughter of that field. The early contributions (in the period 1900–1920) in Finsler geometry are due to mathematicians working in the calculus of variations[1], in particular Bliss, Underhill, Landsberg, Hamel, Carathéodory and his student Finsler. Funk is the author of a quite famous book on the calculus of variations that contains a rich chapter on Finsler geometry [26]. The name "Finsler geometry" has been proposed (somewhat improperly) by Élie Cartan in 1933. The 1950 article [12] by H. Busemann contains some interesting historical remarks on the early development of the subject, and the 1959 book by H. Rund [51] gives a broad overview of the development of Finsler geometry during its first 50 years; this book is rich in references and historical comments.

In 1908, A. Underhill and G. Landsberg introduced a notion of curvature for two-dimensional Finsler manifolds that generalizes the classical Gauss curvature of Riemannian surfaces, and in 1926 L. Berwald generalized this construction to higher dimensional Finsler spaces [7], [33], [62]. This invariant is today called the *flag curvature* and it generalizes the Riemannian sectional curvature. In 1929, Paul Funk proved that the flag curvature of a Hilbert geometry in a (smooth and strongly convex) domain $\mathcal{U} \subset \mathbb{R}^2$ is constant $\mathrm{K} = -1$ and Berwald extended this result in all dimensions and refined Funk's investigation in several aspects, leading to the following characterization of Hilbert geometry [8], [24]:

Theorem (Funk–Berwald). *Let F be a smooth and strongly convex Finsler metric defined in a bounded convex domain $\mathcal{U} \subset \mathbb{R}^n$. Suppose that F is complete with*

[1] Although Bernhard Riemann already considered this possible generalization of his geometry in his Habilitation Dissertation, he did not pursue the subject and Finsler geometry did not emerge as a subject before the early twentieth century.

constant flag curvature $\mathrm{K} = -1$ *and that the associated distance* d *satisfies* (1.1), *then* d *is the Hilbert metric in* $\mathcal{U}$.

The completeness hypothesis means that every geodesic segment can be indefinitely extended in both directions; in other words, every geodesic segment is contained in a line that is isometric to the full real line $\mathbb{R}$.

In fact, Funk and Berwald assumed the Finsler metric to be reversible, that is, $F(p, -\xi) = F(p, \xi)$. But they also needed the (unstated) hypothesis that F is *forward complete*, meaning that every oriented geodesic segment can be extended as a ray. Observe that reversibility together with forward completeness implies the completeness of F, and therefore the way we characterize the Hilbert metrics in the above theorem is slightly stronger than the Funk–Berwald original statement. Note that completeness is necessary: it is possible to locally construct reversible (incomplete) Finsler metrics for which the condition (1.1) holds and which are not restrictions of Hilbert metrics.

The Funk–Berwald Theorem is quite remarkable, and it has been an important landmark in Finsler geometry. Our goal in this chapter is to develop all the necessary concepts and tools to explain this statement precisely. The actual proof is given in Section 12.

The rest of the chapter is organized as follows. In Section 2 we explain what a Finsler manifold is and give some basic definitions in the subject with a few elementary examples. In Section 3, we introduce a very natural example of a Finsler structure on a convex domain, discovered by Funk, and which we called the *tautological Finsler structure*. We also compute the distances and the geodesics for the tautological structure. In Section 4, we introduce Hilbert's Finsler structure as the symmetrization of the tautological structure and compute its distances and geodesics.

In Section 5, we introduce the *fundamental tensor* g_{ij} of a Finsler metric (assuming some smoothness and strong convexity hypothesis). In Section 6, we compute the geodesic equations and we introduce the notion of *spray* and the exponential map. In Section 7, we discuss various characterizations of *projectively flat* Finsler metrics, that is, Finsler metrics for which the affine lines are geodesics. In Section 8, we introduce the Hilbert differential form and the notion of Hamel potential, which is a tool to compute distances in a general projectively flat Finsler space.

In Section 9, we introduce the curvature of a Finsler manifold based on the classical Riemannian curvature of some associated (osculating) Riemannian metric. The curvature of projectively flat Finsler manifolds is computed in Sections 10 and 11. The characterization of Hilbert geometries is given in Section12 and the chapter ends with two appendices: one on further developments of the subject and one on the Schwarzian derivative.

We tried to make this chapter as self-contained as possible. The material we present here is essentially built on Berwald's paper [8], but we have also used a number of more recent sources. In particular we found the books [53], [54] by Z. Shen and [15] by S.-S. Chern and Z. Chen quite useful.

2 Finsler manifolds

We start with the main definition of this chapter:

Definition 2.1. 1. A *weak Finsler structure* on a smooth manifold M is a lower semi-continuous function $F\colon TM \to [0,\infty]$ such that for every point $x \in M$, the restriction $F_x = F_{|T_xM}$ is a weak Minkowski norm, that is, it satisfies the following properties:

i) $F(x,\xi_1+\xi_2) \leq F(x,\xi_1) + F(x,\xi_2)$,

ii) $F(x,\lambda\xi) = \lambda F(x,\xi)$ for all $\lambda \geq 0$,

for any $x \in M$ and $\xi_1, \xi_2 \in T_xM$.

2. If $F\colon TM \to [0,\infty)$ is finite and continuous and if $F(x,\xi) > 0$ for any $\xi \neq 0$ in the tangent space T_xM, then one says that F is a *Finsler structure* on M.

We shall mostly be interested in Finsler structures, but extending the theory to the case of weak Finsler structures can be useful in some arguments. The notion of weak Finsler structure appears for instance in [19], [29], [44]. Another situation where weak Finsler structures appear naturally is the field of *sub-Finsler* geometry.

The function F itself is called the *Lagrangian* of the Finsler structure; it is also called the (weak) *metric*, since it is used to measure the length of a tangent vector.

The weak Finsler metric is called *reversible* if $F(x,\cdot)$ is actually a norm, that is, if it satisfies

$$F(x,-\xi) = F(x,\xi)$$

for all points x and any tangent vector $\xi \in T_xM$. The Finsler metric is said to be *of class* C^k if the restriction of F to the slit tangent bundle $TM^0 = \{(x,\xi) \in TM \mid \xi \neq 0\} \subset TM$ is a function of class C^k. If it is C^∞ on TM^0, then one says F is smooth.

Some of the most elementary examples of Finsler structures are the following:

Examples 2.2. i) A weak Minkowski norm $F_0\colon \mathbb{R}^n \to \mathbb{R}$ defines a weak Finsler structure F on $M = \mathbb{R}^n$ by

$$F(x,\xi) = F_0(\xi).$$

See [46] for a discussion of weak Minkowski norms. This Finsler structure is constant in x and is thus translation-invariant. Such structures are considered to be the flat spaces of Finsler geometry.

ii) A Riemannian metric g on the manifold M defines a Finsler structure on M by $F(x,\xi) = \sqrt{\mathrm{g}_x(\xi,\xi)}$.

iii) If g is a Riemannian metric on M and $\theta \in \Omega^1(M)$ is a 1-form whose g-norm is everywhere smaller than 1, then one can define a smooth Finsler structure on M by

$$F(x,\xi) = \sqrt{\mathrm{g}_x(\xi,\xi)} + \theta_x(\xi).$$

Such structures are called *Randers metrics*.

iv) If θ is an arbitrary 1-form on the Riemannian manifold (M, g), then one can define two new Finsler structures on M by

$$F_1(x,\xi) = \sqrt{g_x(\xi,\xi)} + |\theta_x(\xi)|, \quad F_2(x,\xi) = \sqrt{g_x(\xi,\xi)} + \max\{0, \theta_x(\xi)\}.$$

v) If F is a Finsler structure on M, then the *reverse Finsler structure* $F^*\colon TM \to [0,\infty)$ is defined by

$$F^*(x,\xi) = F(x,-\xi).$$

Additional examples will be given below.

A number of concepts from Riemannian geometry naturally extend to Finsler geometry, in particular one defines the *length* of a smooth curve $\beta\colon [0,1] \to M$ as

$$\ell(\beta) = \int_0^1 F(\beta(s), \dot{\beta}(s))dt.$$

We then define the *distance* $d_F(x,y)$ between two points p and q to be the infimum of the length of all smooth curves $\beta\colon [0,1] \to M$ joining these two points (that is, $\beta(0) = p$ and $\beta(1) = q$). This distance satisfies the axioms of a weak metric, see [46]. Together with the distance comes the notion of completeness.

Definition 2.3. A sequence $\{x_i\} \subseteq M$ is a *forward Cauchy sequence* if for any $\varepsilon > 0$, there exists an integer N such that $d(x_i, x_{i+k}) < \varepsilon$ for any $i \geq N$ and $k \geq 0$. The Finsler manifold (M, F) is said to be *forward complete* if every forward Cauchy sequence converges. We similarly define *backward Cauchy sequence* by the condition $d(x_{i+k}, x_i) < \varepsilon$, and the corresponding notion of *backward complete*. The Finsler manifold (M, F) is a *complete Finsler manifold* if it is both backward and forward complete.

A weak Finsler structure on the manifold M can also be seen as a "field of convex sets" sitting in the tangent bundle TM of that manifold. Indeed, given a Finsler structure F on the manifold M, we define its *domain* $\mathcal{D}_F \subset TM$ to be the set of all vectors with finite F-norm. The *unit domain* $\Omega \subset \mathcal{D}_F$ is the bundle of all tangent unit balls

$$\Omega = \{(x,\xi) \in TM \mid F(x,\xi) < 1\}.$$

The unit domain contains the zero section in TM and its restriction $\Omega_x = \Omega \cap T_xM$ to each tangent space is a bounded and convex set. It is called the *tangent unit ball* of F at x, while its boundary

$$\mathcal{I}_x = \{\xi \in T_xM \mid F(x,\xi) = 1\} \subset T_xM$$

is called the *indicatrix* of F at x. We know from Theorem 3.12 of [46] that the Lagrangian $F\colon TM \to \mathbb{R}$ can be recovered from $\Omega \subset TM$ via the formula

$$F(x,\xi) = \inf\{t > 0 \mid \tfrac{1}{t}\xi \in \Omega\}. \tag{2.1}$$

Sometimes, a weak Finsler structure is defined by specifying its unit domain, and the Lagrangian is then obtained from (2.1). Let us give two elementary examples; more can be found in [43]:

Example 2.4. Given a bounded open set $\Omega_0 \subset \mathbb{R}^n$ that contains the origin, one naturally defines a Finsler structure on $\mathbb{R}^n$ by parallel transporting Ω_0. That is, $\Omega \subset T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ is defined as

$$\Omega = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi \in \Omega_0\}.$$

The space $\mathbb{R}^n$ equipped with this Finsler structure is characterized by the property that its Lagrangian F is invariant with respect to the translations of $\mathbb{R}^n$, i.e. F is independent of the point x:

$$F(x, \xi) = F_0(\xi).$$

Such a Finsler structure on $\mathbb{R}^n$ is of course the same thing as a Minkowski norm, see [46].

Example 2.5. Let F be an arbitrary weak Finsler structure on a manifold M with unit domain $\Omega \subset TM$. If $Z\colon M \to TM$ is a continuous vector field such that $F(x, Z(x)) < 1$, for any point x (equivalently $Z(M) \subset \Omega$), a new weak Finsler structure can be defined as

$$\Omega_Z = \{(x, \xi) \in TM \mid \xi \in (\Omega_x - Z(x))\}. \tag{2.2}$$

Here, $\Omega_x - Z(x)$ is the translate of $\Omega_x \subset T_xM$ by the vector $-Z(x)$. The corresponding Lagrangian is given by

$$F_Z(x, \xi) = \inf\{t > 0 \mid \tfrac{1}{t}\xi \in (\Omega_x - Z(x))\}, \tag{2.3}$$

and for $\xi \neq 0$, it is computable from the identity

$$F\left(x, \frac{\xi}{F_Z(x, \xi)} + Z(x)\right) = 1. \tag{2.4}$$

This weak Finsler structure F_Z is called the *Zermelo transform* of F with respect to the vector field Z.

We end this section with a word on Berwald spaces. Recall first that in Riemannian geometry, the tangent space of the manifold at every point is isometric to a fixed model, which is a Euclidean space. Furthermore, using the Levi-Civita connection, one defines parallel transport along any piecewise C^1 curve, and this parallel transport induces an isometry between the tangent spaces at any point along the curve. In a general Finsler manifold, neither of these facts holds. This motivates the following definition:

Definition 2.6. A weak Finsler manifold (M, F) is said to be *Berwald* if there exists a torsion free linear connection ∇ (called an *associated connection*) on M whose associated parallel transport preserves the Lagrangian F. That is, if $\gamma\colon [0,1] \to M$ is a smooth path connecting the point $x = \gamma(0)$ to $y = \gamma(1)$ and $P_\gamma\colon T_xM \to T_yM$ is the associated ∇-parallel transport, then

$$F(y, P_\gamma(\xi)) = F(x, \xi)$$

for any $\xi \in T_xM$.

This definition is a slight generalization of the usual definition, compare with [15], Proposition 4.3.3. It is known that the connection associated to a smooth Berwald metric can be chosen to be the Levi-Civita connection of some Riemannian metric [37], [60].

3 The tautological Finsler structure

Definition 3.1. Let us consider a proper convex set $\mathcal{U} \subset \mathbb{R}^n$. This will be our ground manifold. The *tautological weak Finsler structure* F_f on $\mathcal{U}$ is the Finsler structure for which the unit ball at a point $x \in \mathcal{U}$ is the domain $\mathcal{U}$ itself, but with the point x as center. The unit domain of the tautological weak Finsler structure is thus defined as

$$\Omega = \{(x,\xi) \in T\mathcal{U} \mid \xi \in (\mathcal{U} - x)\} \subset T\mathcal{U} = \mathcal{U} \times \mathbb{R}^n,$$

and the Lagrangian is given by

$$F_f(x,\xi) = \inf\{t > 0 \mid \xi \in t(\mathcal{U} - x)\} = \inf\big\{t > 0 \mid \big(x + \tfrac{\xi}{t}\big) \in \mathcal{U}\big\}.$$

Equivalently, F_f is given by $F_f(x,\xi) = 0$ if the ray $x + \mathbb{R}_+\xi$ is contained in $\mathcal{U}$, and

$$F_f(x,\xi) > 0 \quad \text{and} \quad \left(x + \frac{\xi}{F_f(x,\xi)}\right) \in \partial\mathcal{U}$$

otherwise.

The convex set $\mathcal{U}$ can be recovered from the Lagrangian as follows:

$$\mathcal{U} = \{z \in \mathbb{R}^n \mid F_f(x, z - x) < 1\}. \tag{3.1}$$

By construction, this formula is independent of x. The tautological weak structure has been introduced by Funk in 1929, see [24]. We will often call it the *Funk weak metric* on $\mathcal{U}$ (whence the index f in the notation F_f).

Remark 3.2. If the convex set $\mathcal{U}$ contains the origin, then the tautological weak structure F_f is the Zermelo transform of the Minkowski norm with unit ball $\mathcal{U}$ for the position vector field $Z_x = x$.

Example 3.3. Suppose $\mathcal{U}$ is the Euclidean unit ball $\{x \in \mathbb{R}^n \mid \|x\| < 1\}$, where $\|\cdot\|$ is the Euclidean norm. Then

$$\big(x + \tfrac{\xi}{F_f}\big) \in \partial\mathcal{U} \iff \big\|x + \tfrac{\xi}{F_f}\big\|^2 = 1.$$

Rewriting this condition as

$$F_f^2(1 - \|x\|^2) - 2F \cdot \langle x, \xi\rangle - \|\xi\|^2 = 0,$$

the non-negative root of this quadratic equation is

$$F_f(x,\xi) = \frac{\langle x,\xi\rangle + \sqrt{\langle x,\xi\rangle^2 + (1 - \|x\|^2)\|\xi\|^2}}{(1 - \|x\|^2)}, \tag{3.2}$$

which is the Lagrangian of the tautological Finsler structure in the Euclidean unit ball. Observe that this is a Randers metric.

Example 3.4. Consider a half-space $\mathcal{H} = \{x \in \mathbb{R}^n \mid \langle \nu, x\rangle < \tau\} \subset \mathbb{R}^n$ where ν is a non-zero vector and $\tau \in \mathbb{R}$; here $\langle\ ,\ \rangle$ is the standard scalar product in $\mathbb{R}^n$. We then have

$$x + \frac{\xi}{F} \in \partial\mathcal{H} \iff \Big\langle \nu, x + \frac{\xi}{F}\Big\rangle = \tau,$$

which implies

$$F_f(x,\xi) = \max\left(\frac{\langle \nu,\xi\rangle}{\tau - \langle \nu, x\rangle}, 0\right). \tag{3.3}$$

We now compute the distance between two points in the tautological Finsler structure:

Theorem 3.5. *The tautological distance in a proper convex domain $\mathcal{U} \subset \mathbb{R}^n$ is given by*

$$\varrho_f(p,q) = \log\left(\frac{|a-p|}{|a-q|}\right), \tag{3.4}$$

where a is the intersection of the ray starting at p in the direction of q with the boundary $\partial\mathcal{U}$:

$$a = \partial\mathcal{U} \cap (p + \mathbb{R}_+(q-p)).$$

If the ray is contained in $\mathcal{U}$, then a is considered to be a point at infinity and $\varrho_f(p,q) = 0$.

Definition 3.6. The distance (3.4) is called the *Funk metric* in $\mathcal{U}$. In his paper [24], Funk introduced the tautological Finsler structure while the distance (3.4) appears as an interesting geometric object in the 1959 memoir [67] by Eugene Zaustinsky, a student of Busemann. We refer to [44], [67] and Chapter 2 of this volume [47] for expositions of the Funk geometry.

Proof of Theorem 3.5. We follow the proof in [44]. Consider first the special case where $\mathcal{U} = \mathcal{H}$ is the half-space $\{x \in \mathbb{R}^n \mid \langle \nu, x\rangle < \tau\} \subset \mathbb{R}^n$ for some vector $\nu \neq 0$. Then the tautological Lagrangian is given by (3.3) and the length of a curve β joining p to q is

$$\begin{aligned}\ell(\beta) &= \int_0^1 \max\left\{0, \frac{\langle \nu, \dot\beta(s)\rangle}{s - \langle \nu, \beta(s)\rangle}\right\} dt \\ &= \int_0^1 \max\left\{0, \frac{(\tau - \langle \nu, \beta(s)\rangle)'}{|\tau - \langle \nu, \beta(s)\rangle|}\right\} dt \\ &\geq \max\left\{0, \log\left(\frac{\tau - \langle \nu, p\rangle}{\tau - \langle \nu, q\rangle}\right)\right\},\end{aligned}$$

with equality if and only if $\langle \nu, \dot\beta(s)\rangle$ has almost everywhere constant sign. It follows that the tautological distance in the half-space $\mathcal{H}$ is given by

$$\varrho_f(p,q) = \max\left\{0, \log\left(\frac{\tau - \langle \nu, p\rangle}{\tau - \langle \nu, q\rangle}\right)\right\}. \tag{3.5}$$

We can rewrite this formula in a different way. Suppose that the ray L^+ starting at p in the direction of q meets the hyperplane $\partial\mathcal{H}$ at a point a. Then $\langle \nu, p\rangle = \tau$ and

$$\tau - \langle \nu, p\rangle = \langle \nu, a - p\rangle = |a - p| \cdot \langle \nu, \xi\rangle,$$

where $\xi = \frac{a-p}{|a-p|}$. We also have $\tau - \langle \nu, q\rangle = |a - q| \cdot \langle \nu, \xi\rangle$; therefore

$$\varrho_f(p,q) = \log\left(\frac{|a-p|}{|a-q|}\right). \tag{3.6}$$

If the ray L^+ is contained in the half-space $\mathcal{U}$, then $\varrho_f(p,q) = 0$ and the above formula still holds in the limit sense if we consider the point a to be at infinity.

For the general case, we will need two lemmas on general tautological Finsler structures.

Lemma 3.7. *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two convex domains in $\mathbb{R}^n$. If F_1 and F_2 are the Lagrangians of the corresponding tautological structures, then*

$$\mathcal{U}_1 \subset \mathcal{U}_2 \iff F_1(x,\xi) \geq F_2(x,\xi)$$

for all $(x,\xi) \in T\mathcal{U}$.

Proof. We have indeed

$$F_1(x,\xi) = \inf\{t > 0 \mid \xi \in t(\mathcal{U}_1 - x)\} \geq \inf\{t > 0 \mid \xi \in t(\mathcal{U}_2 - x)\} = F_2(x,\xi).$$

□

We also have the following *reproducing formula* for the tautological Lagrangian.

Lemma 3.8. *Let $\mathcal{U}$ be a convex domain in $\mathbb{R}^n$ and $p, q \in \mathcal{U}$. If $q = p + t\xi$ for some $\xi \in \mathbb{R}^n$, and $t \geq 0$, then*

$$F_f(q,\xi) = \frac{F_f(p,\xi)}{1 - tF_f(p,\xi)}. \tag{3.7}$$

Proof. If $\xi = 0$, then there is nothing to prove. Assume $\xi \neq 0$ and denote by L_p^+ and L_q^+ the rays in the direction ξ starting at p and q. We obviously have $L_p^+ \subset \mathcal{U}$ if and only if $L_q^+ \subset \mathcal{U}$. But this means that

$$F_f(p,\xi) = 0 \iff F_f(q,\xi) = 0,$$

so Equation (3.7) holds in this case. If on the other hand the rays are not contained in $\mathcal{U}$, then we have

$$L_p^+ \cap \partial\mathcal{U} = L_q^+ \cap \partial\mathcal{U}.$$

We denote by $a = a(x,\xi)$ this intersection point; then by definition of F we have

$$a = q + \frac{\xi}{F_f(q,\xi)} = p + \frac{\xi}{F_f(p,\xi)}. \tag{3.8}$$

Since $q = p + t\xi$, we have

$$\frac{\xi}{F_f(q,\xi)} = \left(\frac{1}{F_f(p,\xi)} - t\right)\xi,$$

from which (3.7) follows, since $\xi \neq 0$. □

To complete the proof of Theorem 3.5, we need to compute the tautological distance between two points p and q in a general convex domain $\mathcal{U} \in \mathbb{R}^n$. We first compute the length of the affine segment $[p,q]$ which we parametrize as

$$\beta(t) = p + t\xi, \quad \xi = \frac{y - x}{|y - x|},$$

where $0 \leq t \leq |y - x|$. Let us denote as above by a the point $L_p^+ \cap \partial\mathcal{U}$ where L_p^+ is the ray with origin p in the direction q. Then

$$F_f(p,\xi) = \frac{1}{|a - p|},$$

and using (3.7) we have

$$F_f(\beta(t),\dot{\beta}(t)) = F_f(p + t\xi,\xi) = \frac{F_f(p,\xi)}{1 - tF_f(p,\xi)} = \frac{\frac{1}{|a-p|}}{1 - \frac{t}{|a-p|}} = \frac{1}{|a - p| - t}.$$

The length of β is then

$$\ell(\beta) = \int_0^{|q-p|} \frac{dt}{|a - p| - t} = \log\left(\frac{1}{|a - p| - |q - p|}\right) - \log\left(\frac{1}{|a - p|}\right).$$

But $q \in [p,a]$, therefore $|a-p| - |q-p| = |a-q|$ and we finally have

$$\ell(\beta) = \log\left(\frac{|a-p|}{|a-q|}\right).$$

This proves that

$$\varrho_f(p,q) \le \log\left(\frac{|a-p|}{|a-q|}\right).$$

In fact we have equality. To see this, choose a supporting hyperplane for $\mathcal{U}$ at a, and let $\mathcal{H}$ be the corresponding half-space containing $\mathcal{U}$ (recall that a hyperplane in $\mathbb{R}^n$ is said to *support* the convex set $\mathcal{U}$ if it meets the closure of that set and $\mathcal{U}$ is contained in one of the half-space bounded by that hyperplane). Using Lemma 3.7 and Equation (3.6), we obtain

$$\varrho_f(p,q) \ge \log\left(\frac{|a-p|}{|a-q|}\right).$$

The proof of Theorem 3.5 follows now from the two previous inequalities. □

Remark 3.9. 1. From Equation (3.8), one sees that $a - p = \frac{\xi}{F_f(p,\xi)}$ and $a - q = \frac{\xi}{F_f(q,\xi)}$. Therefore, the Funk distance can also be written as

$$\varrho_f(p,q) = \log\left(\frac{F_f(q,q-p)}{F_f(p,q-p)}\right). \tag{3.9}$$

2. The proof of Theorem 3.5 given here is taken from [44]. Another interesting proof is given in [66].

Proposition 3.10. *The tautological distance ϱ_f in a proper convex domain $\mathcal{U}$ satisfies the following properties.*

a) *The distance ϱ_f is projective, that is, for any point $z \in [p,q]$ we have*

$$\varrho_f(p,q) = \varrho_f(p,z) + \varrho_f(z,p).$$

b) *ϱ_f is invariant under affine transformations.*

c) *ϱ_f is forward complete.*

d) *ϱ_f is not backward complete.*

The proof is easy, see also Chapter 2 of this volume [47] for more on Funk geometry.

Proposition 3.11. *The unit speed linear geodesic starting at $p \in \mathcal{U}$ in the direction $\xi \in T_p\mathcal{U}$ is the path*

$$\beta_{p,\xi}(s) = p + \frac{(1-e^{-s})}{F_f(p,\xi)} \cdot \xi \tag{3.10}$$

Proof. Let us define β by (3.10); we have then

$$a - p = \frac{\xi}{F(p,\xi)}, \quad \text{and} \quad a - \beta(s) = \frac{\mathrm{e}^{-s}}{F(p,\xi)} \cdot \xi.$$

Therefore,

$$\varrho_f(p, \beta(s)) = \log\left(\frac{|a-p|}{|a-\beta(s)|}\right) = \log\left(\frac{1}{\mathrm{e}^{-s}}\right) = s. \qquad \square$$

We can also prove the proposition using the fact that (3.10) parametrizes an affine segment and is therefore a minimizer for the length. We then only need to check that the curve has unit speed. Indeed we have

$$\dot\beta(s) = \frac{\mathrm{e}^{-s}}{F_f(p,\xi)} \cdot \xi,$$

and using Equation (3.7), we thus obtain

$$\begin{aligned} F_f(\beta(s), \dot\beta(s)) &= \frac{\mathrm{e}^{-s}}{F_f(p,\xi)} \cdot F\left(p + \frac{(1-\mathrm{e}^{-s})}{F_f(p,\xi)} \cdot \xi, \xi\right) \\ &= \frac{\mathrm{e}^{-s}}{F_f(p,\xi)} \cdot \frac{F_f(p,\xi)}{\left(1 - \frac{(1-\mathrm{e}^{-s})}{F_f(p,\xi)} F_f(p,\xi)\right)} \\ &= 1. \end{aligned}$$

The tautological Finsler structure F_f in a convex domain $\mathcal{U}$ is not reversible. We can thus define the *reverse tautological Finsler structure* F_f^* to be the Finsler structure whose Lagrangian is defined as

$$F_f^*(p,\xi) = F_f(p,-\xi).$$

We then have the following

Proposition 3.12. *The distance ϱ_f^* associated to the reverse tautological Finsler structure in a proper convex domain $\mathcal{U} \subset \mathbb{R}^n$ is given by*

$$\varrho_f^*(p,q) = \varrho_f(q,p) = \log\left(\frac{|b-q|}{|b-p|}\right), \tag{3.11}$$

where b is the intersection of the ray starting at q in the direction p with $\partial\mathcal{U}$.

The proof is obvious. Observe in particular that ϱ_f^* is also a projective metric.

4 The Hilbert metric

Definition 4.1. The *Hilbert Finsler structure* F_h in a convex domain $\mathcal{U}$ is the arithmetic symmetrisation of the tautological Finsler structure:

$$F_h(x,\xi) = \frac{1}{2}(F_f(p,\xi) + F_f(p,-\xi)).$$

By construction, the Hilbert Finsler structure is reversible. Its tangent unit ball at a point $p \in \mathcal{U}$ is obtained from the tautological unit ball by the following procedure from convex geometry: first, take the polar dual of $(\mathcal{U} - p)$, then symmetrize this convex set and finally take again the polar dual of the result. This procedure is called the *harmonic symmetrization* of $\mathcal{U}$ based at p, see [45] for the details.

Example 4.2. Symmetrizing the metric (3.2) we obtain the Hilbert metric in the unit ball $\mathbb{B}^n$:

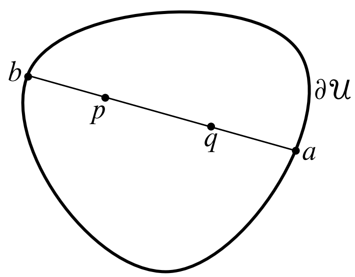
$$F_h(x,\xi) = \frac{\sqrt{(1-\|x\|^2)\|\xi\|^2 + \langle x,\xi\rangle^2}}{(1-\|x\|^2)}. \tag{4.1}$$

Observe that this is a Riemannian metric. We shall prove later on that it has constant sectional curvature $\mathrm{K} = -1$. This metric is the Klein model for hyperbolic geometry.

Using the fact that the affine segment joining two points p and q in $\mathcal{U}$ has minimal length for both the Funk metric F_f and its reverse F_f^*, it is easy to prove the following

Proposition 4.3. *The Hilbert distance ϱ_h in a convex domain $\mathcal{U} \subset \mathbb{R}^n$ is obtained by symmetrizing the Funk distance* (3.4) *in that domain:*

$$\varrho_h(p,q) = \frac{1}{2}(\varrho_f(p,q) + \varrho_f(q,p)) = \frac{1}{2}\log\left(\frac{|a-p|}{|a-q|}\cdot\frac{|b-q|}{|b-p|}\right). \tag{4.2}$$



Using (3.9), one can also write the Hilbert distance as

$$\varrho_h(p,q) = \frac{1}{2}\log\left(\frac{F_f(q,q-p)}{F_f(p,q-p)}\cdot\frac{F_f(p,p-q)}{F_f(q,p-q)}\right). \tag{4.3}$$

We also have the following properties:

Proposition 4.4. *The Hilbert distance ϱ_h in a proper convex domain $\mathcal{U}$ satisfies the following*

a) *The distance ϱ_h is projective: for any point $z \in [p,q]$ we have $\varrho_h(p,q) = \varrho_h(p,z) + \varrho_h(z,p)$.*

b) *ϱ_h is invariant under projective transformations.*

c) *ϱ_h is (both forward and backward) complete.*

The proof is elementary. For an introduction to Hilbert geometry, we refer to Sections 28 and 50 in [14] and Section 18 in [13]. We now describe the geodesics in Hilbert geometry.

Proposition 4.5. *The unit speed linear geodesic starting at $p \in \mathcal{U}$ in the direction $\xi \in T_p\mathcal{U}$ is the path*

$$\beta_{p,\xi}(s) = p + \varphi(s) \cdot \xi, \tag{4.4}$$

where φ is given by

$$\varphi(s) = \frac{(\mathrm{e}^s - \mathrm{e}^{-s})}{F_f(p,\xi)\mathrm{e}^s + F_f(p,-\xi)\mathrm{e}^{-s}}. \tag{4.5}$$

Proof. To simplify notation we write $F = F_f(p,\xi)$ and $F^* = F_f(p,-\xi)$. From (3.8) we have $a - p = \frac{\xi}{F}$ and $b - q = -\frac{\xi}{F^*}$, therefore

$$\varrho_h(p,\beta(s)) = \frac{1}{2}\log\left(\frac{|a-p|}{|a-\beta(s)|}\cdot\frac{|b-\beta(s)|}{|b-p|}\right) = \frac{1}{2}\log\left(\frac{1+F^*\cdot\varphi(s)}{1-F\cdot\varphi(s)}\right).$$

From (4.5), we have

$$\begin{aligned}\frac{1+F^*\cdot\varphi(s)}{1-F\cdot\varphi(s)} &= \frac{(F\cdot\mathrm{e}^s + F^*\cdot\mathrm{e}^{-s}) + (F^*\cdot\mathrm{e}^s - F^*\cdot\mathrm{e}^{-s})}{(F\cdot\mathrm{e}^s + F^*\cdot\mathrm{e}^{-s}) - (F\cdot\mathrm{e}^s - F\cdot\mathrm{e}^{-s})}\\ &= \frac{(F+F^*)\cdot e^s}{(F+F^*)\cdot\mathrm{e}^{-s}} = e^{2s},\end{aligned}$$

and we conclude that

$$\varrho_h(p,\beta(s)) = s.$$

□

Corollary 4.6. *The metric balls in a Hilbert geometry are convex sets.*

Proof. We see from the previous proposition that the ball of radius r around the point $p \in \mathcal{U}$ is the set of points $z \in \mathcal{U}$ such that

$$\mathrm{e}^r F_f(p,z-p) + \mathrm{e}^{-r} F_f(p,p-z) \leq (\mathrm{e}^r - \mathrm{e}^{-r}),$$

which is a convex set. □

Note that another proof of this proposition is given in Chapter 2 of this volume [47].

5 The fundamental tensor

Our goal in this section is to write down and study an ordinary differential equation for the geodesics in a Finsler manifold (M, F). To this aim we need to assume the following condition:

Definition 5.1. A Finsler metric F on a smooth manifold M is said to be *strongly convex* if it is smooth on the slit tangent bundle TM^0 and if the *vertical Hessian* of F^2 at a point $(p, \xi) \in T^0M$

$$g_{p,\xi}(\eta_1, \eta_2) = \frac{1}{2} \frac{\partial^2}{\partial u_1 \partial u_2}\Big|_{u_1=u_2=0} F^2(p, \xi + u_1\eta_1 + u_2\eta_2) \tag{5.1}$$

is positive definite. The bilinear form (5.1) is called the *fundamental tensor* of the Finsler metric.

This condition is called the *Legendre–Clebsch condition* in the calculus of variations. Geometrically it means that the indicatrix at any point is a hypersurface of strictly positive Gaussian curvature.

Classical Finsler geometry has sometimes the reputation of being an "impenetrable forest of tensors",[2] and we shall need to venture a few steps into this wilderness. It will be convenient to work with local coordinates on TM; more precisely, if $U \subset M$ is the domain of some coordinate system $x^1, x^2, \dots, x^n$, then any vector $\xi \in TU$ can be written as $\xi = y^i \frac{\partial}{\partial y^i}$. The $2n$ functions on TU given by $x^1, \dots, x^n, y^1, \dots, y^n$ are called *natural coordinates* on TU. The restriction of the Lagrangian on TU is thus given by a function of $2n$ variables $F(x, y)$. The Legendre-Clebsch condition states that $F(x, y)$ is smooth (although C^3 would suffice) on $\{y \neq 0\}$ and that the matrix given by

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y) \tag{5.2}$$

is positive definite for any $y \neq 0$. The fundamental tensor shares some formal properties with a Riemannian metric, but it is important to remember that it is not defined on U (nor on the manifold M), but on the slit tangent bundle T^0U.

Manipulating tensors in Finsler geometry needs to be done with care. In general, a *tensor* on a Finsler manifold (M, F) is a field of multilinear maps on TM which smoothly depends on a point $(x, y) \in TM^0$ (and not only on a point $x \in M$ as in Riemannian geometry). In the case of the fundamental tensor, note that[3]

$$g_y(u, v) = g_{x,y}(u, v) = g_{ij}(x, y)u^i v^j,$$

where $u = u_i(x)\frac{\partial}{\partial x^i}$ and $v = v_j(x)\frac{\partial}{\partial x^j}$ are elements in T_xM.

[2] This comment on the subject goes back to the paper [12] by Busemann. The first sentence of this nice paper is *"The term* Finsler space *evokes in most mathematicians the picture of an impenetrable forest whose entire vegetation consists of tensors."* A quick glance at [51] will probably convince the reader.

[3] We use the summation convention: terms with repeated indices are summed from 1 to n.

Given a point $(x, y) \in TM^0$, we have a canonical element in $T_x M$, namely the vector y itself, and we can evaluate a given tensor on that canonical vector. In particular we have the following basic fact.

Lemma 5.2. *The Lagrangian and the fundamental tensor are related by*

$$F(x, y) = \sqrt{g_y(y, y)} = \sqrt{g_{ij}(x, y) y^i y^j}.$$

Proof. Recall that if $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is a smooth positively homogenous function of degree r on $\mathbb{R}^n$, that is, $f(\lambda y) = \lambda^r f(y)$ for $\lambda \geq 0$, then its partial derivatives are positively homogenous functions of degree $(r-1)$ and $r \cdot f(y) = y^i \frac{\partial f}{\partial y^i}$ (see [46]). Applying this fact twice to the function $y \mapsto \frac{1}{2} F^2(x, y)$ proves the lemma. □

Observe that the same argument also shows that the fundamental tensor $g_{ij}(x, y)$ is 0-homogenous with respect to y. This type of argument plays a central role in Finsler geometry and anyone venturing into the subject will soon become an expert in recognizing how homogeneity is being used (sometimes in a hidden way) in calculations, or she will quit the subject.

6 Geodesics and the exponential map

We now consider a curve $\beta \colon [a, b] \to M$ of class C^1 in the Finsler manifold (M, F). Recall that its *length* is defined as

$$\ell(\beta) = \int_a^b F(\beta(s), \dot{\beta}(s)) ds. \tag{6.1}$$

We also define the *energy* of the curve β by

$$E(\beta) = \int_a^b F^2(\beta(s), \dot{\beta}(s)) ds. \tag{6.2}$$

The following basic inequality between these functionals holds:

Lemma 6.1. *For any curve $\beta \colon [a, b] \to M$ of class C^1, we have*

$$\ell(\beta)^2 \leq (b-a)\, E(\beta),$$

with equality if and only if β has constant speed, i.e. $t \mapsto F(\beta(s), \dot{\beta}(s))$ is constant.

Proof. Let us denote by $f(s) = F(\beta(s), \dot{\beta}(s))$ the speed of the curve. We have by the Cauchy–Schwarz inequality

$$\ell(\beta) = \int_a^b 1 \cdot f(s) ds \leq \left(\int_a^b 1^2 ds \right)^{1/2} \left(\int_a^b v(t)^2 ds \right)^{1/2} = \sqrt{(b-a)} \sqrt{E(\beta)}.$$

Furthermore, the equality holds if and only if $f(s)$ and 1 are collinear as elements of $L^2(a,b)$, that is, if and only if the speed $f(s)$ is constant. □

Corollary 6.2. *The curve $\beta\colon [a,b] \to M$ is a minimal curve for the energy if and only if it minimizes the length and has constant speed.*

Definition 6.3. The curve $\beta\colon [a,b] \to M$ is a *geodesic* if it is a critical point of the energy functional.

Sometimes the terminology varies, and the term *geodesic* also designates a critical or a minimal curve for the length functional. Note that our notion imposes the restriction that β has constant speed. From the point of view of calculations, this is an advantage since the length is invariant under forward reparametrization.

We now seek to derive the equation satisfied by the geodesics. By the classical theory of the calculus of variations, a curve $s \mapsto \beta(s)$ in some coordinate domain $U \subset M$ is a critical point of the energy functional (6.2) if and only if the following *Euler–Lagrange equations*

$$\frac{d}{ds}\frac{\partial F^2}{\partial y^\mu} = \frac{\partial F^2}{\partial x^\mu} \quad (\mu = 1,\dots,n) \tag{6.3}$$

hold, where $(x(s), y(s)) = (\beta(s), \dot\beta(s))$.

Using the fundamental tensor, one writes $F^2(x,y) = g_{ij}(x,y)y^i y^j$, therefore

$$\frac{\partial F^2}{\partial x^\mu} = \frac{\partial g_{ij}}{\partial x^\mu} y^i y^j \tag{6.4}$$

and

$$\frac{\partial F^2}{\partial y^\mu} = \frac{\partial g_{ij}}{\partial y^\mu} y^i y^j + g_{i\mu} y^i + g_{j\mu} y^j.$$

Observe that the sums $g_{i\mu}y^i$ and $g_{j\mu}y^j$ coincide. Using the homogeneity of F^2 in y and the fact that F is of class C^3 on $y \neq 0$, we obtain

$$\frac{\partial g_{ij}}{\partial y^\mu} y^i = \frac{\partial^3 F^2}{\partial y^\mu \partial y^i \partial y^j} \cdot y^i = \frac{\partial^3 F^2}{\partial y^i \partial y^\mu \partial y^j} \cdot y^i = 0.$$

Therefore,

$$\frac{\partial F^2}{\partial y^\mu} = 2 g_{i\mu} y^i. \tag{6.5}$$

Differentiating this expression with respect to s, we get

$$\frac{d}{ds}\frac{\partial F^2}{\partial y^\mu} = 2\frac{\partial g_{i\mu}}{\partial x^j} \cdot y^i \cdot \frac{dx^j}{ds} + 2 g_{i\mu} \cdot \frac{dy^i}{ds} + 2\frac{\partial g_{i\mu}}{\partial y^j} \cdot y^i \cdot \frac{dy^j}{ds}.$$

We have, as before, $\frac{\partial g_{i\mu}}{\partial y^j} \cdot y^i = 0$ (because $g_{i\mu}$ is 0-homogenous), therefore using $\dot{x}^i = y^i$, we obtain

$$\frac{d}{ds}\frac{\partial F^2}{\partial y^\mu} = 2\frac{\partial g_{i\mu}}{\partial x^j} \cdot y^i y^j + 2g_{i\mu}\dot{y}^i. \tag{6.6}$$

From Equations (6.3), (6.5) and (6.6), we obtain for $\mu = 1, \dots, n$,

$$\sum_i g_{i\mu} \cdot \dot{y}^i = \frac{1}{2}\sum_{i,j}\left(\frac{\partial g_{ij}}{\partial x^\mu} - 2\frac{\partial g_{i\mu}}{\partial x^j}\right) y^i y^j.$$

Multiplying this identity by $g^{k\mu}(x, y)$, where $g^{k\mu}g_{\mu i} = \delta_i^k$, and summing over μ, one gets

$$\dot{y}^k = \frac{1}{2}\sum_{i,j,\mu} g^{k\mu}\left(\frac{\partial g_{ij}}{\partial x^\mu} - 2\frac{\partial g_{i\mu}}{\partial x^j}\right) y^i y^j. \tag{6.7}$$

Proposition 6.4. *The C^2 curve $\beta(s) \in (M, F)$ is a geodesic for the Finsler metric F if only if in local coordinates, we have*

$$\ddot{x}^k + \gamma_{ij}^k \dot{x}^i \dot{x}^j = 0, \tag{6.8}$$

where $\beta(s) = (x^1(s), \dots, x^n(s))$ is a local coordinate expression for the curve and

$$\gamma_{ij}^k(x, y) = \frac{1}{2}g^{k\mu}\left(\frac{\partial g_{i\mu}}{\partial x^j} + \frac{\partial g_{j\mu}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\mu}\right),$$

are the formal Christoffel symbols *of F.*

Proof. A direct calculation shows that Equation (6.8) is equivalent to Equation (6.7) with $y^k = \dot{x}^k$. □

Another way to write the geodesic equation is to introduce the functions

$$\begin{aligned} G^k(x, y) &= \frac{1}{2}\gamma_{ij}^k(x, y)\, y^i y^j \\ &= \frac{1}{4}g^{k\mu}\left(2\frac{\partial g_{i\mu}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\mu}\right) y^i y^j \\ &= \frac{1}{2}g^{k\mu}\left(\frac{\partial^2 F}{\partial y^\mu \partial x^j} y^j - \frac{\partial F}{\partial x^\mu}\right), \end{aligned}$$

so the geodesic equation can be written as

$$\dot{y}^k + 2G^k(x, y) = 0, \quad \dot{x}^k = y^k \quad (k = 1, \dots n). \tag{6.9}$$

The vector field on TU defined by

$$G = y^k \frac{\partial}{\partial x^k} - 2G^k(x, y)\frac{\partial}{\partial y^k} \tag{6.10}$$

is in fact independent on the choice of the coordinates x^j on U and is therefore globally defined on the tangent bundle TM. This vector field is called the *spray* of the Finsler manifold. A significant part of a Finsler geometry is contained in the behavior of its spray (see [53]). Observe that a curve $\beta(s) \in M$ is geodesic if and only if its lift $(\beta(s), \dot{\beta}(s)) \in TM$ is an integral curve for the spray G.

Observe the rather surprising analogy between the equation (6.8) and the Riemannian geodesic equation. This is due to the fact that the 0-homogeneity of $g_{i,j}(x, y)$ in y implies that all the non-Riemannian terms in the calculation of the Euler–Lagrange equation end up vanishing. The important difference between the Finslerian and the Riemannian cases lies in the fact that the formal Christoffel symbols γ_{ij}^k are not functions of the coordinates x^i only. Equivalently, the spray coefficients are generally not quadratic polynomials in the coordinates y^i. In fact, it is known that the spray coefficients $G^k(x, y)$ of a Finsler metric F are quadratic polynomials in the coordinates y^i if and only if F is Berwald.

On a smooth manifold M with a strongly convex Finsler metric, one can define an exponential map as it is done in Riemannian geometry. Because the coefficients $G^k(x, y)$ of the geodesic equation are Lipschitz continuous, for any point $p \in M$ and any vector $\xi \in T_pM$, there is locally a unique solution to the geodesics equation

$$s \mapsto \sigma_\xi(s) \in M, \quad -\epsilon < s < \epsilon,$$

with initial conditions $\sigma_\xi(0) = p$, $\dot{\sigma}_\xi(0) = \xi$. Observe that

$$\sigma_{\lambda\xi}(s) = \sigma_\xi(\lambda s), \quad \lambda \geq 0,$$

so if ξ is small enough, then $\sigma_\xi(1)$ is well defined and we denote it by

$$\exp_p(\xi) = \sigma_\xi(1).$$

We then have the following

Theorem 6.5. *The set $\mathcal{O}_p$ of vectors $\xi \in T_pM$ for which $\exp_p(\xi)$ is defined is a neighborhood of $0 \in T_pM$. The map $\exp_p \colon \mathcal{O}_p \to M$ is of class C^1 in the interior of $\mathcal{O}_p$ and its differential at 0 is the identity. In particular, the exponential is a diffeomorphism from a neighborhood of $0 \in T_pM$ to a neighborhood of p in M.*

If (M, F) is connected and forward complete, then $\exp_p$ is defined on all of T_pM and is a surjective map

$$\exp_p \colon T_pM \to M.$$

The coefficients $G^k(x, y)$ in the geodesic equation are in general not C^1 at $y = 0$, therefore the proof of this theorem is more delicate than its Riemannian counterpart. See [6] for the details.

Finally note that $\exp_p$ has no reason to be of class C^2. In fact a result of Akbar-Zadeh states that $\exp_p$ is a C^2 map near the origin if and only if (M, F) is Berwald.

7 Projectively flat Finsler metrics

Definition 7.1. 1. A Finsler structure F on a convex set $\mathcal{U} \subset \mathbb{R}^n$ is *projectively flat* if every affine segment $[p,q] \subset \mathcal{U}$ can be parametrized as a geodesic.

2. A Finsler manifold (M,F) is *locally projectively flat* if each point admits a neighborhood that is isometric to a convex region in $\mathbb{R}^n$ with a projectively flat Finsler structure.

Examples 7.2. Basic examples are the following.

a) Minkowski metrics are obviously projectively flat.

b) The Funk and the Hilbert metrics are projectively flat Finsler metrics. In particular the Klein metric (4.1) in $\mathbb{B}^n$ is a projectively flat Riemannian metric.

c) The canonical metric on the sphere $S^n \subset \mathbb{R}^{n+1}$ is locally projectively flat. Indeed, the central projection of the half-sphere $S^n \cap \{x_n < 0\}$ on the hyperplane $\{x_n = -1\}$ with center the origin maps the great circles in S^n on affine lines on that hyperplane (in cartography, this map is called the *gnomonic representation of the sphere*). In formulas, the map $\mathbb{R}^n \to S^n$ is given by $x \mapsto \frac{(x,-1)}{\sqrt{1+\|x\|^2}}$, and the metric can be written as

$$F_x(x,\xi) = \frac{\sqrt{(1+\|x\|^2)\|\xi\|^2 - \langle x,\xi\rangle^2}}{(1+\|x\|^2)}. \tag{7.1}$$

This is the projective model for the spherical metric.

A classic result, due to E. Beltrami, says that a Riemannian manifold is locally projectively flat if and only if it has constant sectional curvature, see Theorem 10.6 below. In Finsler geometry, there are more examples and the Finsler version of Hilbert's IV$^{\text{th}}$ problem is to determine and study all conformally flat (complete) Finsler metrics in a convex domain.

A first result on projectively flat Finsler metrics is given in the next proposition:

Proposition 7.3. *A smooth and strongly convex Finsler metric F on a convex domain $\mathcal{U} \subset \mathbb{R}^n$ is projectively flat if only if its spray coefficients satisfy*

$$G^k(x,y) = P(x,y) \cdot y^k$$

for some scalar function $P : T\mathcal{U} \to \mathbb{R}$.

Proof. The proof is standard, see e.g. [15]. Suppose that the affine segments are geodesics. This means that

$$\beta(s) = p + \varphi(s)\xi$$

satisfy the geodesic equation (6.9) for any $p \in \mathcal{U}$, $\xi \in \mathcal{U}$ and some (unknown) function $\varphi(s)$. Along β we have

$$y^k = \dot{x}^k = \dot{\varphi}(s)\xi^k, \quad \dot{y}^k = \ddot{\varphi}(s)\xi^k,$$

therefore Equation (6.9) can be written as

$$\ddot{\varphi}(s)\xi^k + 2\dot{\varphi}^2(s)G^k(\beta(s), \xi) = 0,$$

which implies that $G^k(x, y) = P(x, y) \cdot y^k$ with

$$P(x, \xi) = -\frac{\ddot{\varphi}(0)}{2\dot{\varphi}(0)^2}. \tag{7.2}$$

□

The function $P(x, \xi)$ in the previous proposition is called the *projective factor*. It can be computed from (7.2) if the geodesics are explicitly known. For instance we have the

Proposition 7.4. *The projective factor of the tautological Finsler structure F_f in the convex domain $\mathcal{U}$ is given by*

$$P_f(x, y) = \frac{1}{2}F_f(x, y).$$

For the Hilbert Finsler structure F_h, we have

$$P_h(x, y) = \frac{1}{2}(F_f(x, y) - F_f(x, -y)).$$

Proof. The geodesics for F_f are given by $\beta(s) = p + \varphi(s)y$ with

$$\varphi(s) = \frac{1}{F_f(p, \xi)}(1 - \mathrm{e}^{-s}).$$

Therefore,

$$P_f(x, y) = -\frac{\ddot{\varphi}(0)}{2\dot{\varphi}(0)^2} = \frac{1}{2}F_f(p, \xi).$$

The geodesics for the Hilbert metric $F_h(x, y) = \frac{1}{2}(F_f(x, y) + F_f(x, -y))$ are the curves $\beta(s) = p + \varphi(s)y$ with

$$\varphi(s) = \frac{(\mathrm{e}^s - \mathrm{e}^{-s})}{F_f(p, \xi)\mathrm{e}^s + F_f(p, -\xi)\mathrm{e}^{-s}}.$$

The derivative of this function is

$$\dot{\varphi}(s) = \frac{2(F_f(p, \xi) + F_f(p, -\xi))}{(F_f(p, \xi)\mathrm{e}^s + F_f(p, -\xi)\mathrm{e}^{-s})^2},$$

and

$$\ddot{\varphi}(s) = -4\,\frac{(F_f(p, \xi) + F_f(p, -\xi))(F_f(p, \xi)\mathrm{e}^s - F_f(p, -\xi)\mathrm{e}^{-s})}{(F_f(p, \xi)\mathrm{e}^s + F_f(p, -\xi)\mathrm{e}^{-s})^3}.$$

Therefore,

$$P_h(x, y) = -\frac{\ddot{\varphi}(0)}{2\dot{\varphi}(0)^2} = \frac{1}{2}(F_f(x, y) - F_f(x, -y)).$$

□

It is clear that the distance associated to a projectively flat metric is projective in the sense of Definition 2.1 in [46]. It is also clear that the sum of two projective (weak) distances is again projective. This suggests that one can write down a linear condition on F which is equivalent to projective flatness. This is the content of the next proposition which goes back to the work [31] of G. Hamel.

Proposition 7.5. *Let $F : T\mathcal{U} \to \mathbb{R}$ be a smooth Finsler metric on the convex domain $\mathcal{U}$. The following conditions are equivalent.*

(a) *F is projective flat.*

(b) $y^k \dfrac{\partial^2 F}{\partial x^k \partial y^m} - \dfrac{\partial F}{\partial x^m} = 0$, *for* $1 \le m \le n$.

(c) $\dfrac{\partial^2 F}{\partial x^j \partial y^m} = \dfrac{\partial^2 F}{\partial y^j \partial x^m}$, *for* $1 \le j, m \le n$.

(d) $y^k \dfrac{\partial^2 F}{\partial x^m \partial y^k} = y^k \dfrac{\partial^2 F}{\partial x^k \partial y^m}$.

Proof. The Euler–Lagrange equation for the length functional (6.1) is

$$0 = \frac{\partial F}{\partial x^m} - \frac{d}{dt}\frac{\partial F}{\partial y^m} = \frac{\partial F}{\partial x^m} - \frac{\partial^2 F}{\partial x^k \partial y^m}\dot{x}^k - \frac{\partial^2 F}{\partial y^k \partial y^m}\dot{y}^k,$$

and this can be written as

$$\frac{\partial^2 F}{\partial y^k \partial y^m}\dot{y}^k = \frac{\partial F}{\partial x^m} - \frac{\partial^2 F}{\partial x^k \partial y^m} y^k.$$

Now F is projectively flat if and only if $x(t) = p + t\xi$ is a solution (recall that the length is invariant under reparametrization), which is equivalent to $\dot{y}^k = 0$. The equivalence (a) $\Leftrightarrow$ (b) follows.

To prove (b) $\Rightarrow$ (c), we differentiate (b) with respect to y^j. This gives

$$A_{jm} = y^k \frac{\partial^3 F}{\partial y^j \partial x^k \partial y^m} + \frac{\partial^2 F}{\partial x^j \partial y^m} - \frac{\partial^2 F}{\partial y^j \partial x^m} = 0,$$

and thus

$$0 = \frac{1}{2}(A_{jm} - A_{mj}) = \frac{\partial^2 F}{\partial x^j \partial y^m} - \frac{\partial^2 F}{\partial y^j \partial x^m},$$

which is equivalent to (c).

(c) $\Rightarrow$ (d) is obvious.

Finally, to prove (d) $\Rightarrow$ (b), we use the homogeneity of F in y. We have $F(x, y) = y^k \frac{\partial F}{\partial y^k}$, therefore condition (d) implies

$$\frac{\partial F}{\partial x^m} = y^k \frac{\partial^2 F}{\partial x^m \partial y^k} = y^k \frac{\partial^2 F}{\partial x^k \partial y^m},$$

which is equivalent to (b). □

Remark 7.6. In [50] A. Rapcsàk gave a generalization of the previous conditions for the case of a pair of projectively equivalent Finsler metrics, that is, a pair of metrics having the same geodesics up to reparametrization.

Example 7.7. We know that the tautological (Funk) Finsler metric F_f in a convex domain $\mathcal{U}$ is projectively flat. Using the previous proposition we have more examples:

i) the reverse Funk metric $F_f^*(x,\xi) = F_f(x,-\xi)$,

ii) the Hilbert metric $F_h(x,\xi) = \frac{1}{2}(F_f(x,-\xi) + F_f(x,-\xi))$, and

iii) the metric $F_f(x,\xi) + F_0(\xi)$, where F_0 is an arbitrary (constant) Minkowski norm

are projectively flat. More generally if F_1, F_2 are projectively flat, then so is the sum $F_1 + F_2$. Assuming either F_1 or F_2 to be (forward) complete, the sum is also a (forward) complete solution to Hilbert's IV$^{\text{th}}$ problem.

The following result gives a general formula computing the projective factor of a projectively flat Finsler metric:

Lemma 7.8. *Let $F(x,y)$ be a smooth projectively flat Finsler metric on some domain in $\mathbb{R}^n$. Then the following equations hold:*

$$2FP = y^k \frac{\partial F}{\partial x^k} \tag{7.3}$$

and

$$\frac{\partial F}{\partial x^m} = P\frac{\partial F}{\partial y^m} + F\frac{\partial P}{\partial y^m}, \tag{7.4}$$

where $P(x,y)$ is the projective factor.

Proof. Along a geodesic $(x(s), y(s))$, we have

$$\dot{y}^k = -2G^k(x,y) = -2P(x,y)y^k, \quad \dot{x}^k = y^k.$$

Since the Lagrangian $F(x(s), y(s))$ is constant in s, we obtain the first equation

$$0 = \frac{dF}{ds} = \frac{\partial F}{\partial x^k}y^k + \frac{\partial F}{\partial y^k}\dot{y}^k = \frac{\partial F}{\partial x^k}y^k - 2P\frac{\partial F}{\partial y^k}y^k = \frac{\partial F}{\partial x^k}y^k - 2PF.$$

Differentiating this equation and using (b) in Proposition 7.5, we obtain

$$2\Big(P\frac{\partial F}{\partial y^m} + F\frac{\partial P}{\partial y^m}\Big) = \frac{\partial}{\partial y^m}\Big(y^k\frac{\partial F}{\partial x^k}\Big) = \frac{\partial F}{\partial x^m} + y^k\frac{\partial^2 F}{\partial x^k \partial y^m} = 2\frac{\partial F}{\partial x^m}. \qquad \square$$

A first consequence is the following description of Minkowski metrics:

Corollary 7.9. *A strongly convex projectively flat Finsler metric on some domain in $\mathbb{R}^n$ is the restriction of a Minkowski metric if and only if the associated projective factor $P(x,y)$ vanishes identically.*

Proof. Suppose $F(x,y)$ is locally Minkowski, then $\frac{\partial F}{\partial x^m} = 0$ and therefore the spray coefficient satisfies $P(x,y)y^k = G^k(x,y) = 0$. Conversely, if $P(x,y) \equiv 0$, then the second equation in the lemma implies that $\frac{\partial F}{\partial x^m} = 0$. □

Another consequence is the following result about the projective factor of a reversible Finsler metric.

Corollary 7.10. *Let $F(x,y)$ be a strongly convex projectively flat Finsler metric. Suppose F is reversible, then its projective factor satisfies*

$$P(x,-y) = -P(x,y).$$

Proof. This is obvious from the first equation in Lemma 7.8. □

8 The Hilbert form

We now introduce the Hilbert 1-form of a Finsler manifold and show its relation with Hamel's condition:

Definition 8.1. The *Hilbert* 1*-form* on a smooth Finsler manifold (M,F) is the 1-form on TM^0 defined in natural coordinates as

$$\omega = \frac{\partial F}{\partial y^j} dx^j = g_{ij}(x,y) y^i dx^j.$$

From the homogeneity of F, we have $F(x,\xi) = \omega(x,\xi)$. Note also that $\omega(\eta) = 0$ for any vector $\eta \in TTM^0$ that is tangent to some level set $F(x,y) = \text{const}$. These two conditions characterize the Hilbert form which is therefore independent of the choice of coordinates, see also [18]. Observe also that the length of a smooth non-singular curve $\beta\colon [a,b] \to M$ is

$$\ell(\beta) = \int_a^b F(\tilde{\beta}) = \int_{\tilde{\beta}} \omega,$$

where $\tilde{\beta}\colon [a,b] \to TM^0$ is the natural lift of β. We then have the

Proposition 8.2 ([4], [16]). *The smooth Finsler metric $F\colon T\mathcal{U} \to \mathbb{R}$ on the convex domain $\mathcal{U}$ is projectively flat if and only if the Hilbert form is d_x-closed, that is, if we have*

$$d_x\omega = \frac{\partial^2 F}{\partial x^i \partial y^j} dx^i \wedge dx^j = 0.$$

Equivalently, F is projectively flat if and only if the Hilbert form is d_x-exact. This means that there exists a function $h\colon T\mathcal{U}^0 \to \mathbb{R}$ such that

$$\omega = d_x h = \frac{\partial h}{\partial x^j} dx^j.$$

Proof. The first assertion is a mere reformulation of the Hamel Condition (c) in Proposition7.5. The second assertion is proved using the same argument as in the proof of the Poincaré Lemma. Indeed, using Condition (c) from Proposition 7.5, we compute:

$$\begin{aligned}\frac{d}{dt}\big(t\cdot\omega_{(tx,y)}\big) &= \frac{d}{dt}\Big(t\frac{\partial F}{\partial y^i}(tx,y)dx^i\Big)\\ &= \frac{\partial F}{\partial y^i}(tx,y)dx^i + t\frac{\partial^2 F}{\partial x^k\partial y^i}x^k dx^i\\ &= \frac{\partial F}{\partial y^i}(tx,y)dx^i + t\frac{\partial^2 F}{\partial x^i\partial y^k}x^k dx^i\\ &= d_x\Big(x^k\frac{\partial F}{\partial y^k}(tx,y)\Big) = d_x\big(\omega_{(tx,y)}(x)\big).\end{aligned}$$

Suppose now that $0\in\mathcal{U}$ and set

$$h(x,y) = \int_0^1 \omega_{(tx,y)}(x)dt = \int_0^1 \Big(x^k\frac{\partial F}{\partial y^k}(tx,y)\Big)dt,$$

then by the previous calculation we have

$$d_x h = \int_0^1 d_x(\omega_{(tx,y)})dt = \int_0^1 \frac{d}{dt}(t\cdot\omega_{(tx,y)})dt = \omega_{(x,y)}. \qquad \square$$

We shall call such a function h a *Hamel potential* of the projective Finsler metric F. Observe that a Hamel potential is well defined up to adding a function of y. The above proof shows that one can chose $h(x,y)$ to be 0-homogenous in y. This potential allows us to compute distances.

Corollary 8.3. *The distance d_F associated to the projective Finsler metric F is given by*

$$d_F(p,q) = h(q,q-p) - h(p,q-p),$$

where $h(x,y)$ is a Hamel potential for F.

Proof. Let $\beta(t) = p + t(q-p)$, $(t\in[0,1])$. Then

$$d_F(p,q) = \int_0^1 F(\beta,\dot\beta)dt = \int_{\tilde\beta}\omega = h(q,q-p) - h(p,q-p). \qquad \square$$

Example 8.4. If F_f is the tautological Finsler structure in $\mathcal{U}$, then from the previous corollary and Equation (3.9) we deduce that a Hamel potential is given by

$$h(x,y) = -\log(F_f(x,y)).$$

If one prefers a 0-homogenous potential, then a suitable choice is

$$h(x,y) = \log\Big(\frac{F_f(p,y)}{F_f(x,y)}\Big),$$

where $p \in \mathcal{U}$ is some fixed point.

Remark 8.5. A consequence of this example is that the tautological Finsler structure in a domain $\mathcal{U}$ satisfies the following equation:

$$\frac{\partial F_f}{\partial x^j} = F_f \frac{\partial F_f}{\partial y^j}. \tag{8.1}$$

Indeed, since $h(x, y) = -\log(F_f(x, y))$ is a Hamel potential, the Hilbert form is

$$\frac{\partial F_f}{\partial y^j} dx^j = \omega = d_x h = \frac{1}{F_f} \frac{\partial F_f}{\partial x^j} dx^j,$$

from which (8.1) follows immediately. This equation plays an important role in the study of projectively flat metrics with constant curvature, see e.g. [9].

An intrinsic discussion of geodesics based on the Hilbert form is given in [3], [16], [17], [18].

Remark 8.6. J. C. Álvarez Paiva and G. Berck gave a nice generalization of Proposition 8.2 above to a Hamel-type characterization of higher dimensional Lagrangians whose minimal submanifolds are k-flats, see Theorem 4.1 in [4].

9 Curvature in Finsler geometry

A notion of curvature for Finsler surfaces already appeared in the beginning of the last century. This notion was extended in all dimensions by L. Berwald in 1926 [7]. This curvature is an analogue of the sectional curvature in Riemannian geometry and it is best explained using the notion of *osculating Riemannian metric* introduced by Varga [64]. See also Auslander [5] and the book of Rund [51], p. 84.

Definition 9.1. 1. Let (M, F) be a smooth manifold with a strongly convex Finsler metric and let $(x_0, y_0) \in TM^0$. A vector field V defined in some neighborhood $\mathcal{O} \subset M$ of the point x_0 is said to be a *geodesic extension* of y_0 if $V_{x_0} = y_0$ and if the integral curves of V are geodesics of the Finsler metric F (in particular V does not vanish throughout the neighborhood $\mathcal{O}$).

2. The *osculating Riemannian metric* g_V of F in the direction of V is the Riemannian metric on $\mathcal{O}$ defined by the Fundamental tensor at the point $(x, y) = (x, V_x) \in T\mathcal{O}^0$. In local coordinates we have

$$\mathrm{g}_V = g_{ij}(x) dx^i dx^j = g_{ij}(x, V(x)) dx^i dx^j = \frac{1}{2} \frac{\partial^2 F(x, V_x)}{\partial y^i \partial y^j} dx^i dx^j.$$

Let us fix a point $(x_0, y_0) \in TM^0$ and a geodesic extension of y_0. We shall denote by Riem_V the $(1, 3)$ Riemann curvature tensor of the osculating metric g_V. Recall that

$$\mathrm{Riem}_V(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z,$$

where ∇ is the Levi-Civita connection of g_V.

Definition 9.2. The *Riemann curvature* of g_V is the field of endomorphisms ((1,1)-tensor) $\mathrm{R}_V : TM \to TM$ defined as

$$\mathrm{R}_V(W) = \mathrm{Riem}_V(W,V)V.$$

A basic fact of Finsler geometry is the following result:

Proposition 9.3. *Let (M,F) be a smooth manifold with a strongly convex Finsler metric and let (x,y) be a point in TM^0. Then the Riemann curvature R_V at $(x,y) \in TM^0$ is well defined independently of the choice of a geodesic extension V of y.*

Proof. Choose some local coordinate system and let us write $R^i{}_k(x,y)$ for the components of the tensor R_V:

$$\mathrm{R}_V = R^i{}_k(x,y)\, dx^k \otimes \frac{\partial}{\partial x^i}.$$

Then we have the formula

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}, \tag{9.1}$$

where the $G^i = G^i(x,y)$ are the spray coefficients of F. We refer to Lemma 6.1.1 and Proposition 6.2.2 in [54] for a proof. Formula (9.1) is also obtained in [15], p. 43, where it is seen as a consequence of the structure equations for the Chern connection, see also [53], Proposition 8.4.3. Since the spray coefficients $G^i(x,y)$ only depend on the fundamental tensor and its partial derivatives, it follows that $R^i{}_k$ depends only on $(x,y) = (x,V_x) \in TM^0$ and not on the choice of a geodesic field extending y. □

This proposition implies that we can write

$$\mathrm{R}_y = \mathrm{R}_V = R^i{}_k(x,y)\, dx^k \otimes \frac{\partial}{\partial x^i}$$

for the Riemann curvature at a point $(x,y) \in TM^0$, where V is an arbitrary geodesic extension of y. We then have the following important

Corollary 9.4. *Let $\sigma \subset T_xM$ be a 2-plane containing the non-zero vector $y \in T_xM$. Choose a local geodesic extension V of y, then the sectional curvature $\mathrm{K}_{\mathrm{g}_V}(\sigma)$ of σ for the osculating Riemannian metric g_V is independent of the choice of V.*

Proof. By definition of the sectional curvature in Riemannian geometry, we have

$$\mathrm{K}_{\mathrm{g}_V}(\sigma) = \frac{\mathrm{g}_V(\mathrm{R}_V(W),W)}{\mathrm{g}_V(y,y)\,\mathrm{g}_V(W,W) - \mathrm{g}_V(y,W)^2}, \tag{9.2}$$

where $W \in \sigma$ and $y = V_x$ are linearly independent (and thus $\sigma = \operatorname{span}\{V, W\}$), and R_V is the Riemann curvature of g_V. This quantity is independent of the geodesic extension V, by the previous proposition. □

Definition 9.5. The pair (y, σ) with $0 \neq y \in \sigma$ is called a *flag* in M, and $\mathrm{K}_{\mathrm{g}_V}(\sigma)$ is called the *flag curvature* of (y, σ), and denoted by $\mathrm{K}(y, \sigma)$. The vector $y \in \sigma$ is sometimes suggestively called the *flagpole* of the flag.

In local coordinates, the flag curvature can be written as

$$\mathrm{K}(y, \sigma) = \frac{R_{mk}(x, y)\, w^k w^m}{F^2(x, y) g_{ij}(x, y) w^i w^j - (g_{rs}(x, y) w^r y^s)^2}, \tag{9.3}$$

where $W = w^k \frac{\partial}{\partial x^k} \in \sigma$ and y are linearly independent and $R_{mk} = g_{im} R^i{}_k$.

Example 9.6. The flag curvature of a Minkowski metric is zero. Indeed, the fundamental tensor is constant and coincides with any osculating metric which is thus flat. Note that conversely, there are many examples of Finsler metrics with vanishing flag curvature which are not locally isometric to a Minkowski metric. The first example has been given in Section 7 of [8].

We also have a notion of Ricci curvature:

Definition 9.7. The *Ricci curvature* of the Finsler metric F at $(x, y) \in TM^0$ is defined as

$$\operatorname{Ric}(x, y) = \operatorname{Trace}(\mathrm{R}_y) = F^2(x, y) \cdot \sum_{i=2}^{n} \mathrm{K}(y, \sigma_i)$$

where $e_1, e_2, \dots, e_n \in T_x M$ is an orthonormal basis relative to the inner product g_y such that $e_1 = \frac{y}{F(x,y)}$.

The geometric meaning of the Finslerian flag curvature presents both similarities and striking differences with its Riemannian counterpart. The Riemann curvature R_y plays an important role in the Finsler literature. It appears naturally in the second variation formula for the length of geodesics and in the theory of Jacobi fields (see [54], Lemma 6.1.1, or [51], Chapters IV.4 and IV.5). This leads to natural formulations of comparison theorems in Finsler geometry which are similar to their Riemannian counterparts. In particular we have

- In 1952, L. Auslander proved a Finsler version of the Cartan–Hadamard Theorem. If the Finsler manifold (M, F) is forward complete and has non-positive flag curvature, then the exponential map is a covering map [5], [39].
- Auslander also proved a Bonnet–Myers Theorem: A forward complete Finsler manifold (M, F) with Ricci curvature $\operatorname{Ric}(x, y) \geq (n-1)F^2(x, y)$ for all $(x, y) \in TM^0$ is compact with diameter $\leq \pi$, see [5], [39].

- In 2004, H.-B. Rademacher proved a sphere theorem: a compact, simply-connected Finsler manifold of dimension $n \geq 3$ such that $F(p,-\xi) \leq \lambda F(p,\xi)$ for any $(p,\xi) \in TM$ and with flag curvature satisfying
$$\left(1 - \frac{1}{1+\lambda}\right)^2 < \mathrm{K} \leq 1$$
is homotopy equivalent (and thus homeomorphic) to a sphere [49] (see also [20] for an earlier result in this direction). Note that if F is reversible, i.e. $\lambda = 1$, then we have the analog of the familiar $\frac{1}{4}$-pinching sphere theorem of Riemannian geometry.

We also mention the Finslerian version of the Schur Lemma:

Lemma 9.8 (The Schur Lemma). *Let (M,F) be a smooth manifold of dimension $n \geq 3$ with a strongly convex Finsler metric. Suppose that at each point $p \in M$ the flag curvature is independent of the flag, that is,*
$$\mathrm{K}(y,\sigma) = \kappa(p)$$
for every flag (y,σ) at p, where $\kappa\colon M \to \mathbb{R}$ is an arbitrary function. Then K *is a constant, that is, κ is independent of p.*

We refer to Lemma 3.10.2 of [6] for a proof. This result is already stated without proof in the work of Berwald (see the footnote on p. 468 in [8]).

Finally we should warn the reader that unlike the situation in Riemannian geometry, the flag curvature does not control the purely metric notions of curvature such as the notions of non-positive (or non-negative) curvature in the sense of Busemann or Alexandrov. In particular we shall prove below that Hilbert geometries have constant negative flag curvature, yet they satisfy the Busemann or Alexandrov curvature condition if and only if the convex domain is an ellipsoid, see [28].

10 The curvature of projectively flat Finsler metrics

The Riemann curvature of a general projectively flat Finsler metric was computed by Berwald in [8]. We can state the result in the following form:

Theorem 10.1. *Let $F(x)$ be a strongly convex projectively flat Finsler metric on some domain $\mathcal{U}$ of $\mathbb{R}^n$. Then the Riemann curvature at a point $(x,y) \in T\mathcal{U}^0$ is given by*
$$\mathrm{R}_y = \mathcal{R}(x,y)\,\mathrm{Proj}_{y^\perp}, \tag{10.1}$$
where
$$\mathcal{R}(x,y) = \left(P^2 - y^j \tfrac{\partial P}{\partial x^j}\right) \tag{10.2}$$
and $\mathrm{Proj}_{y^\perp}\colon T_x\mathcal{U} \to T_x\mathcal{U}$ *is the orthogonal projection onto the hyperplane $y^\perp \subset T_x\mathcal{U}$ with respect to the fundamental tensor* g_y *at (x,y).*

Proof. We basically follow the proof in [15], p. 110. Since F is projectively flat, we have $G^i = P(x,y)y^i$ and equation (9.1) gives

$$\mathrm{R}_y = R^i{}_k(x,y)\,dx^k \otimes \frac{\partial}{\partial x^i},$$

with

$$R^i{}_k = 2\frac{\partial(Py^i)}{\partial x^k} - \frac{\partial^2(Py^i)}{\partial x^j \partial y^k}y^j + 2(Py^j)\frac{\partial^2(Py^i)}{\partial y^j \partial y^k} - \frac{\partial(Py^i)}{\partial y^j}\frac{\partial(Py^j)}{\partial y^k}.$$

Using the homogeneity in y for $P(x,y)$ and $\frac{\partial y^i}{\partial y^k} = \delta^i_k$, we calculate that

$$R^i{}_k = \mathcal{R}\,\delta^i_k + \mathcal{T}_k\,y^i,$$

where $\mathcal{R} = \left(P^2 - y^j\frac{\partial P}{\partial x^j}\right)$, and

$$\begin{aligned}\mathcal{T}_k &= 2P\frac{\partial P}{\partial y^k} - \frac{\partial P}{\partial y^k} - \frac{\partial^2 P}{\partial y^k \partial x^j}y^i y^k + 3\left(\frac{\partial P}{\partial x^k} - P\frac{\partial P}{\partial y^k}\right)\\ &= \frac{\partial\,\mathcal{R}}{\partial y^k} + 3\left(\frac{\partial P}{\partial x^k} - P\frac{\partial P}{\partial y^k}\right).\end{aligned}$$

Observe that

$$y^k\,\mathcal{T}_k = -\,\mathcal{R}, \tag{10.3}$$

therefore $R^i{}_k = \mathcal{R}\,\delta^i_k + \mathcal{T}_k\,y^i$, which we write as

$$R_{mk} = g_{mi}R^i{}_k = g_{mi}(\mathcal{R}\,\delta^i_k + \mathcal{T}_k\,y^i) = \mathcal{R}\,g_{mk} + g_{mi}y^i\,\mathcal{T}_k\,.$$

We shall compute $\mathcal{T}_m$ using a trick. From the symmetry $R_{mk} = R_{km}$, we find

$$0 = (R_{mk} - R_{km}) = (g_{ki} - g_{mi})y^i\,\mathcal{T}_m,$$

therefore $g_{mi}y^i\,\mathcal{T}_k = g_{ki}y^i\,\mathcal{T}_m$. Using (10.3), one gets

$$g_{mi}y^i\,\mathcal{R} = -g_{mi}y^i y^k\,\mathcal{T}_k = -g_{ki}y^i y^k\,\mathcal{T}_m = -F^2\,\mathcal{T}_m, \tag{10.4}$$

that is,

$$\mathcal{T}_k = -\,\mathcal{R}(x,y)\frac{g_{kj}(x,y)}{F^2(x,y)}\,y^j.$$

Let $\mathrm{Proj}_y : T_x\mathcal{U} \to T_x\mathcal{U}$ denotes the orthogonal projection on the line $\mathbb{R}y \subset T_xM$, then we have

$$\mathrm{Proj}_y\left(\frac{\partial}{\partial x^k}\right) = \frac{\mathrm{g}_y(\partial_k, y)}{\mathrm{g}_y(y,y)}\cdot y = \frac{g_{kj}(x,y)\,y^j}{F^2(x,y)}\cdot y = \frac{\mathcal{T}_k\,y^i}{\mathcal{R}}\,\frac{\partial}{\partial x^i},$$

and finally

$$\mathrm{R}_y = \mathcal{R}(x,y)\cdot(\mathrm{Id} - \mathrm{Proj}_y) = \mathcal{R}(x,y)\cdot\mathrm{Proj}_{y^\perp}\,. \qquad \square$$

Remark 10.2. The coefficient $\mathcal{T}_k$ is also expressed as $\mathcal{T}_k = -\frac{\mathcal{R}}{F}\frac{\partial F}{\partial y^k}$. This is equivalent to (10.4) since we have by homogeneity $F\frac{\partial F}{\partial y^k} = g_{ki}\,y^i$.

Corollary 10.3 (Berwald [8]). *The flag curvature of a projectively flat strongly convex Finsler metric F at a point $(x, y) \in TM^0$ is given by*

$$\mathrm{K}(y,\sigma) = \frac{1}{F^2}\Big(P^2 - y^j \frac{\partial P}{\partial x^j}\Big), \tag{10.5}$$

where $P = P(x, y)$ is the projective factor and $\sigma \subset TM$ is an arbitrary 2-plane containing y. The Ricci curvature of a projectively Finsler metric F is given by

$$\mathrm{Ric}(x, y) = (n-1)\Big(P^2 - y^j \frac{\partial P}{\partial x^j}\Big). \tag{10.6}$$

Proof. From the previous proposition, we have, for any $\mathrm{w} \in T_x\mathcal{U}$:

$$\mathrm{R}_y(\mathrm{w}) = \mathcal{R}(x,y)\Big(\mathrm{w} - \frac{\mathrm{g}_y(\mathrm{w},y)}{\mathrm{g}_y(y,y)}y\Big) = \frac{\mathcal{R}(x,y)}{\mathrm{g}_y(y,y)}\big(\mathrm{g}_y(y,y)\mathrm{w} - \mathrm{g}_y(\mathrm{w},y)y\big)$$

where $\mathcal{R}(x,y) = \big(P^2 - y^j \frac{\partial P}{\partial x^j}\big)$. Let $\sigma \subset T_x\mathcal{U}$ be a 2-plane containing y and choose a vector $\mathrm{w} \in T_x\mathcal{U}$ such that $\sigma = \mathrm{span}(y, \mathrm{w})$, then

$$\mathrm{K}(y,\sigma) = \frac{\mathrm{g}_y(\mathrm{R}_y(\mathrm{w}), y)}{\mathrm{g}_y(y,y)\,\mathrm{g}_y(\mathrm{w},\mathrm{w}) - \mathrm{g}_y(\mathrm{w},y)^2} = \frac{\mathcal{R}(x,y)}{\mathrm{g}_y(y,y)} = \frac{\mathcal{R}(x,y)}{F^2(x,y)}.$$

For the Ricci curvature we have

$$\mathrm{Ric}(x,y) = \mathrm{Trace}(\mathrm{R}_y) = \mathcal{R}(x,y)\,\mathrm{Trace}(\mathrm{Proj}_{y^\perp}) = (n-1)\,\mathcal{R}(x,y). \qquad \square$$

Remark 10.4. Observe in particular that the flag curvature of a projectively flat Finsler metric at a point $(x, y) \in TM^0$ depends only on (x, y) and not on the 2-plane $\sigma \in TM^0$. Such metrics are said to be *of scalar flag curvature*, because the flag curvature is given by a scalar function $\mathrm{K}\colon TM^0 \to \mathbb{R}$. In that case we write the flag curvature as

$$\mathrm{K}(x,y) = \mathrm{K}(y,\sigma).$$

If the flag curvature is independent of y, then it is in fact constant:

Proposition 10.5. *Let (M, F) be a smooth manifold with a strongly convex Finsler metric. If the flag curvature $\mathrm{K}(x, y)$ is independent of $y \in T_xM$ for any $x \in M$, then the flag curvature is actually a constant.*

We omit the proof. In dimension ≥ 3 this is a particular case of the Schur Lemma. Berwald gave an independent proof in all dimensions in [8], §9.

In the same spirit, we show how Corollary 10.3 can be used to prove the Beltrami Theorem of Riemannian geometry in dimension at least 3.

Theorem 10.6 (Beltrami). *A connected Riemannian manifold (M, g) is locally projectively flat if and only if it has constant sectional curvature.*

Proof. If $\dim(M) = 2$, then a direct proof of the fact that its curvature is constant is not difficult, see e.g. Busemann [13], p. 85. We can thus assume $\dim(M) \geq 2$. Fix a point p and consider two 2-planes $\sigma, \sigma' \subset T_pM$ and choose non-zero vectors $y \in \sigma$ and $y' \in \sigma'$. Let us also set $\tau = \text{span}\{y, y'\} \subset T_pM$. Because (M, g) is Riemannian, the flag curvature $\mathrm{K}_p(y, \sigma)$ of the flag (y, σ) is independent of the choice of the flagpole $y \in \sigma$, and because M is projectively flat, Corollary 10.3 says that the flag curvature of a flag (y, σ) is independent the 2-plane σ containing y. Therefore,

$$\mathrm{K}_p(\sigma) = \mathrm{K}_p(y, \sigma) = \mathrm{K}(p, y) = \mathrm{K}_p(\tau, y) = \mathrm{K}(p, y') = \mathrm{K}_p(\sigma', y') = \mathrm{K}_p(\sigma').$$

The sectional curvature of (M, g) at a point p is thus independent of the choice of a 2-plane $\sigma \subset T_p$. Because M is connected we then conclude from the Schur Lemma that (M, g) has constant sectional curvature. □

Remark 10.7. For a modern Riemannian proof, see [36], or [52] (Chapter 8, Theorem 4.2) for a proof from the point of view of Cartan geometry. The converse of this theorem is classical. Suppose (M, g) has constant sectional curvature K. Rescaling the metric if necessary, we may assume $K = 0, +1$ or -1. If $K = 0$, then (M, g) is locally isometric to the Euclidean space $\mathbb{R}^n$, which is flat. If $K = +1$, then (M, g) is locally isometric to the standard sphere S^n, which is projectively flat by Example 7.2 c. And if $K = -1$, then (M, g) is locally isometric to the hyperbolic space $\mathbb{H}^n$, which is isometric to the unit ball $\mathbb{B}^n$ with its Klein metric (4.1).

11 The flag curvature of the Funk and the Hilbert geometries

The flag curvature of the Hilbert metric was computed in 1929 by Funk in dimension 2 and by Berwald in all dimensions, see [8], [24]. In 1983, T. Okada proposed a more direct computation [41], and in 1995, D. Egloff related these curvatures to the Reeb field of the Finsler manifold [21], [22].

The original Funk–Berwald computation is based on the following important relation between the flag curvature and the exponential map of a projectively flat Finsler manifold:

Proposition 11.1. *Let $F : T\mathcal{U} \to \mathbb{R}$ be a strongly convex projectively flat Finsler metric on the convex domain $\mathcal{U}$ of $\mathbb{R}^n$. Then for any geodesic $\beta(s) = p + \varphi(s)\xi$, we have*

$$\dot{\varphi}(s)^2 \cdot \mathcal{R}(\beta(s), \xi) = \tfrac{1}{2}\{\varphi(s), s\}|_{s=0}$$

where $\{\varphi(s), s\}$ is the Schwarzian derivative[4] of φ with respect to s.

[4] See Appendix B.

Proof. Let us write $x(s) = \beta(s) = p + \varphi(s)\xi$ and $y(s) = \dot{x}(s) = \dot{\varphi}(s)\xi$. The geodesic equation (6.9) implies

$$\dot{y}^k + 2G^k(x,y) = \dot{y}^k + 2P^k(x,y)y^k = (\ddot{\varphi}(s) + 2\dot{\varphi}(s)P(x,y))\xi^k = 0$$

for $k = 1, 2, \dots, n$. We thus have

$$\frac{\ddot{\varphi}(s)}{\dot{\varphi}(s)} = -2P(x,y),$$

and therefore

$$\frac{d}{ds}\left(\frac{\ddot{\varphi}}{\dot{\varphi}}\right) = -2\frac{d}{ds}P(x,y) = -2\frac{\partial P}{\partial x^k}y^k - 2\frac{\partial P}{\partial y^k}\dot{y}^k.$$

Since $y^k = \dot{x}^k = \dot{\varphi}(s)\xi^k$ and $\dot{y}^k = \ddot{\varphi}(s)\xi^k$, we have

$$\dot{y}^k = \frac{\ddot{\varphi}}{\dot{\varphi}} \cdot y^k = -2P(x,y)\, y^k$$

and

$$\frac{d}{ds}\left(\frac{\ddot{\varphi}}{\dot{\varphi}}\right) = -2\frac{\partial P}{\partial x^k}\dot{x}^k + 4P\,\frac{\partial P}{\partial y^k}y^k = -2\frac{\partial P}{\partial x^k}y^k + 4P^2,$$

because $P(x,y)$ is homogenous of degree 1 in y. We thus obtain

$$\{\varphi, s\} = \frac{d}{ds}\left(\frac{\ddot{\varphi}}{\dot{\varphi}}\right) - \frac{1}{2}\left(\frac{\ddot{\varphi}}{\dot{\varphi}}\right)^2 = 2\left(P^2 - \frac{\partial P}{\partial x^k}y^k\right) = 2\,\mathcal{R}(x,y) = 2\dot{\varphi}(s)^2\,\mathcal{R}(x,\xi).$$

□

A first consequence of this proposition is the following local characterization of reversible Minkowski metrics:

Corollary 11.2. *Let $F(x,y)$ be a strongly convex reversible Finsler metric. Then F is locally Minkowski if and only if it is projectively flat with flag curvature* $\mathrm{K} = 0$.

Proof. The geodesics of a (strongly convex) Minkowski metric F are the affinely parametrized straight lines $\beta(s) = p + (as + b)\xi$. Therefore, F is projectively flat and its flag curvature vanishes since

$$\{as + b, s\} = 0.$$

Conversely, suppose that $F(x,y)$ is a strongly convex projectively flat reversible Finsler metric with flag curvature $\mathrm{K} = 0$. The geodesics are then of the type $\beta(s) = p + \varphi(s)\xi$, with $\{\varphi(s), s\} = 0$. Using Lemma B.2, one obtains

$$\varphi(s) = \frac{As + B}{Cs + D} \quad (AD - BC \neq 0).$$

Assuming the initial conditions $\beta(0) = p$ and $\dot{\beta}(0) = \xi$, we get $B = 0$ and $A = D$. Since F is reversible, $\beta^{-}(s) = p - \varphi(s)\xi$ is also a geodesic and it has the same speed as β, therefore $\varphi(-s) = -\varphi(s)$ and thus

$$\varphi(s) = \frac{As}{Cs + A} = -\varphi(-s) = \frac{As}{-Cs + A}.$$

This implies $C = 0$, therefore $\varphi(s) = s$ and the projective factor $P(p, \xi) = 0$. We conclude form Corollary 7.9 that F is locally Minkowski. □

This result is due to Berwald, who gave a different proof. Berwald also gave a counterexample in the non-reversible case, see [8], §7, and [55].

Another consequence of the proposition is the following calculation of the flag curvature of the Funk and Hilbert geometries:

Corollary 11.3. *The flag curvature of the Funk metric in a strongly convex domain $\mathcal{U} \subset \mathbb{R}^n$ is constant equal to $-\frac{1}{4}$ and the flag curvature of the Hilbert metric in $\mathcal{U}$ is constant equal to -1.*

Proof. These geometries are projectively flat with geodesics $\beta(s) = p + \varphi(s)\xi$ and from the previous proposition, we know that

$$\mathrm{K}(p, \xi) = \frac{\mathcal{R}(p, \xi)}{F^2(p, \xi)} = \frac{1}{2} \frac{\{\varphi(s), s\}|_{s=0}}{\dot{\varphi}(0)^2 F^2(p, \xi)}.$$

For the Funk metric, the function $\varphi(s)$ is given by (3.10) and we have $\{\varphi, s\} = -\frac{1}{2}$ and $\dot{\varphi}(0)F(p, \xi) = 1$, therefore $\mathrm{K} = -\frac{1}{4}$. For the Hilbert metric, the function $\varphi(s)$ is given by (4.5). By a calculation, $\{\varphi(s), s\} = -2$ and $\dot{\varphi}(0)F(p, \xi) = 1$, and we thus obtain $\mathrm{K} = -1$. □

Remark 11.4. Following Okada [41], we can also compute these curvatures directly from Corollary 10.3 and Equation (8.1). For the Funk metric, we have $P(x, y) = \frac{1}{2}F_f(x, y)$, therefore equation (8.1), gives

$$\frac{\partial P}{\partial x^j} y^j = 2P \frac{\partial P}{\partial y^j} y^j = 2P^2,$$

and we thus have $\mathrm{K} = \frac{1}{4P^2}\left(P^2 - \frac{\partial P}{\partial x^j} y^j\right) = -\frac{1}{4}$. For the Hilbert metric $F_h = \frac{1}{2}(F_f + F_f^*)$, the projective factor is $P(x, y) = \frac{1}{2}(F_f - F_f^*)$, therefore

$$\frac{\partial P}{\partial x^j} = \frac{1}{2}\left(\frac{\partial F_f}{\partial x^j} - \frac{\partial F_f^*}{\partial x^j}\right) = \frac{1}{2}\left(F_f \frac{\partial F_f}{\partial y^j} + F_f^* \frac{\partial F_f^*}{\partial y^j}\right),$$

and thus

$$\frac{\partial P}{\partial x^j} y^j = \frac{1}{2} F_f F_h = \frac{1}{2}(F_f^2 + F_f^{*2}).$$

It follows that

$$\begin{aligned} P^2 - \frac{\partial P}{\partial x^j} y^j &= \frac{1}{4}(F_f - F_f^*)^2 - \frac{1}{2}(F_f^2 + F_f^{*2}) \\ &= -\frac{1}{4}(F_f + F_f^*)^2 \\ &= -F_h^2. \end{aligned}$$

And we conclude that $\mathrm{K} = \frac{1}{F_h^2}\big(P^2 - \frac{\partial P}{\partial x^j} y^j\big) = -1$.

Remark 11.5. The previous corollary provides us also with the following values for the Ricci curvature: $\mathrm{Ric}(x,y) = -\frac{n-1}{4} F^2(x,y)$ for the Funk metric and $\mathrm{Ric}(x,y) = -(n-1)F^2(x,y)$ for the Hilbert metric. The recent paper [40] by S. I. Ohta computes the more general *weighted Ricci curvatures* for these metrics.

This curvature computation allows us to provide a simple proof of the following important result which is due to I. J. Schoenberg and D. Kay, see Corollary 4.6 of [27].

Theorem 11.6. *The Hilbert metric F_h in a bounded domain $\mathcal{U} \subset \mathbb{R}^n$ with smooth strongly convex boundary is Riemannian if and only if $\mathcal{U}$ is an ellipsoid.*

Proof. If $\mathcal{U}$ is an ellipsoid, then it is affinely equivalent to the unit ball $\mathbb{B}^n$, therefore F_h is equivalent to the Klein metric (4.2), which is Riemannian. Suppose conversely that F_h is Riemannian, that is, $F_h = \sqrt{\mathrm{g}}$ for some Riemannian metric in $\mathcal{U}$. Then $(\mathcal{U}, \mathrm{g})$ is a complete, simply-connected Riemannian manifold with constant sectional curvature $\mathrm{K} = -1$, therefore $(\mathcal{U}, \mathrm{g})$ is isometric to the hyperbolic space $\mathbb{B}^n$ with its Klein metric and it follows that $\mathcal{U}$ is an ellipsoid. □

Corollary 11.7. *Let $\mathcal{U} \subset \mathbb{R}^n$ be a bounded smooth strongly convex domain. Assume that there exists a discrete group Γ of projective transformations leaving $\mathcal{U}$ invariant and acting freely with compact quotient $M = \mathcal{U}/\Gamma$ (the convex domain $\mathcal{U}$ is then said to be* ***divisible****). Then $\mathcal{U}$ is an ellipsoid.*

Proof. The Hilbert metric induces a smooth and strongly convex Finsler metric F on the quotient $M = \mathcal{U}/\Gamma$. The compact Finsler manifold (M, F) has constant negative flag curvature, it is therefore Riemannian by Theorem A.2. It now follows from the previous theorem that the universal cover $\mathcal{U} = \widetilde{M}$ is an ellipsoid. □

This corollary is a special case of a result of Benzécri [10]. There are several generalizations of this result and divisible convex domains have been the subject of intensive research in recent years, see [28], Section 7, and [34] for a discussion.

12 The Funk–Berwald characterization of Hilbert geometries

We are now in a position to prove our characterization theorem for Hilbert metrics.

Theorem 12.1. *Let F be a strongly convex Finsler metric on a bounded convex domain $\mathcal{U} \subset \mathbb{R}^n$. Then F is the Hilbert metric of that domain if and only if F is projectively flat, complete and has constant flag curvature* $\mathrm{K} = -1$.

Proof. Since the Hilbert metric in a bounded convex domain $\mathcal{U} \subset \mathbb{R}^n$ is complete with flag curvature $\mathrm{K} = -1$, we can reformulate the theorem as follows: *Let F_1 and F_2 be two strongly convex Finsler metrics with constant flag curvature* $\mathrm{K} = -1$ *on a bounded convex domain $\mathcal{U} \subset \mathbb{R}^n$. Suppose that F_1 and F_2 are complete and projectively flat, then $F_1 = F_2$.*

To prove that statement, let us consider two complete and projectively flat strongly convex Finsler metrics F_1 and F_2 in $\mathcal{U}$ such that

$$\mathrm{K}_{F_1}(x,y) = \mathrm{K}_{F_2}(x,y) = -1$$

for any $(x,y) \in T\mathcal{U}$. Fix a point $p \in \mathcal{U}$ and a non-zero vector $\xi \in T_p\mathcal{U}$, and let

$$\beta_i(s) = p + \varphi_i(s)\xi$$

be the unit speed linear geodesic for the metric F_i starting at p in the direction ξ for $i = 1,2$. In particular we have $\varphi_1(0) = \varphi_2(0) = 0$ and $\dot{\varphi}_i(s) > 0$.

Since F_i is forward and backward complete, we have

$$\beta_i(\pm\infty) = \lim_{s\to\pm\infty} \beta_i(s) \in \partial\mathcal{U} \quad (i = 1,2).$$

By the definitions of the tautological and reverse tautological Finsler structures, we obtain from Equation (3.10) that

$$\varphi_i(+\infty) = \frac{1}{F_f(p,\xi)} \quad \text{and} \quad \varphi_i(-\infty) = \frac{-1}{F_f^*(p,\xi)} \tag{12.1}$$

for $i = 1,2$. In particular $\varphi_1(+\infty) = \varphi_2(+\infty)$ and $\varphi_1(-\infty) = \varphi_2(-\infty)$.

Because β_1 and β_2 are unit speed geodesics, we have from Proposition 11.1 that

$$\frac{1}{2}\{\varphi_i, s\} = F_i^2(\beta_i(s), \beta_i(s)) \cdot \mathrm{K}_{F_i}(\beta_i(s), \dot{\beta}_i(s)) = -1.$$

In particular $\{\varphi_1, s\} = \{\varphi_2, s\}$. From Lemma B.2, we have therefore

$$\varphi_2(s) = \frac{A\varphi_1(s) + B}{C\varphi_1(s) + D}.$$

Since $\varphi_1(0) = \varphi_2(0) = 0$ we have $B = 0$; and Equation (12.1) implies the relations

$$A - D = \frac{C}{F_f(p,\xi)} = \frac{-C}{F_f^*(p,\xi)}.$$

It follows that $C = 0$ and $A = D$ and therefore $\varphi_1(s) = \varphi_2(s)$ for every $s \in \mathbb{R}$, which finally implies that $F_1(p,\xi) = F_2(p,\xi)$ for all $(p,\xi) \in T\mathcal{U}$. □

Appendices

A On Finsler metrics with constant flag curvature

The previous discussion suggests the following program: *Describe all Finsler metrics with constant flag curvature.*

This program is quite broad and it is not clear whether a complete answer will be at hand in the near future, but many examples and partial classifications have been obtained. Let us only mention a few classical results. For the flat case, we have the following:

Theorem A.1. *Let (M, F) be a smooth manifold with a strongly convex Finsler metric of constant flag curvature* $\mathrm{K} = 0$. *Suppose that either*

i) *F is reversible and locally projectively flat, or*

ii) *F is Randers, or*

iii) *F is Berwald, or*

iv) *M is compact.*

Then (M, F) is flat, that is, it is locally isometric to a Minkowski space.

The first case is Corollary 11.2. The second case is due to Shen [59], while the case of a Berwald metric is classical (see Theorem 2.3.2 in [15]). Finally the compact case is due to Akbar-Zadeh [2].

Another notable result is

Theorem A.2 (Akbar-Zadeh [2]). *On a smooth compact manifold, every strongly convex Finsler metric with negative flag curvature is a Riemannian metric.*

A proof is also given in [53], p. 162. For positive constant curvature, we have the following recent result by Kim and Min [32]:

Theorem A.3. *Any strongly convex reversible Finsler metric with positive constant flag curvature is Riemannian.*

In the case of projectively flat reversible Finsler structures on the 2-sphere, the result is due to Bryant [11], who in fact classified all projectively flat Finsler structures on the 2-sphere with flag curvature $\mathrm{K} = +1$.

Non-reversible projectively flat Randers metrics with constant curvature have been classified by Shen in [59]. For further examples and discussions, we refer to Chapter 8 of [15], §11.2 of [53], Chapter 9 of [54], [58], [38] and the recent survey [27] by Guo, Mo and Wang.

B On the Schwarzian derivative

The Schwarzian derivative is a third order non-linear differential operator that is invariant under the group of one-dimensional projective transformations. It appeared first in complex analysis in the context of the Schwarz–Christoffel transformations formulae. It is defined as follows.

Definition B.1. Let φ be a C^3 function of the real variable s, or a holomorphic function of the complex variable s. Its *Schwarzian derivative* is defined as

$$\{\varphi(s), s\} = \frac{\dddot{\varphi}(s)}{\dot{\varphi}(s)} - \frac{3}{2}\left(\frac{\ddot{\varphi}(s)}{\dot{\varphi}(s)}\right)^2 = \frac{d}{ds}\left(\frac{\ddot{\varphi}(s)}{\dot{\varphi}(s)}\right) - \frac{1}{2}\left(\frac{\ddot{\varphi}(s)}{\dot{\varphi}(s)}\right)^2,$$

where the dots represent differentiation with respect to s.

For instance $\{e^{\lambda s}, s\} = -\frac{1}{2}\lambda^2$, $\{\frac{1}{s}, s\} = 0$ and $\{\tan(\lambda s), s\} = 2\lambda^2$. The fundamental property of the Schwarzian derivative is contained in the following

Lemma B.2. (i) *Let $u(s)$ and $v(s)$ be two linearly independent solutions to the equation*

$$\ddot{w}(s) + \frac{1}{2}\rho(s)\, w(s) = 0. \tag{B.1}$$

If $v \neq 0$, then $\varphi(s) = u(s)/v(s)$ is a solution to

$$\{\varphi(s), s\} = \rho(s).$$

(ii) *Let $\varphi_1(s)$ and $\varphi_2(s)$ be two non-singular functions of s. Then*

$$\{\varphi_1(s), s\} = \{\varphi_2(s), s\}$$

if and only if there exist constants $A, B, C, D \in \mathbb{R}$ with $AD - BC \neq 0$ such that:

$$\varphi_2 = \frac{A\varphi_1 + B}{C\varphi_1 + D}.$$

Proof. The proof of (i) follows from a direct calculation. To prove (ii), we first observe that $\{-\varphi, s\} = \{\varphi, s\}$ and we may therefore assume $\dot{\varphi}_i > 0$. Let us set

$$u_i(s) = \frac{\varphi_i(s)}{\sqrt{\dot{\varphi}_i(s)}}, \quad v_i(s) = \frac{1}{\sqrt{\dot{\varphi}_i(s)}},$$

for $i = 1, 2$. A calculation shows that u_1, v_1 and u_2, v_2 are two fundamental systems of solutions to the equation (B.1), where $\rho(s) = \{\varphi_1, s\} = \{\varphi_2, s\}$. Since the solutions to that equation form a two-dimensional vector space, we have

$$u_2 = Au_1 + Bv_1, \quad v_2 = Cu_1 + Dv_1 \quad (AD - BC \neq 0),$$

and therefore

$$\varphi_2 = \frac{u_2}{v_2} = \frac{Au_1 + Bv_1}{Cu_1 + Dv_1} = \frac{A\varphi_1 + B}{C\varphi_1 + D}. \qquad \square$$

Example B.3. Using this lemma, one immediately gets that the general solution to the equation

$$\{\varphi(s), s\} = 2\lambda,$$

where $\lambda \in \mathbb{R}$ is a constant, is given respectively by

$$\frac{Ae^{\sqrt{-\lambda}\cdot s} + Be^{-\sqrt{-\lambda}\cdot s}}{Ce^{\sqrt{-\lambda}\cdot s} + De^{-\sqrt{-\lambda}\cdot s}}, \quad \frac{As + B}{Cs + D}, \quad \text{and} \quad \frac{A\sin(\sqrt{\lambda}\cdot s) + B\cos(\sqrt{\lambda}s)}{C\sin(\sqrt{\lambda}\cdot s) + D\cos(\sqrt{\lambda}\cdot s)},$$

depending on $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

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