

Chapter 2

From Funk to Hilbert geometry

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1 Introduction

The Funk metric associated to an open convex subset of a Euclidean space is a *weak* metric in the sense that it does not satisfy all the axioms of a metric: it is not symmetric, and we shall also allow the distance between two points to be zero; see Chapter 1 in this volume [19], where such metrics are introduced. Weak metrics often occur in the calculus of variations and in Finsler geometry and the study of such metrics has been revived recently in low-dimensional topology and geometry by Thurston who introduced an asymmetric metric on Teichmüller space, which became the subject of intense research. In this chapter, we shall sometimes use the expression “metric” instead of “weak metric” in order to simplify.

The Funk weak metric $F_\Omega(x, y)$ is associated to an open convex subset Ω of a Euclidean space \mathbb{R}^n . It is an important metric for the subject treated in this Handbook, because the Hilbert metric H_Ω of the convex set Ω is the *arithmetic symmetrization* of its Funk metric. More precisely, for any x and y in Ω , we have

$$H_\Omega(x, y) = \frac{1}{2} (F_\Omega(x, y) + F_\Omega(y, x)).$$

Independently of its relation with the Hilbert metric, the Funk metric is a nice example of a weak metric, and there are many natural questions for a given convex subset Ω of \mathbb{R}^n that one can ask and solve for such a metric, regarding its geodesics, its balls, its isometries, its boundary structure, and so on. Of course, the answer depends on the shape of the convex set Ω , and it is an interesting aspect of the theory to study the influence of the properties of the boundary $\partial\Omega$ (its degree of smoothness, the fact that it is a polyhedron, a strictly convex hypersurface, etc.) on the Funk geometry of this set. The same questions can be asked for the Hilbert geometry, and they are addressed in several chapters of this volume. The Funk geometry is studied in Busemann's book *The geometry of geodesics* [2], although the name "Funk metric" is not used there, but it is used by Busemann in his later papers and books, see e.g. [3], and in the memoir [28] by Zaustinsky (who was a student of Busemann).

We studied some aspects of this metric in [17], following Busemann's ideas. But a systematic study of this metric is something which seems to be still missing in the literature, and the aim of this chapter is somehow to fill this gap.

Euclidean segments in Ω are geodesics for the Funk metric, and in the case where the domain Ω is strictly convex (that is, if there is no nonempty open Euclidean segment in $\partial\Omega$), the Euclidean segments are the unique geodesic segments. For a metric d on a subset of Euclidean space, the property of having the Euclidean segments d -geodesics is the subject of Hilbert's Problem IV (see Chapter 15 in this volume [16]). One version of this problem asks for a characterization of non-symmetric metrics on subsets of \mathbb{R}^n for which the Euclidean segments are geodesics.

The Funk metric is also a basic example of a Finsler structure, perhaps one of the most basic. In the paper [17], we introduced the notions of *tautological* and *reversible tautological* Finsler structures of the domain Ω . The Funk metric F_Ω is the length metric induced by the tautological weak Finsler structure, and the Hilbert metric H_Ω is the length metric induced by the *reversible tautological* Finsler structure of the domain Ω . This is in fact how Funk introduced his metric in 1929, see [9], [24]. The reversible Finsler structure is obtained by the process of *harmonic symmetrization* at the level of the convex sets in the tangent spaces that define these structures, cf. [18]. This gives another relation between the Hilbert and the Funk metrics. The Finsler geometry of the Funk metric is studied in some detail in Chapter 3 [24] of this volume. A useful variational description of the Funk metric is studied in [27]. There are also interesting non-Euclidean versions of Funk geometry, see Chapter 13 in this volume ([20]).

In the present chapter, we start by recalling the definition of the Funk metric, and we give several basic properties of this metric, some of which are new, or at least formulated in a new way. In Section 3, we introduce what we call the *reverse Funk metric*, that is, the metric ${}^rF_\Omega$ defined by ${}^rF_\Omega(x, y) = F(y, x)$. This metric is much less studied than the Funk metric. In Section 4, we give the formula for the Funk metric for the two main examples, namely, the cases of convex polytopes and the Euclidean unit ball. In Section 5, we study the geometry of balls in the Funk metric. Since the metric is non-symmetric, one has to distinguish between *forward* and *backward* balls. We show that any forward ball is the image of Ω by a Euclidean homothety. This gives a rigidity property, namely, that the local geometry of the convex set determines the convex set up to a scalar factor. In Section 6, we show that the topologies defined by the Funk and the reverse Funk metrics coincide with the Euclidean topology. In Section 7, we give a proof of the triangle inequality for the Funk metric and at the same time we study its geodesics and its convexity properties. In Section 8, we study the property of the nearest point projections of a point in Ω on a convex subset of Ω equipped with the Funk metric, and the perpendicularity properties. In the case where Ω is strictly convex, the nearest point projection is unique. There is a formulation of perpendicularity to hyperplanes in Ω in terms of properties of support hyperplanes for Ω . In Section 9, we study the infinitesimal (Finsler) structure associated to a Funk metric. In Section 10, we study the isometries of a Funk metric. In the case where Ω is bounded and strictly convex, its isometry group coincides with the subgroup of affine transformations of \mathbb{R}^n that leave Ω invariant. In Section 11, we propose a projective viewpoint and a generalization of the Funk metric and in Section 12, we present some basic properties of the Hilbert metric which are direct consequences of the fact that it is a symmetrization of the Funk metric. In the last section we give some new perspectives on the Funk metric, some of which are treated in other chapters of this volume. Appendix A contains two classical theorems of Euclidean geometry (the theorems of Menelaus and Ceva). Menelaus' Theorem is used in Appendix B to give the classical proof of the triangle inequality for the Funk metric.

2 The Funk metric

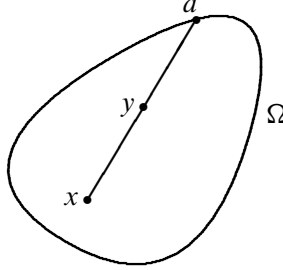
In this section, Ω is a proper convex domain in \mathbb{R}^n , that is, Ω is convex, open, non-empty, and $\Omega \neq \mathbb{R}^n$. We denote by $\bar{\Omega}$ the closure of Ω in \mathbb{R}^n and by $\partial\Omega = \bar{\Omega} \setminus \Omega$ the topological boundary of Ω .

In the case of an unbounded domain, it will be convenient to add points at infinity to the boundary $\partial\Omega$. To do so, we consider \mathbb{R}^n as an affine space in the projective space $\mathbb{R}\mathbb{P}^n$ and we denote by $H_\infty = \mathbb{R}\mathbb{P}^n \setminus \mathbb{R}^n$ the hyperplane at infinity. We then denote by $\tilde{\Omega}$ the closure of Ω in $\mathbb{R}\mathbb{P}^n$ and by $\tilde{\partial}\Omega = \tilde{\Omega} \setminus \Omega$. Observe that $\partial\Omega = \tilde{\partial}\Omega \setminus H_\infty$.

For any two points $x \neq y$ in \mathbb{R}^n , we denote by $[x, y]$ the closed affine segment joining them. We also denote by $R(x, y)$ the affine ray starting at x and passing

through y and by $\tilde{R}(x, y)$ its closure in $\mathbb{R}\mathbb{P}^n$. Finally we set

$$a_\Omega(x, y) = \tilde{R}(x, y) \cap \tilde{\partial}\Omega \in \mathbb{R}\mathbb{P}^n.$$



We now define the Funk metric.

Definition 2.1 (The Funk metric). The *Funk metric* on Ω , denoted by F_Ω , is defined for x and y in Ω , by $F_\Omega(x, x) = 0$ and by

$$F_\Omega(x, y) = \log \left(\frac{|x - a|}{|y - a|} \right)$$

if $x \neq y$, where $a = a_\Omega(x, y) \in \mathbb{R}\mathbb{P}^n$. Here $|p - q|$ is the Euclidean distance between the points q and p in \mathbb{R}^n . It is understood in this formula that if $a \in H_\infty$, then $F_\Omega(x, y) = 0$. This is consistent with the convention $\infty/\infty = 1$.

Let us begin with a few basic properties of the Funk metric.

Proposition 2.2. *The Funk metric in a convex domain $\Omega \neq \mathbb{R}^n$ satisfies the following properties:*

- (a) $F_\Omega(x, y) \geq 0$ and $F_\Omega(x, x) = 0$ for all $x, y \in \Omega$.
- (b) $F_\Omega(x, z) \leq F_\Omega(x, y) + F_\Omega(y, z)$ for all $x, y, z \in \Omega$.
- (c) F_Ω is projective, that is, $F_\Omega(x, z) = F_\Omega(x, y) + F_\Omega(y, z)$ whenever z is a point on the affine segment $[x, y]$.
- (d) The weak metric F_Ω is non symmetric, that is, $F_\Omega(x, y) \neq F_\Omega(y, x)$ in general.
- (e) The weak metric F_Ω is separating, that is, $x \neq y \implies F_\Omega(x, y) > 0$, if and only if the domain Ω is bounded.
- (f) The weak metric F_Ω is unbounded.

Property (a) and (b) say that F_Ω is a *weak metric*.

Proof. Property (a) follows from the fact that $y \in [x, a_\Omega(x, y)]$, therefore

$$\frac{|x - a_\Omega(x, y)|}{|y - a_\Omega(x, y)|} \geq 1$$

and we have equality if $y = x$. The triangle inequality (b) is not completely obvious. The classical proof by Hilbert (who wrote it for the Hilbert metric) is given in Appendix B and a new proof is given in Section 7.

To prove (c), observe that if $y \in [x, z]$ and $x \neq y \neq z$, then $a_\Omega(x, y) = a_\Omega(x, z) = a_\Omega(y, z)$. Denoting this common point by a , we have

$$F_\Omega(x, y) + F_\Omega(y, z) = \log \frac{|x-a|}{|y-a|} + \log \frac{|y-a|}{|z-a|} = \log \frac{|x-a|}{|z-a|} = F_\Omega(x, z).$$

Property (d) is easy to check, see Section 3 for more details.

Property (e) follows immediately from the definition and the fact that a convex domain Ω in \mathbb{R}^n is unbounded if and only if it contains a ray, see [19], Remark 3.13.

To prove (f), we recall that we always assume $\Omega \neq \mathbb{R}^n$, and therefore $\partial\Omega \neq \emptyset$. Let x be a point in Ω and a a point in $\partial\Omega$ and consider the open Euclidean segment (x, a) contained in Ω . For any sequence x_n in this segment converging to a (with respect to the Euclidean metric), we have $F_\Omega(x, x_n) = \log \frac{|x-a|}{|x_n-a|} \rightarrow \infty$ as $n \rightarrow \infty$. \square

It is sometimes useful to see the Funk metric from other viewpoints. If x, y and z are three aligned points in \mathbb{R}^n with $z \neq y$, then there is a unique $\lambda \in \mathbb{R}$ such that $x = z + \lambda(y - z)$. We call this number the *division ratio* or *affine ratio* of x with respect to z and y and denote it suggestively by $\lambda = zx/zy$. Note also that $\lambda > 1$ if and only if $y \in [x, z]$. We extend the notion of affine ratio to the case $z \in H_\infty$ by setting $zx/zy = 1$ if $z \in H_\infty$. We then have

$$F_\Omega(x, y) = \log(\lambda),$$

where λ is the affine ratio of x with respect to $a = a_\Omega(x, y)$ and y .

Proposition 2.3. *Let x and y be two points in the convex domain Ω and set $a = a_\Omega(x, y)$. Let $h: \mathbb{R}^n \rightarrow \mathbb{R}$ be an arbitrary linear form such that $h(y) \neq h(x)$. Then*

$$F_\Omega(x, y) = \log \left(\frac{h(a) - h(x)}{h(a) - h(y)} \right).$$

Proof. We first observe that a linear function $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is either constant on a given ray $R(x, y)$, or it is injective on that ray. If $a = a_\Omega(x, y) \in H_\infty$, then

$$h(a) = \lim_{t \rightarrow \infty} h(x + t(y - x)) = h(x) + t(h(y) - h(x)) = \pm\infty,$$

and the proposition is true since $\log(\infty/\infty) = \log(1) = 0 = F_\Omega(x, y)$. If $a \notin H_\infty$, then the division ratio $\lambda = ax/ay > 1$ and we have $(a - x) = \lambda(a - y)$ and $F_\Omega(x, y) = \log(\lambda)$. Since $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is linear, then $(h(a) - h(x)) = \lambda(h(a) - h(y))$ and the proposition follows at once. \square

We now recall some important notions from convex geometry. For these and other classical results on convex geometry, the reader can consult the books by Eggleston [7], Fenchel [8], Valentine [26] and Rockafellar [22].

Definition 2.4. Let $\Omega \subset \mathbb{R}^n$ be a proper convex domain and $a \in \partial\Omega$ be a (finite) boundary point. We assume that Ω contains the origin 0. A *supporting functional* at a for Ω is a linear function $h: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $h(a) = 1$ and $h(x) < 1$ for any point x in Ω ¹. The hyperplane $H = \{z \mid h(z) = 1\}$ is said to be a *support hyperplane* at a for Ω . If Ω is unbounded, then the hyperplane at infinity H_∞ is also considered to be a support hyperplane. A basic fact is that any boundary point of a convex domain admits one or several supporting functional(s).

Let us denote by \mathcal{S}_Ω the set of all supporting functionals of Ω . Then

$$p_\Omega(x) = \sup\{h(x) \mid h \in \mathcal{S}_\Omega\}$$

is the *Minkowski functional* of Ω . This is the unique weak Minkowski norm such that

$$\Omega = \{x \in \mathbb{R}^n \mid p_\Omega(x) < 1\}.$$

We have the following consequences of the previous proposition:

Corollary 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain containing the origin and let h be a supporting functional for Ω . We then have*

$$F_\Omega(x, y) \geq \log \left(\frac{1 - h(x)}{1 - h(y)} \right)$$

for any $x, y \in \Omega$. Furthermore we have equality if and only if either $a = a_\Omega(x, y) \in H_\infty$ and $h(x) = h(y)$, or $a \notin H_\infty$ and $h(a) = 1$.

Proof. If $h(x) = h(y)$ there is nothing to prove, we thus assume that $h(x) \neq h(y)$. Note that this condition means that the line through x and y is not parallel to the supporting hyperplane $H = \{h = 1\}$.

We first assume that $a = a_\Omega(x, y) \notin H_\infty$. The hypothesis $h(x) \neq h(y)$ implies $h(a) \neq h(y)$. Because the function $a \mapsto \log \frac{|x-a|}{|y-a|}$ is strictly monotone decreasing and $h(a) \leq 1$, we have by Proposition 2.3:

$$F_\Omega(x, y) = \log \left(\frac{h(a) - h(x)}{h(a) - h(y)} \right) \geq \log \left(\frac{1 - h(x)}{1 - h(y)} \right).$$

The equality holds if and only if $h(a) = 1$.

Suppose now that $a \in H_\infty$; this means that $F_\Omega(x, y) = 0$ and the ray $R(x, y)$ is contained in Ω . In particular, we have

$$h(x) + \lambda h(y - x) = h(x + \lambda \cdot (y - x)) < 1 \quad \text{for all } \lambda \geq 0,$$

which implies $h(y) - h(x) = h(y - x) \leq 0$. Since we assumed $h(x) \neq h(y)$, we have $h(y) < h(x)$ and therefore

$$F_\Omega(x, y) = 0 > \log \left(\frac{1 - h(x)}{1 - h(y)} \right). \quad \square$$

¹The reader should not mistake the notion of supporting functional for that of *support function* $\mathfrak{h}_\Omega: \mathbb{R}^n \rightarrow \mathbb{R}$ defined as $\mathfrak{h}_\Omega(x) = \sup\{h(x) \mid h \in \mathcal{S}_\Omega\}$.

Corollary 2.6. *If $\Omega \subset \mathbb{R}^n$ is a convex domain containing the origin, then*

$$F_{\Omega}(x, y) = \max \left\{ 0, \sup_{h \in \mathcal{S}_{\Omega}} \log \left(\frac{1-h(x)}{1-h(y)} \right) \right\}.$$

Proof. We first assume that $F_{\Omega}(x, y) > 0$. Then $a = a_{\Omega}(x, y) \notin H_{\infty}$. Let us denote by $\Phi(x, y)$ the right hand side in the above formula. Then Proposition 2.3 implies that $F_{\Omega}(x, y) \leq \Phi(x, y)$ and Corollary 2.5 implies the converse inequality.

If $F_{\Omega}(x, y) = 0$, then $R(x, y) < 0$. For any supporting function h , we then have $h(x + t(y - x)) = h(x) + t(h(y) - h(x)) < 1$ for any $t > 0$. This implies that $h(y) \leq h(x)$ and it follows that $\Phi(x, y) = 0$. \square

Notice that for bounded domains, the previous formula reduces to

$$F_{\Omega}(x, y) = \sup_{h \in \mathcal{S}_{\Omega}} \log \left(\frac{1-h(x)}{1-h(y)} \right).$$

This can also be reformulated as follows (compare to Theorem 1 of [27]):

Corollary 2.7. *The Funk metric in a bounded convex domain $\Omega \subset \mathbb{R}^n$ is given by*

$$F_{\Omega}(x, y) = \sup_H \log \left(\frac{\text{dist}(x, H)}{\text{dist}(y, H)} \right),$$

where the supremum is taken over the set of all support hyperplanes H for Ω and $\text{dist}(x, H)$ is the Euclidean distance from x to H .

The next consequence of Proposition 2.3 is the following relation between the division ratio of three aligned points in a convex domain and the Funk distances between them.

Corollary 2.8. *Let x, y and z be three aligned points in the convex domain $\Omega \subset \mathbb{R}^n$. Suppose that $F_{\Omega}(x, y) > 0$ and $z = x + t(y - x)$ for some $t \geq 0$. Then*

$$t = \frac{e^{F_{\Omega}(x, y)} \cdot (e^{F_{\Omega}(x, z)} - 1)}{e^{F_{\Omega}(x, z)} \cdot (e^{F_{\Omega}(x, y)} - 1)}, \quad (2.1)$$

$$F_{\Omega}(x, z) = F_{\Omega}(x, y) - \log \left(e^{F_{\Omega}(x, y)} + t \cdot (1 - e^{F_{\Omega}(x, y)}) \right). \quad (2.2)$$

Proof. Choose a supporting functional h for Ω at the point $a = a_{\Omega}(x, y) = a_{\Omega}(x, z)$. Then we have, from Corollary 2.5:

$$e^{F_{\Omega}(x, y)} = \frac{1-h(x)}{1-h(y)} \quad \text{and} \quad e^{F_{\Omega}(x, z)} = \frac{1-h(x)}{1-h(z)}.$$

Therefore

$$\frac{e^{F_{\Omega}(x, z)} - 1}{e^{F_{\Omega}(x, z)}} = \left(\frac{1-h(z)}{1-h(x)} \right) \left(\frac{1-h(x)}{1-h(z)} - 1 \right) = \frac{h(z) - h(x)}{1-h(x)},$$

and likewise

$$\frac{e^{F_{\Omega}(x,y)} - 1}{e^{F_{\Omega}(x,y)}} = \frac{h(y) - h(x)}{1 - h(x)}.$$

Thus

$$\frac{e^{F_{\Omega}(x,y)} \cdot (e^{F_{\Omega}(x,z)} - 1)}{e^{F_{\Omega}(x,z)} \cdot (e^{F_{\Omega}(x,y)} - 1)} = \frac{h(z) - h(x)}{h(y) - h(x)} = \frac{h(z - x)}{h(y - x)} = t.$$

This proves Equation (2.1). To prove the equation (2.2), we now resolve

$$\frac{e^{F_{\Omega}(x,z)} - 1}{e^{F_{\Omega}(x,z)}} = t \cdot \frac{e^{F_{\Omega}(x,y)} - 1}{e^{F_{\Omega}(x,y)}}$$

for $e^{F_{\Omega}(x,z)}$. This gives us

$$e^{F_{\Omega}(x,z)} = \frac{e^{F_{\Omega}(x,y)}}{e^{F_{\Omega}(x,y)} + t(1 - e^{F_{\Omega}(x,y)})},$$

which is equivalent to (2.2). \square

It is useful to observe that computing the Funk distance between two points in Ω is a one-dimensional operation. More precisely, if $S = [a_1, a_2] \subset \mathbb{R}^n$ is a compact segment in \mathbb{R}^n containing the points x and y in its interior with $y \in [x, a_2]$, we shall write

$$F_S(x, y) = \log \frac{|x - a_2|}{|y - a_2|}.$$

Although S is not an open set, $F_S(x, y)$ clearly corresponds to the one-dimensional Funk metric in the relative interior of S .

Proposition 2.9. *The Funk distance between two points x and y in Ω is given by*

$$F_{\Omega}(x, y) = \inf \{F_S(x, y) \mid S \text{ is a segment in } \Omega \text{ containing } x \text{ and } y\}.$$

Proof. We identify S with a segment in \mathbb{R} with $b < x \leq y < a$ and observe that the function $a \mapsto \log \frac{|x-a|}{|y-a|}$ is strictly monotone decreasing. \square

This result makes an analogy between the Funk metric and the Kobayashi metric in complex geometry, see [13]. It has the following immediate consequences:

Corollary 2.10. (i) *If $\Omega_1 \subset \Omega_2$ are convex subsets of \mathbb{R}^n , then $F_{\Omega_1} \geq F_{\Omega_2}$ with equality if and only if $\Omega_1 = \Omega_2$.*

(ii) *Let Ω_1 and Ω_2 be two open convex subsets of \mathbb{R}^n . Then, for every x and y in $\Omega_1 \cap \Omega_2$, we have $F_{\Omega_1 \cap \Omega_2}(x, y) = \max(F_{\Omega_1}(x, y), F_{\Omega_2}(x, y))$.*

(iii) *Let Ω be a nonempty open convex subset of \mathbb{R}^n , let $\Omega' \subset \Omega$ be the intersection of Ω with an affine subspace of \mathbb{R}^n , and suppose that $\Omega' \neq \emptyset$. Then, $F_{\Omega'}$ is the metric induced by F_{Ω} on Ω' as a subspace of (Ω, F_{Ω}) .*

3 The reverse Funk metric

Definition 3.1. The *reverse Funk metric* in a proper convex domain Ω is defined as

$${}^rF_{\Omega}(x, y) = F_{\Omega}(y, x) = \log \left(\frac{|y - b|}{|x - b|} \right),$$

where $b = a_{\Omega}(y, x)$.

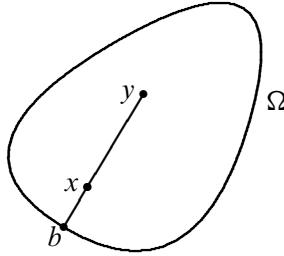


Figure 1. The reverse Funk metric.

Proposition 3.2. *The reverse Funk metric in a convex domain $\Omega \neq \mathbb{R}^n$ is a projective weak metric. It is unbounded and non-symmetric and it is separating if and only if the domain Ω is bounded.*

The proof is a direct consequence of Proposition 2.2. □

An important difference between the Funk metric and the reverse Funk metric is the following:

Proposition 3.3. *Let Ω be a bounded convex domain in \mathbb{R}^n and x be a point in Ω . Then the function $y \rightarrow {}^rF_{\Omega}(x, y)$ is bounded.*

Proof. Define λ_x and δ by

$$\lambda_x = \inf_{b \in \partial\Omega} |x - b|, \quad \text{and} \quad \delta = \sup_{a, b \in \partial\Omega} |a - b|.$$

Observe that δ is the Euclidean diameter of Ω , thus $\delta < \infty$ since Ω is bounded. We also have $\lambda_x > 0$. The proposition follows from the inequality

$${}^rF_{\Omega}(x, y) \leq \log \left(\frac{\delta}{\lambda_x} \right). \quad \square$$

In particular the reverse Funk metric ${}^rF_{\Omega}$ is not bi-Lipschitz equivalent to the Funk metric F_{Ω} .

4 Examples

Example 4.1 (Polytopes). An (open) *convex polytope* in \mathbb{R}^n is defined to be an intersection of finitely many half-spaces:

$$\Omega = \{x \in \mathbb{R}^n \mid \phi_j(x) < s_j, 1 \leq j \leq k\},$$

where $\phi_j: \mathbb{R}^n \rightarrow \mathbb{R}$ is a nontrivial linear form for all j . The Funk distance between two points in such a polytope is given by

$$F_\Omega(x, y) = \max \left\{ 0, \max_{1 \leq j \leq k} \log \left(\frac{s_j - \phi_j(x)}{s_j - \phi_j(y)} \right) \right\}.$$

The proof is similar to that of Corollary 2.6. As a special case, let us mention that the Funk metric in \mathbb{R}_+^n is given by

$$F_{\mathbb{R}_+^n}(x, y) = \max_{1 \leq i \leq n} \max \left\{ 0, \log \frac{x_i}{y_i} \right\}.$$

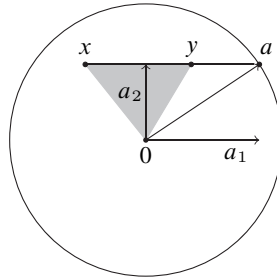
Observe that the map $x = (x_i) \mapsto u = (u_i)$, where $u_i = \log(x_i)$, is an isometry from the space $(\mathbb{R}_+^n, F_{\mathbb{R}_+^n})$ to \mathbb{R}^n with the weak Minkowski distance

$$\delta(u, v) = \max_{1 \leq i \leq n} \max \{0, u_i - v_i\}.$$

Example 4.2 (The Euclidean unit ball). The following is a formula for the Funk metric in the Euclidean unit ball $B \subset \mathbb{R}^n$:

$$F_B(x, y) = \log \left(\frac{\sqrt{|y-x|^2 - |x \wedge y|^2} + |x|^2 - \langle x, y \rangle}{\sqrt{|y-x|^2 - |x \wedge y|^2} - |y|^2 + \langle x, y \rangle} \right), \quad (4.1)$$

where $|x \wedge y| = \sqrt{|x|^2|y|^2 - \langle x, y \rangle^2}$ is the area of the parallelogram with sides $\vec{0x}$, $\vec{0y}$.



Proof. If $x = y$, there is nothing to prove, so we assume that $x \neq y$. Let us set $a = a_B(x, y) = R(x, y) \cap \partial B$. Using Proposition 2.3 with the linear form $h(z) = \langle y - x, z \rangle$ we get

$$F_B(x, y) = \log \left(\frac{\langle y - x, a \rangle - \langle y - x, x \rangle}{\langle y - x, a \rangle - \langle y - x, y \rangle} \right) = \log \left(\frac{\langle y - x, a \rangle + |x|^2 - \langle x, y \rangle}{\langle y - x, a \rangle - |y|^2 + \langle x, y \rangle} \right).$$

So we just need to compute $\langle y - x, a \rangle$. This is an exercise in elementary Euclidean geometry. Let us set $u = \frac{y-x}{|y-x|}$ and

$$a_1 = \langle u, a \rangle u, \quad a_2 = a - a_1.$$

Then $a = a_1 + a_2$ and a_1 is a multiple of $y - x$ while a_2 is the orthogonal projection of the origin O of \mathbb{R}^n on the line through x and y . In particular the altitude of the triangle Oxy is equal to $|a_2|$, therefore

$$\text{Area}(Oxy) = \frac{1}{2}|x \wedge y| = \frac{1}{2}|a_2| \cdot |y - x|.$$

Observe now that $\langle u, a \rangle > 0$ and $|a|^2 = |a_1|^2 + |a_2|^2 = 1$, we thus have

$$\begin{aligned} \langle y - x, a \rangle^2 &= |y - x|^2 \cdot \langle u, a \rangle^2 \\ &= |y - x|^2 \cdot |a_1|^2 \\ &= |y - x|^2 \cdot (1 - |a_2|^2) \\ &= |y - x|^2 - |x \wedge y|^2. \end{aligned}$$

The desired formula follows immediately. \square

5 The geometry of balls in the Funk metric

Since we are dealing with non-symmetric distances, we need to distinguish between *forward* and *backward* balls. For a point x in Ω and $\rho > 0$, we set

$$B^+(x, \rho) = \{y \in B \mid F_\Omega(x, y) < \rho\} \quad (5.1)$$

and we call it the *forward open ball* (or *right open ball*) centered at x of radius ρ . In a symmetric way, we set

$$B^-(x, \rho) = \{y \in B \mid F_\Omega(y, x) < \rho\} \quad (5.2)$$

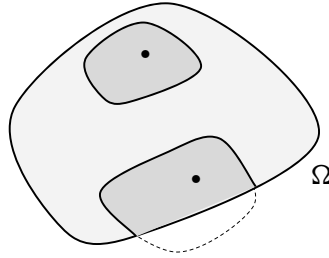
and we call it the *backward open ball* (also called the *left open ball*) centered at x of radius ρ .

Note that the open backward balls of the Funk metric are the open forward balls of the reverse Funk metric, and vice versa.

We define closed forward and closed backward balls by replacing the inequalities in (5.1) and (5.2) by non strict inequalities, and in the same way we define forward and backward spheres by replacing the inequalities by equalities. In Funk geometry, the backward and forward balls have in general quite different shapes and different properties.

Proposition 5.1. *Let Ω be a proper convex open subset of \mathbb{R}^n equipped with its Funk metric F_Ω , let x be a point in Ω and let ρ be a nonnegative real number. We have:*

- The forward open ball $B^+(x, \rho)$ is the image of Ω by the Euclidean homothety of center x and dilation factor $(1 - e^{-\rho})$.
- The backward open ball $B^-(x, \rho)$ is the intersection of Ω with the image of Ω by the Euclidean homothety of center x and dilation factor $(e^\rho - 1)$, followed by the Euclidean central symmetry centered at x .



A forward and a backward ball in Funk geometry. The forward ball is always relatively compact in Ω , while the closure of the backward ball may meet the boundary $\partial\Omega$ if its radius is large enough.

Proof. Let $y \neq x$ be a point in Ω . If $F_\Omega(x, y) = 0$, then the ray $R(x, y)$ is contained in Ω , and for any z on that ray, we have $F_\Omega(x, z) = 0$. Therefore the ray is also contained in $B(x, \rho)$. If $F_\Omega(x, y) = 0$, then $a = a_\Omega(x, y) \notin H_\infty$ and we have the following equivalent conditions for any point y on the segment $[x, a]$:

$$\begin{aligned} y \in B^+(x, \rho) &\iff \log \frac{|x - a|}{|y - a|} < \rho \\ &\iff |x - a| < e^\rho |y - a| = e^\rho (|x - a| - |y - x|) \\ &\iff |y - x| < (1 - e^{-\rho}) |x - a|. \end{aligned}$$

This proves the first statement. The proof of the second statement is similar; let us set $b = a_\Omega(y, x)$, then for any point $y \in [x, a]$ we have $x \in [b, y]$, therefore

$$\begin{aligned} y \in B^-(x, \rho) &\iff \log \frac{|y - b|}{|x - b|} < \rho \\ &\iff e^\rho |x - b| > |y - b| = |y - x| + |x - b| \\ &\iff (e^\rho - 1) |x - b| > |y - x|. \end{aligned} \quad \square$$

Thus, for instance, if Ω is the interior of a Euclidean ball in \mathbb{R}^n , then any forward ball for the Funk metric $B^+(x_0, \delta)$ is also a Euclidean ball. However, its Euclidean center is not the center for the Funk metric (unless x_0 is the center of Ω). Considering Example 4.2, if Ω is the Euclidean unit ball and $B^+(x_0, \rho) \subset \Omega$ is the Funk ball of radius ρ and center x_0 in Ω , then $y \in B^+(x_0, \rho)$ if and only if $F_\Omega(x_0, y) \leq \rho$. Using

Formula (4.1), we compute that this is equivalent to

$$\|y\|^2 - 2e^{-\rho}\langle y, x_0 \rangle + e^{-2\rho}\|x_0\|^2 \leq (1 - e^{-\rho})^2.$$

This set describes a Euclidean ball with center $z_0 = e^{-\rho}x_0$ and Euclidean radius $r = (1 - e^{-\rho})$.

We deduce the following “local-implies-global” property of Funk metrics. The meaning of the statement is clear, and it follows directly from Proposition 5.1.

Corollary 5.2. *We can reconstruct the boundary $\partial\Omega$ of Ω from the local geometry at any point of Ω .*

Corollary 5.3. *For any points x and x' in a convex domain Ω equipped with its Funk metric and for any two positive real numbers δ and δ' , the forward balls $B^+(x, \delta)$ and $B^+(x', \delta')$ are either homothetic or a translation of each other.*

Proof. This follows from Proposition 5.1 and the fact that the set of Euclidean transformations which are either homotheties or translations form a group (sometimes called *the group of dilations*, see e.g. [6]). \square

Remark 5.4. The previous corollary also holds for backward balls $B^-(x, \delta)$ of small enough radii.

Remarks 5.5. In the case where the convex set Ω is unbounded, the forward and backward open balls of its Funk metric are always noncompact.

If Ω is bounded, then for any $x \in \Omega$ and for ρ large enough we have $B^-(x, \rho) = \Omega$. This follows from Proposition 3.3. In particular, the closed backwards balls are not compact for large radii.

The forward open balls are geodesically convex if and only if Ω is strictly convex.

Remark 5.6. The property for a weak metric on a subset Ω of \mathbb{R}^n to have all the right spheres homothetic is also shared by the Minkowski weak metrics on \mathbb{R}^n . Indeed, it is easy to see that in a Minkowski weak metric, any two right open balls are homothetic. (Any two right spheres of the same radius are translates of each other, and it is easy to see from the definition that any two spheres centered at the same point are homothetic, the center of the homothety being the center of the balls.) Thus, Minkowski weak metrics share with the Funk weak metrics the property stated in Proposition 5.1. Busemann proved that in the setting of *Desarguesian spaces*, these are the only examples of spaces satisfying this property (see the definition of a Desarguesian space and the statement of this result in Section 6 of Chapter 1 in this volume ([19])). We state this as the following:

Theorem 5.7 (Busemann [3]). *A Desarguesian space in which all the right spheres of positive radius around any point are homothetic is either a Funk space or a Minkowski space.*

6 On the topology of the Funk metric

Proposition 6.1. *The topology induced by the Funk or reverse Funk metric in a bounded convex domain Ω in \mathbb{R}^n coincides with the Euclidean topology in that domain.*

Proof. The proof consists in comparing the balls in the Euclidean and the Funk (or reverse Funk) geometries. Let us fix a point x in Ω . Then there exists $0 < \lambda_x \leq \Lambda_x < \infty$ such that for any $\xi \in \partial\Omega$ we have

$$\lambda_x \leq |\xi - x| \leq \Lambda_x.$$

If we denote by $B^+(x, \rho)$ the forward ball with center x and radius ρ in the Funk metric, then Proposition 5.1 implies that

$$y \in \partial B^+(x, \rho) \implies (1 - e^{-\rho})\lambda_x \leq |y - x| \leq (1 - e^{-\rho})\Lambda_x.$$

In other words, if ${}^E B(x, \delta)$ denotes the Euclidean ball with center x and radius δ , then

$${}^E B(x, (1 - e^{-\rho})\lambda_x) \subset B^+(x, \rho) \subset {}^E B(x, (1 - e^{-\rho})\Lambda_x). \quad (6.1)$$

This implies that the families of balls $B^+(x, \rho)$ and ${}^E B(x, \delta)$ are sub-bases for the same topology.

For the backward balls $B^-(x, \rho)$, the second part of Proposition 5.1 implies the following

$$y \in \partial B^-(x, \rho) \implies (e^\rho - 1)\lambda_x \leq |y - x| \leq (e^\rho - 1)\Lambda_x,$$

provided $(e^\rho - 1) \leq 1$. This implies that for $\rho \leq \log(2)$ we have

$${}^E B(x, (e^\rho - 1)\lambda_x) \subset B^-(x, \rho) \subset {}^E B(x, (e^\rho - 1)\Lambda_x),$$

and therefore the family of backward balls $B^-(x, \rho)$ also generates the Euclidean topology. \square

For general convex domains, bounded or not, we have the following weaker result on the topology:

Proposition 6.2. *For any convex domain Ω in \mathbb{R}^n , F_Ω is a continuous functions on $\Omega \times \Omega$.*

Proof. We first consider the case where Ω is bounded. Suppose first that x and y are distinct points in Ω and let x_n, y_n be sequences in Ω converging to x and y respectively. Taking subsequences if necessary, we may assume that $x_n \neq y_n$ for all n . Then $a_n = a_\Omega(x_n, y_n)$ is well defined and this sequence converges to $a = a_\Omega(x, y)$.

Since $a \neq y$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} F_{\Omega}(x_n, y_n) &= \lim_{n \rightarrow \infty} \log \left(\frac{|x_n - a_n|}{|y_n - a_n|} \right) \\ &= \log \left(\frac{|x - a|}{|y - a|} \right) \\ &= F_{\Omega}(x, y). \end{aligned}$$

Assume now that $x = y$ and let $x_n, y_n \in \Omega$ be sequences converging to x such that $x_n \neq y_n$ for all n . We have

$$\begin{aligned} F_{\Omega}(x_n, y_n) &= \log \left(\frac{|x_n - a_n|}{|y_n - a_n|} \right) \\ &= \log \left(\frac{|(y_n - a_n) + (x_n - y_n)|}{|y_n - a_n|} \right) \\ &\leq \log \left(1 + \frac{|x_n - y_n|}{|y_n - a_n|} \right). \end{aligned}$$

Since $y_n \in \Omega$ converges to a point x in Ω , we have $\delta = \sup_{b \in \partial\Omega} |y_n - b|^{-1} < \infty$. We then have

$$F_{\Omega}(x_n, y_n) \leq \log(1 + \delta|x_n - y_n|) \rightarrow 0,$$

since $|x_n - y_n| \rightarrow 0$.

If Ω is unbounded, we set $\Omega_R = \Omega \cap {}^E B(x, R)$ where ${}^E B(x, R)$ is the Euclidean ball of radius R centered at the origin. It is easy to check that F_{Ω_R} converges uniformly to F_{Ω} on every compact subset of $\Omega \times \Omega$ as $R \rightarrow \infty$. The continuity of F_{Ω} follows therefore from the proof for bounded convex domains. \square

For the next result we need some more definitions:

Definition 6.3. Let δ be a weak metric defined on a set X . A sequence $\{x_k\}$ in X is *forward bounded* if

$$\sup \delta(x_k, x_m) < \infty$$

where the supremum is taken over all pairs k, m satisfying $m \geq k$. Note that this definition corresponds to the usual notion in the case of a usual (symmetric) metric space. We then say that the weak metric space (X, δ) is *forward proper*, or *forward boundedly compact* if every forward bounded sequence has a converging subsequence. The sequence $\{x_k\}$ is *forward Cauchy* if

$$\lim_{k \rightarrow \infty} \sup_{m \geq k} \delta(x_k, x_m) = 0.$$

The weak metric space (X, δ) is *forward complete* if every forward Cauchy sequence converges. We define *backward properness* and *backward completeness* in a similar way.

Proposition 6.4. *The Funk metric in a convex domain $\Omega \subset \mathbb{R}^n$ is forward proper (and in particular forward complete) if and only if Ω is bounded. The Funk metric is never backward complete.*

Proof. For a convex domain, the ball inclusions (6.1) immediately imply that forward complete balls are relatively compact; this implies forward properness. If Ω is unbounded, then it contains a ray and such a ray contains a divergent sequence $\{x_k\}$ such that $F_\Omega(x_k, x_m) = 0$ for any $m \geq k$ therefore F_Ω is not complete.

To prove that the Funk metric is never backward complete, we consider an affine segment $[a, b] \in \mathbb{R}^n$ with $a \neq b$ and such that $[a, b] \cap \partial\Omega = \{a, b\}$. Set $x_k = b + \frac{1}{k}(a - b)$. If $m \geq k$, then

$$F_\Omega(x_k, x_m) = \log \frac{|x_m - a|}{|x_k - a|},$$

which converges to 0 as $k \rightarrow \infty$. Since the sequence $\{x_k\}$ has no limit in Ω , we conclude that F_Ω is not backward complete. \square

Remark 6.5. The previous proposition also says that in a bounded convex domain, the Funk metric is forward complete and the reverse Funk metric is not.

7 The triangle inequality and geodesics

In this section, we prove the triangle inequality for the Funk metric and give a necessary and sufficient condition for the equality case. We also describe all the geodesics of this metric.

7.1 On the triangle inequality

Theorem 7.1. *If x, y and z are three points in a proper convex domain Ω , then the triangle inequality*

$$F_\Omega(x, y) + F_\Omega(y, z) \geq F_\Omega(x, z) \tag{7.1}$$

holds. Furthermore we have equality $F_\Omega(x, y) + F_\Omega(y, z) = F_\Omega(x, z)$ if and only if the three points

$$a_\Omega(x, y), a_\Omega(y, z), a_\Omega(x, z) \in \tilde{\partial}\Omega \tag{7.2}$$

are aligned in $\mathbb{R}\mathbb{P}^n$.

Before proving this theorem, let us first recall a few additional definitions from convex geometry: Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Then it is known that its closure $\bar{\Omega}$ is also convex. A convex subset $D \subset \bar{\Omega}$ is a *face* of $\bar{\Omega}$ if for any $x, y \in D$ and any

$0 < \lambda < 1$ we have

$$(1 - \lambda)x + \lambda y \in D \implies [x, y] \subset D.$$

The empty set and $\bar{\Omega}$ are also considered to be faces. A face $D \subset \bar{\Omega}$ is called *proper* if $D \neq \bar{\Omega}$ and $D \neq \emptyset$. A face D is said to be *exposed* if there is a supporting hyperplane H for Ω such that $D = H \cap \bar{\Omega}$. Recall that a support hyperplane is a hyperplane H that meets $\partial\Omega$ and $H \cap \Omega = \emptyset$. It is easy to prove that every proper face is contained in an exposed face. In fact every maximal proper face is exposed.

A point $x \in \partial\bar{\Omega}$ is an *exposed point* of Ω if $\{x\}$ is an exposed face, that is, if there exists a hyperplane $H \subset \mathbb{R}^n$ such that $H \cap \bar{\Omega} = \{x\}$. If Ω is bounded, then $\bar{\Omega}$ is the closure of the convex hull of its exposed points (Straszewicz's Theorem).

A point $x \in \bar{\Omega}$ is an *extreme point* if $\bar{\Omega} \setminus \{x\}$ is still a convex set. Such a point belongs to the boundary $\partial\Omega$ and if Ω is bounded, then $\bar{\Omega}$ is the convex hull of its extreme points (Krein–Milman's Theorem). Every exposed point is an extreme point, but the converse does not hold in general. The following result immediately follows from the definitions:

Lemma 7.2. *The following are equivalent conditions for a convex domain $\Omega \subset \mathbb{R}^n$:*

- (i) *Every boundary point is a extreme point.*
- (ii) *Every boundary point is an exposed point.*
- (iii) *The boundary $\partial\Omega$ does not contain any non-trivial segment.*

If one of these conditions holds, then Ω is said to be *strictly convex*. The following result will play an important role in the proof of Theorem 7.1:

Lemma 7.3. *Let Ω be bounded convex domain and x, y, z three points in Ω . Then the following are equivalent:*

- (a) *There exists a proper face $D \subset \tilde{\partial}\Omega$ such that*

$$a_\Omega(x, y), a_\Omega(y, z), a_\Omega(x, z) \in D.$$

- (b) *The three points $a_\Omega(x, y)$, $a_\Omega(y, z)$ and $a_\Omega(x, z)$ are aligned in \mathbb{RP}^n .*

Proof. Let us set $a = a_\Omega(x, y)$, $b = a_\Omega(y, z)$ and $c = a_\Omega(x, z)$. If x, y and z are aligned, then $a = b = c$. Otherwise, a, b and c belong to the 2-plane Π containing x, y, z . Therefore if a, b, c belong to a proper face D , then those three points are contained in the interval $\Pi \cap D$. This proves the implication (a) \implies (b). The converse implication (b) \implies (a) is obvious. \square

Proof of Theorem 7.1. We first consider $F_\Omega(x, z) = 0$. In this case the inequality (7.1) is trivial and we have $c \in H_\infty$. We then have equality in (7.1) if and only if $F_\Omega(x, y) = F_\Omega(y, z) = 0$, which is equivalent to $a \in H_\infty$ and $b \in H_\infty$ (the points a and b are as in the proof of the previous lemma). It then follows from Lemma 7.3 that a, b, c lie on some line (at infinity).

We now consider the case $F_\Omega(x, z) > 0$, that is, $c \notin H_\infty$. Choose a supporting functional h at the point c (that is, $h(c) = 1$). We then have from Corollary 2.5:

$$\begin{aligned} F_\Omega(x, z) &= \log \left(\frac{1 - h(x)}{1 - h(z)} \right) \\ &= \log \left(\frac{1 - h(x)}{1 - h(y)} \right) + \log \left(\frac{1 - h(y)}{1 - h(z)} \right) \\ &\leq F_\Omega(x, y) + F_\Omega(y, z). \end{aligned}$$

This proves the triangle inequality. Using again Corollary 2.5, we see that we have equality if and only if

$$F_\Omega(x, y) = \log \left(\frac{1 - h(x)}{1 - h(y)} \right)$$

and

$$F_\Omega(y, z) = \log \left(\frac{1 - h(y)}{1 - h(z)} \right),$$

and this holds if and only if one of the following cases holds:

Case 1. We have $a \notin H_\infty$ and $b \notin H_\infty$. In that case, $h(a) = h(b) = 1 = h(c)$. The three points a, b, c belong to the face $D = \partial\Omega \cap \{h = 1\}$ and we conclude by Lemma 7.3 that a, b, c lie on some line.

Case 2. We have $a \in H_\infty$ and $b \notin H_\infty$. In that case $h(b) = 1 = h(c)$ and $h(x) = h(y)$. This implies that the line through x and y is parallel to the hyperplane $\{h = 1\}$ and therefore the point

$$a \in \tilde{R}(x, y) \cap H_\infty \subset \tilde{\partial}\Omega$$

belongs to the support hyperplane $\{h = 1\}$. Since $h(b) = 1$, the three points a, b, c belong to the face $D = \partial\Omega \cap \{h = 1\}$ and we conclude by Lemma 7.3.

Case 3. We have $a \notin H_\infty$ and $b \in H_\infty$. The argument is the same as in Case 2.

To complete the proof, we need to discuss the case $a \in H_\infty$ and $b \in H_\infty$. In this case, we would have $h(x) = h(y)$ and $h(y) = h(z)$ and this is not possible. Indeed, we have $c = x + \lambda(z - x)$ for some λ and the equality $h(z) = h(x)$ would lead to the contradiction

$$1 = h(c) = h(x + \lambda(z - x)) = h(x) + \lambda(h(z) - h(x)) = h(x) < 1.$$

We thus proved in all cases that the equality holds in (7.1) if and only if the points a, b and c are aligned in $\mathbb{R}\mathbb{P}^n$. \square

Corollary 7.4. *Let x and z be two points in a proper convex domain $\Omega \subset \mathbb{R}^n$. Suppose that $a_\Omega(x, z) \in \partial\Omega$ is an exposed point. Then for any point $y \notin [x, z]$ we have $F_\Omega(x, z) < F_\Omega(x, y) + F_\Omega(y, z)$.*

7.2 Geodesics and convexity in Funk geometry

We now describe geodesics in Funk geometry. Let us start with a few definitions.

Definitions 7.1. A *path* in a weak metric space (X, d) is a continuous map $\gamma: I \rightarrow X$, where I is an interval of the real line. The *length* of path $\gamma: [a, b] \rightarrow X$ is defined as

$$\text{Length}(\gamma) = \sup \sum_{i=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all subdivisions $a = t_0 < t_1 < \dots < t_N = b$. Note that in the case where the weak metric d is non-symmetric, the order of the arguments is important. The path $\gamma: [a, b] \rightarrow X$ is a *geodesic* if $d(\gamma(a), \gamma(b)) = \text{Length}(\gamma)$. The weak metric space (X, d) is said to be a *weak geodesic metric space* if there exists a geodesic path connecting any pair of points. It is said to be *uniquely geodesic* if this geodesic path is unique up to reparametrization. A subset $A \subset X$ is said to be *geodesically convex* if given any two points in A , any geodesic path joining them is contained in A .

Lemma 7.5. *The path $\gamma: [a, b] \rightarrow X$ is geodesic if and only if for any t_1, t_2, t_3 in $[a, b]$ satisfying $t_1 \leq t_2 \leq t_3$ we have $d(\gamma(t_1), \gamma(t_3)) = d(\gamma(t_1), \gamma(t_2)) + d(\gamma(t_2), \gamma(t_3))$.*

The proof is an easy consequence of the definitions. \square

Let us now consider a proper convex domain $\Omega \subset \mathbb{R}^n$ and a proper face $D \subset \tilde{\partial}\Omega$. For any point $p \in \Omega$, we denote by

$$C_p(D) = \{v \in \mathbb{R}^n \mid v = 0 \text{ or } \bar{R}(p, p+v) \cap D \neq \emptyset\}. \quad (7.3)$$

Here $\bar{R}(p, p+v)$ is the extended ray through p and $p+v$ in $\mathbb{R}\mathbb{P}^n$. (Recall that the projective space $\mathbb{R}\mathbb{P}^n$ is considered here as a completion of the Euclidean space \mathbb{R}^n obtained by adding a hyperplane at infinity; the completion of the ray is then its topological completion.) Observe that $C_p(D)$ is a cone in \mathbb{R}^n at the origin, its translate $p + C_p(D)$ is the cone over D with vertex at p . We then have the following

Theorem 7.6. *Let $\gamma: [0, 1] \rightarrow \Omega$ be a path in a proper convex domain of \mathbb{R}^n . Then γ is a geodesic for the Funk metric in Ω if and only if there exists a face $D \subset \tilde{\partial}\Omega$ such that for any $t_1 < t_2$ in $[0, 1]$ we have*

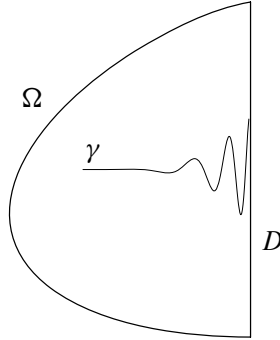
$$\gamma(t_2) - \gamma(t_1) \in C_{\gamma(t_1)}(D).$$

In particular if $a_\Omega(x, y) \in \partial\Omega$ is an exposed point, then there exists a unique (up to reparametrization) geodesic joining x to y , and this geodesic is a parametrization of the affine segment $[x, y]$.

Proof. This is a direct consequence of Theorem 7.1 together with Lemma 7.3 and Lemma 7.5. \square

For smooth curves we have the following

Corollary 7.7. *Let $\gamma: [0, 1] \rightarrow \Omega$ be a C^1 path in a proper convex domain of \mathbb{R}^n . Then γ is a Funk geodesic if and only if there exists a face $D \subset \tilde{\partial}\Omega$ such that $\dot{\gamma}(t) \in C_{\gamma(t)}(D)$ for any $t \in [a, b]$.*



A typical smooth geodesic in Funk geometry: all tangents to the curve meet the same face $D \subset \partial\Omega$.

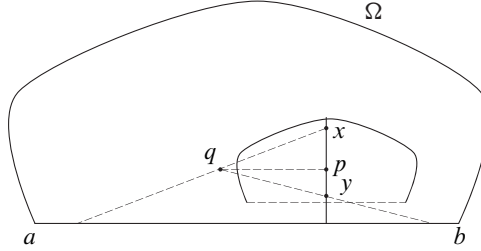
For a subset Ω of \mathbb{R}^n , equipped with a (weak) metric F , we have two notions of convexity: *affine convexity*, saying that for every pair of points in Ω , the affine (or Euclidean) geodesic joining them is contained in Ω , and *geodesic convexity*, saying that for every pair of points in Ω , the F -geodesic joining them is contained in Ω .

From the preceding results, we have the following consequence on geodesic convexity of subsets for the Funk metric.

Corollary 7.8. *Let Ω be a bounded convex domain of \mathbb{R}^n . Then the following are equivalent:*

- (1) Ω is strictly convex.
- (2) Ω is uniquely geodesic for the Funk metric.
- (3) A subset $A \subset \Omega$ is geodesically convex for the Funk metric if and only if A is affinely convex.
- (4) The forward open balls in Ω are geodesically convex with respect to the Funk metric F_Ω .

Proof. (1) \implies (2) immediately follows from Theorem 7.6. (2) \implies (3) is obvious and (3) \implies (4) follows from Proposition 5.1.



We now prove (4) \implies (1) by contraposition. Let us assume that Ω is not strictly convex, so that there exists a non-trivial segment $[a, b] \subset \partial\Omega$. One can then find a supporting functional h for Ω such that $h(a) = h(b) = 1$. Let us choose a segment $[x, y] \subset \Omega$ such that $a_\Omega(x, y) \in [a, b]$ and another segment $[p, q] \subset \Omega$ such that $p \in [x, y]$ and $h(q) = h(p)$. We also assume $x \neq p \neq y$ and $F_\Omega(p, q) > \delta := F_\Omega(p, x) + F_\Omega(p, y)$. Observe that we then have $h(x) < h(p) = h(q) < h(y)$.

If $a_\Omega(x, q) \in [a, b]$ and $a_\Omega(q, y) \in [a, b]$, then the proof is finished since in this case

$$F_\Omega(x, y) = \log \frac{h(x)}{h(y)} = \log \frac{h(x)}{h(q)} \cdot \frac{h(q)}{h(y)} = F_\Omega(x, q) + F_\Omega(q, y).$$

Since $x, y \in B^+(p, \delta)$ while $q \notin B^+(p, \delta)$, we conclude that the forward ball $B^+(p, \delta)$ is not geodesically convex.

If $a_\Omega(x, q) \notin [a, b]$ or $a_\Omega(q, y) \notin [a, b]$, we let $c = a_\Omega(x, y)$ and we consider the Euclidean homothety $f_\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^n$ centered at c with dilation factor $\lambda < 1$. Let $p' = f_\lambda(p)$, $q' = f_\lambda(q)$, $x' = f_\lambda(x)$, $y' = f_\lambda(y)$ and $q' = f_\lambda(q)$. It is now clear that if $\lambda > 0$ is small enough, then $a_\Omega(x', q') \in [a, b]$ and $a_\Omega(q', y') \in [a, b]$. It is also clear that one can find a number δ' such that $x', y' \in B^+(p', \delta')$ while $q' \notin B^+(p', \delta')$. The previous argument shows then that $F_\Omega(x', y') = F_\Omega(x', q') + F_\Omega(q', y')$ and therefore $B^+(p', \delta')$ is not geodesically convex. \square

Remark 7.9. Note the formal analogy between Corollary 7.8 and the corresponding result concerning the geodesic segments of a Minkowski metric on \mathbb{R}^n : if the unit ball of a Minkowski metric is strictly convex, then the only geodesic segments of this metric are the affine segments.

8 Nearest points in Funk geometry

Let $\Omega \subset \mathbb{R}^n$ be a convex set equipped with its Funk metric F .

Definition 8.1. Let x be a point in Ω and let A be a subset of Ω . A point y in A is said to be a *nearest point*, or a *foot* (in Buseman's terminology), for x on A if

$$F_\Omega(x, y) = F_\Omega(x, A) := \inf_{z \in A} F_\Omega(x, z).$$

It is clear from the continuity of the function $y \mapsto F_\Omega(x, y)$ that for any closed non-empty subset $A \subset \Omega$ and any $x \in \Omega$, there exists a nearest point $y \in A$. This point need not be unique in general.

Proposition 8.2. *For a proper convex domain $\Omega \subset \mathbb{R}^n$, the following properties are equivalent:*

- a) Ω is strictly convex.
- b) For any closed convex subset $A \subset \Omega$ and for any $x \in \Omega$, there is a unique nearest point $y \in A$.

Proof. let x be a point in Ω and assume $r = F_\Omega(x, A) > 0$. Suppose that y and z are two nearest points of A for x . For any point w on the segment $[y, z]$ we have $F_\Omega(x, w) \leq r$ because the closed ball $\bar{B}^+(x, r)$ is convex. Since A is also assumed to be convex, we have $w \in A$ and therefore $F_\Omega(x, w) \geq r$. We conclude that $F_\Omega(x, w) = r$, that is, $w \in \partial\bar{B}^+(x, r)$. From Proposition 5.1, we know that if Ω is strictly convex, then $\bar{B}^+(x, r)$ is also strictly convex and we conclude that $y = z$. It follows that we have a unique nearest point on A for x . This proves (a) \implies (b).

To prove (b) \implies (a), we assume by contraposition that Ω is not strictly convex. Again from Proposition 5.1, we know that the forward ball $\bar{B}^+(x, r)$ is not strictly convex. In particular $\partial B^+(x, r)$ contains a non trivial segment $A = [y, z]$. Any point in the convex set A is a nearest point to x and this completes the proof. \square

Proposition 8.3. *Let A be an affinely convex closed subset of a proper convex domain $\Omega \subset \mathbb{R}^n$ and let $x \in \Omega \setminus A$. A point $y \in A$ is a nearest point in A for x if and only if either $F_\Omega(x, y) = 0$ or there exists a hyperplane $\Pi \subset \mathbb{R}^n$ which contains y , which separates A and x and which is parallel to a support hyperplane H for Ω at $a = a_\Omega(x, y)$.*

Proof. We assume $F_\Omega(x, y) > 0$ (otherwise, there is nothing to prove). First, suppose there exists a hyperplane $\Pi \subset \mathbb{R}^n$ containing y and separating A from x and which is parallel to a support hyperplane H for Ω at a . Let h be the corresponding supporting functional. Then we have, from our hypothesis,

$$h(x) < h(y) = \inf_{z \in A} h(z).$$

From Proposition 2.3 and Corollary 2.5 we then have

$$F_\Omega(x, y) = \log \left(\frac{1 - h(x)}{1 - h(y)} \right) = \inf_{z \in A} \log \left(\frac{1 - h(x)}{1 - h(z)} \right) \leq F_\Omega(x, A),$$

therefore y is a nearest point on A for x .

To prove the converse, we assume that y is a nearest point on A for x . Set $r = F_\Omega(x, y) = F_\Omega(x, A)$, then, by definition, the forward open ball $B^+(x, r)$ and the set A are disjoint. Since both sets are affinely convex, there exists a hyperplane Π that separates them. Note that Π is then a support hyperplane at y for the ball $B^+(x, r)$.

We conclude from Proposition 5.1 that Π is parallel to a support hyperplane H for Ω at a . \square

There are several possible notions of perpendicularity in metric spaces. The following definition is due to Busemann (see [2], page 103).

Definition 8.4 (Perpendicularity). Let A be a subset of Ω and p a point in A . A geodesic $\gamma: I \rightarrow \Omega$ is said to be *perpendicular* to A at p if the following two properties hold:

- (1) $p = \gamma(t_0)$ for some $t_0 \in I$,
- (2) for every $t \in I$, p is a nearest point for $\gamma(t)$ on A .

From the previous results we have the following

Corollary 8.5. *Let x be a point in a convex domain Ω and $a \in \partial\Omega$ be a boundary point. If $\Pi \subset \mathbb{R}^n$ is a hyperplane containing x , then the ray $[x, a)$ is perpendicular to $\Pi \cap \Omega$ if and only if Π is parallel to a support hyperplane H_a of Ω at a .*

If $b \in \partial\Omega$ is another boundary point, then the line (a, b) is perpendicular to $\Pi \cap \Omega$ if and only if $\Pi \cap (a, b) \neq \emptyset$ and Π is parallel to both a support hyperplane H_a at a and a support hyperplane H_b at b .

9 The infinitesimal Funk distance

In this section, we consider a convex domain $\Omega \subset \mathbb{R}^n$ and a point p in Ω . We define a weak distance $\Phi_p = \Phi_{\Omega, p}$ on \mathbb{R}^n as the limit

$$\Phi_p(x, y) = \lim_{t \searrow 0} \frac{F_{\Omega, p}(p + tx, p + ty)}{t}.$$

Theorem 9.1. *The weak metric Φ_p at a point p in \mathbb{R}^n is a Minkowski weak metric in \mathbb{R}^n . Its unit ball is the translated domain $\Omega_p = \Omega - p$.*

Proof. Choose a supporting functional h for Ω and set

$$\varphi_h(t) = \log \left(\frac{1 - h(p + tx)}{1 - h(p + ty)} \right) = \log \left(1 + \frac{th(y - x)}{1 - h(p) - th(y)} \right).$$

The first two derivatives of this functions are

$$\varphi'_h(t) = \frac{(1 - h(p))h(y - x)}{(1 - h(p) - th(x))(1 - h(p) - th(y))},$$

and

$$\varphi''_h(t) = \frac{((1 - h(p))(h(x) + h(y)) - 2t \cdot h(x)h(y))(1 - h(p))h(y - x)}{(1 - h(p) - th(x))^2(1 - h(p) - th(y))^2}.$$

We have in particular

$$\varphi'_h(0) = \frac{h(y-x)}{1-h(p)},$$

and the second derivative is uniformly bounded in some neighborhood of p . More precisely, given a relatively compact neighborhood $U \subset \bar{U} \subset \Omega$ of p , one can find a constant C which depends on U but not on h such that

$$|\varphi''_h(t)| \leq C,$$

for any $x, y \in U$ and $|t| \leq 1$ and any support function h . We have, from Taylor's formula,

$$\varphi_h(t) = t \cdot \frac{h(y-x)}{1-h(p)} + t^2 \rho(x, y, h),$$

where $|\rho(x, y, h)| \leq C$. Using Corollary 2.6 we have

$$F_{\Omega, p}(p+tx, p+ty) = \sup_h \varphi_h(t) = \sup_h \frac{th(y-x)}{1-h(p)} + O(t^2),$$

where the supremum is taken over the set \mathcal{S}_Ω of all support functions for Ω . Therefore

$$\Phi_p(x, y) = \sup_{h \in \mathcal{S}_\Omega} \frac{h(y-x)}{1-h(p)}.$$

We then see that $\Phi_p(x, y)$ is weak Minkowski distance (see Chapter 1 in this Handbook).

$$\begin{aligned} \Phi_p(0, y) \leq 1 &\iff \sup_{h \in \mathcal{S}_\Omega} \frac{h(y)}{1-h(p)} \leq 1 \\ &\iff h(y) \leq 1-h(p) \quad \text{for all support functions } h \text{ of } \Omega \\ &\iff h(p+y) \leq 1 \quad \text{for all } h \\ &\iff y \in \bar{\Omega} - p, \end{aligned}$$

this means that the unit ball of Φ_p is the translate of Ω by $-p$. \square

Remark 9.2. The Funk metric of a convex domain Ω is in fact Finslerian, and the previous theorem means that the Finslerian unit ball at any point p coincides with the domain Ω itself with the point p as its center. This is why the Funk metric was termed *tautological* in [17].

The Finslerian approach to Funk geometry is developed in [24].

10 Isometries

It is clear from its definition that the Funk metric is invariant under affine transformation. Conversely, we have the following:

Proposition 10.1. *Let Ω_1 and Ω_2 be two bounded convex domains in \mathbb{R}^n . Assume that there exists a Funk isometry $f : U_1 \rightarrow U_2$, where U_i is an open convex subset of Ω_i , $i = 1, 2$. If Ω_2 is strictly convex, then f is the restriction of a global affine map of \mathbb{R}^n that maps Ω_1 to Ω_2 .*

Proof. Let x and y be two distinct points in U_1 . Then, for any $z \in [x, y]$, we have

$$\begin{aligned} F_{\Omega_2}(f(x), f(y)) - F_{\Omega_2}(f(x), f(z)) - F_{\Omega_2}(f(z), f(y)) \\ = F_{\Omega_1}(x, y) - F_{\Omega_1}(x, z) - F_{\Omega_1}(z, y) = 0. \end{aligned}$$

Since Ω_2 is assumed to be strictly convex, this implies that $f(z)$ belongs to the line through $f(x)$ and $f(y)$. We can thus define the real numbers t and s by

$$z = x + t(y - x) \quad \text{and} \quad f(z) = f(x) + s(f(y) - f(x)).$$

It now follows from Corollary 2.8 and the fact that f is an isometry that $s = t$. We thus have proved that for any $x, y \in U_1$ and any $t \in [0, 1]$ we have

$$f(x + t(y - x)) = f(x) + t(f(y) - f(x)). \quad (10.1)$$

This relation easily implies that f is the restriction of a global affine mapping. \square

We immediately deduce the following:

Corollary 10.2. *The group of Funk isometries of a strictly convex bounded domains $\Omega \subset \mathbb{R}^n$ coincides with the subgroup of the affine group of \mathbb{R}^n leaving Ω invariant.*

Remark 10.3. The conclusion of the corollary may fail for unbounded domains. For instance, if Ω is the upper half plane $\{x_2 > 0\}$ in \mathbb{R}^2 , then

$$F_{\Omega}(x, y) = \max\{0, \log(x_2/y_2)\}$$

and any map $f : \Omega \rightarrow \Omega$ of the type $f(x_1, x_2) = (ax_1 + b, \psi(x_2))$, where $a \neq 0$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is arbitrary, is an isometry.

11 A projective viewpoint on Funk geometry

In this section we consider the following generalization of Funk geometry: We say that a subset $U \subset \mathbb{RP}^n$ is *convex* if it does not contain any full projective line and if the intersection of any projective line $L \subset \mathbb{RP}^n$ with U is a connected set. If U and Ω are connected domains in \mathbb{RP}^n with $\Omega \subset U$, and if x, y are two distinct points in Ω . We denote by $a_{\Omega}(x, y) \in \partial\Omega$ and $\omega_{\Omega}(x, y) \in \partial U$ the boundary points on the line L through x and y appearing in the order a, y, x, ω .

Definition 11.1. The *relative Funk metric* for $\Omega \subset U$ is defined by $F_{\Omega,U}(x, x) = 0$ and by

$$F_{\Omega,U}(x, y) = \log \left(\frac{|y - \omega|}{|x - \omega|} \cdot \frac{|x - a|}{|y - a|} \right),$$

if $x \neq y$.

The relative Funk metric is a projective weak metric, it is invariant under projective transformations in the sense that if $f : \mathbb{RP}^n \rightarrow \mathbb{RP}^n$ is a projective transformation, then

$$F_{\Omega,U}(x, y) = F_{f(\Omega),f(U)}(f(x), f(y)).$$

Observe also that if $U \subset \mathbb{R}^n$ is a proper convex domain, then

$$F_{\Omega,U}(x, y) = F_{\Omega}(x, y) + {}^rF_U(x, y).$$

Lemma 11.2. *In the case $U = \mathbb{R}^n$, we have*

$$F_{\Omega,U}(x, y) = F_{\Omega}(x, y).$$

Proof. If $U = \mathbb{R}^n$, then its boundary is the hyperplane at infinity H_{∞} and thus $\frac{|y-\omega|}{|x-\omega|} = 1$ for any $x, y \in \Omega$. \square

Recall that there is no preferred hyperplane in projective space. Therefore, *the classical Funk geometry is a special case of the relative Funk geometry where the englobing domain $U \subset \mathbb{RP}^n$ is the complement of a hyperplane.* Such a set U is sometimes called an *affine patch*.

12 Hilbert geometry

Definition 12.1. The *Hilbert metric* in a proper convex domain $\Omega \subset \mathbb{R}^n$ is defined as

$$H_{\Omega}(x, y) = \frac{1}{2} (F_{\Omega}(x, y) + {}^rF_{\Omega}(y, x)),$$

where Ω is considered as a subset of an affine patch $U \subset \mathbb{RP}^n$.

This metric is a projective weak metric. Note that for $x \neq y$ we have

$$H_{\Omega}(x, y) = \frac{1}{2} \log \left(\frac{|y - b|}{|x - b|} \cdot \frac{|x - a|}{|y - a|} \right),$$

where $a = a_{\Omega}(x, y)$ and $b = a_{\Omega}(y, x)$. The expression inside the logarithm is the cross ratio of the points b, x, y, a , therefore the Hilbert metric is invariant by projective transformations.

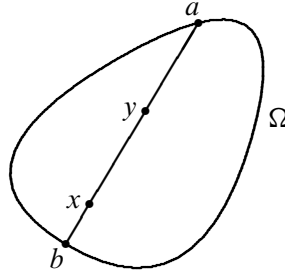


Figure 2. The Hilbert metric.

Note also that the Hilbert metric coincides with half of the relative Funk metric of the domain Ω with respect to itself:

$$H_{\Omega}(x, y) = \frac{1}{2} F_{\Omega, \Omega}(x, y),$$

In the case where Ω is the Euclidean unit ball, the Hilbert distance coincides with the Klein model (also called the *Beltrami–Cayley–Klein model*) of hyperbolic space. We refer to Sections 2.3–2.6 in [25] for a nice introduction to Klein’s model. In the case of a convex polytope defined by the linear inequalities $\phi_j(x) < s_j$, $1 \leq j \leq k$, we have

$$H_{\Omega}(x, y) = \max_{1 \leq i, j \leq k} \frac{1}{2} \cdot \log \left(\frac{s_i - \phi_j(y)}{s_i - \phi_j(x)} \cdot \frac{s_j - \phi_i(x)}{s_j - \phi_i(y)} \right).$$

Applying our investigation on Funk geometry immediately gives a number of results on Hilbert geometry. In particular, applying Proposition 2.2 we get:

Proposition 12.2. *The Hilbert metric in a convex domain $\Omega \neq \mathbb{R}^n$ satisfies the following properties:*

- (a) $H_{\Omega}(x, y) \geq 0$ and $H_{\Omega}(x, x) = 0$ for all $x, y \in \Omega$.
- (b) $H_{\Omega}(x, z) \leq H_{\Omega}(x, y) + H_{\Omega}(y, z)$ for all $x, y, z \in \Omega$.
- (c) H_{Ω} is projective, that is, $H_{\Omega}(x, z) = H_{\Omega}(x, y) + H_{\Omega}(y, z)$ whenever z is a point on the affine segment $[x, y]$.
- (d) The weak metric H_{Ω} is symmetric, that is, $H_{\Omega}(x, y) = H_{\Omega}(y, x)$ for any x and y .
- (e) The weak metric H_{Ω} is separating, that is, $x \neq y \implies H_{\Omega}(x, y) > 0$, if and only if the domain Ω does not contain any affine line.
- (f) The weak metric H_{Ω} is unbounded.

The proof of this proposition easily follows from the definitions and from Proposition 2.2.

Proposition 12.3. *If the convex domain $\Omega \neq \mathbb{R}^n$ does not contain any affine line, then H_Ω is a metric in the classical sense. Furthermore, it is complete.*

A convex domain which does not contain any affine line is called a *sharp* convex domain.

From Theorem 7.1, we deduce the following necessary and sufficient condition for the equality case in the triangle inequality:

Theorem 12.4. *Let x , y and z be three points in a proper convex domain Ω . We have $F_\Omega(x, y) + F_\Omega(y, z) = F_\Omega(x, z)$ if and only if both triples of boundary points $a_\Omega(x, y)$, $a_\Omega(y, z)$, $a_\Omega(x, z)$ and $a_\Omega(y, x)$, $a_\Omega(z, y)$, $a_\Omega(z, x)$ are aligned in \mathbb{RP}^n .*

From Theorem 7.6, we obtain:

Theorem 12.5. *Let $\gamma : [0, 1] \rightarrow \Omega$ be a path in a sharp convex domain of \mathbb{R}^n . Then γ is a geodesic for the Hilbert metric in Ω if and only if there exist two faces $D^-, D^+ \subset \partial\Omega$ such that for any $t_1 < t_2$ in $[0, 1]$ we have $\gamma(t_2) - \gamma(t_1) \in C_{\gamma(t_1)}(D^+)$ and $\gamma(t_1) - \gamma(t_2) \in C_{\gamma(t_2)}(D^-)$.*

Recall that $C_p(D)$ is the cone at p on the face D , see Equation (7.3). If γ is smooth, then it is geodesic if and only if $\dot{\gamma}(t) \in C_{\gamma(t)}(D^+)$ and $-\dot{\gamma}(t) \in C_{\gamma(t)}(D^-)$ for any $t \in [0, 1]$. We then have the following characterization of smooth geodesics in Hilbert geometry which we formulate only for bounded domains for convenience:

Corollary 12.6. *Let $\gamma : [0, 1] \rightarrow \Omega$ be a path of class C^1 in a bounded convex domain of \mathbb{R}^n . Then γ is a geodesic for the Hilbert metric in Ω if and only if there exist two faces $D^-, D^+ \subset \partial\Omega$ such that for any t , the tangent line to the curve γ at t meets the boundary $\partial\Omega$ on $D^+ \cup D^-$.*

Corollary 12.7 (compare [10]). *Assume that Ω is strictly convex, or more generally that all but possibly one of its proper faces are reduced to points. Then the Hilbert geometry in Ω is uniquely geodesic.*

13 Related questions

In this section we briefly discuss some recent developments related to the idea of the Funk distance.

Yamada recently introduced what he called the *Weil–Petersson Funk metric* on Teichmüller space (see [27], and see [15] in this volume). The definition is analogous to one of the definitions of the Funk metric, using in an essential way the non-completeness of the Weil–Petersson metric on Teichmüller space. One can wonder whether there are Funk-like metrics associated to other interesting known symmetric

metrics. This geometry bears some analogies with Thurston's metric on Teichmüller space and the Funk metric. One can ask for a study of the Thurston metric which parallels the study of the Funk metric (that is, study its balls, its convexity properties, orthogonality and projections, etc.).

It should also be of interest to study the geometric properties of the *reverse* Funk metric, that is, the metric ${}^rF_\Omega$ on an open convex set Ω defined by ${}^rF_\Omega(x, y) = F_\Omega(y, x)$. Let us recall that the reverse Funk metric is not equivalent to the Funk metric in any reasonable sense, see Remark 6.5. This is also related to the fact that the forward and backward open balls at some point can be very different, as we already noticed. We note in this respect that the *reverse* metrics of the Thurston weak metric and of the Weil–Petersson Funk weak metric that we mentioned are also very poorly understood.

Finally, there is another symmetrization of the Funk metric, besides the Hilbert metric, namely, its *max-symmetrization*, defined as

$$S(x, y) = \max\{(F(x, y), F(y, x))\},$$

and it should be interesting to study its properties. Note that the max-symmetrization of the Thurston metric is an important metric on Teichmüller space, known as the *length spectrum metric*.

A Menelaus' Theorem

An elementary proof of the triangle inequality for the Funk metric is given by Zaustinsky in [28], and it is recalled in the next appendix. This proof is based on the classical Menelaus' Theorem.² For the convenience of the reader, we give a statement and a proof of this result in the present appendix.

To state Menelaus' Theorem, we recall the notion of *division ratio* of three aligned points. Consider three points A, B, P in \mathbb{R}^n with $A \neq B$. Then P belongs to the line through A and B if and only if $P = tB + (1 - t)A$ for some uniquely defined $t \in \mathbb{R}$. The number t is called the *division ratio* or the *affine ratio* of P relative to B and A . We denote it by $t = \frac{AP}{AB}$. The division ratio is invariant under any affine transformation. Note that if both $A \neq B$ and $A \neq P$, then

$$\frac{AP}{AB} = t \iff \frac{PB}{PA} = \frac{t - 1}{t}.$$

For instance P is the midpoint of $[B, A]$ if and only if $\frac{AP}{AB} = \frac{1}{2}$ or, equivalently, $\frac{PB}{PA} = -1$. Note that the sign is an important component of the division ratio, and in

²This theorem, in the Euclidean and in the spherical case, is quoted by Ptolemy (2nd century A.D.) and it is due to Menelaus (second century A.D.) Its proof is contained in Menelaus' *Spherics*. No Greek version of Menelaus' *Spherics* survived, but there are Arabic versions; cf. the forthcoming English edition [21] from the Arabic original of al-Harawī (10th century).

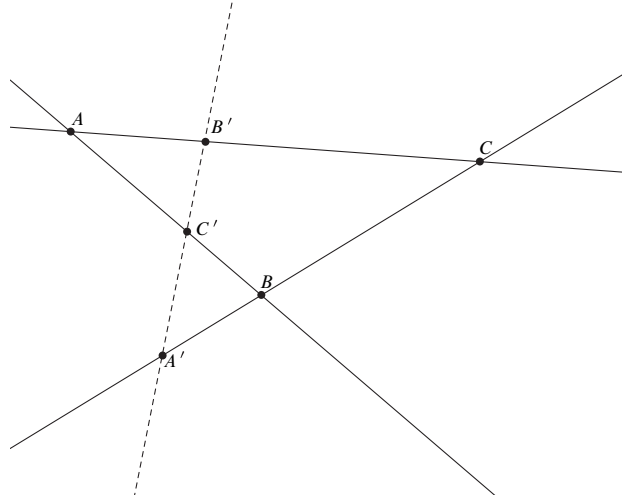
fact we have

$$\frac{AP}{AB} = \pm \frac{|P - A|}{|B - A|}$$

with a minus sign if and only if A lies between B and P .

Proposition A.1 (Menelaus' Theorem). *Let ABC be a non-degenerate Euclidean triangle and let A' , B' , C' be three arbitrary points on the lines containing the sides BC , AC , AB . Assume that $A' \neq C$, $B' \neq A$ and $C' \neq B$. Then, the points A' , B' , C' are aligned if and only if we have*

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = +1.$$



Proof. Although a purely geometric proof is possible, it is somewhat delicate to correctly handle the signs of the division ratios throughout the arguments. We follow below a more algebraic approach. It will be convenient to assume that A , B and C are points in \mathbb{R}^n with $n \geq 3$ and to assume that the origin $0 \in \mathbb{R}^n$ does not belong to the plane π containing the three points A , B , C . By hypothesis, the point C' lies on the line through A and B , therefore

$$C' = \nu A + (1 - \nu)B, \quad \text{with} \quad \frac{C'A}{C'B} = \frac{\nu - 1}{\nu}.$$

Likewise, we have $A' = \lambda B + (1 - \lambda)C$ and $B' = \mu C + (1 - \mu)A$ with $\frac{A'B}{A'C} = \frac{\lambda - 1}{\lambda}$ and $\frac{B'C}{B'A} = \frac{\mu - 1}{\mu}$. Now the point C' lies on the line through A' and B' if and only if

there exists $\rho \in \mathbb{R}$ such that $C' = \rho A' + (1 - \rho)B'$. We then have

$$\begin{aligned} C' &= \rho(\lambda B + (1 - \lambda)C) + (1 - \rho)(\mu C + (1 - \mu)A) \\ &= (1 - \rho)(1 - \mu)A + \rho\lambda B + (\rho(1 - \lambda) + \mu(1 - \rho))C \\ &= vA + (1 - v)B. \end{aligned}$$

Our hypothesis implies that A, B, C correspond to three linearly independent vectors in \mathbb{R}^n , therefore the latter identity implies

$$v = (1 - \rho)(1 - \mu), \quad (1 - v) = \rho\lambda \quad \text{and} \quad (\rho(1 - \lambda) + \mu(1 - \rho)) = 0.$$

Thus,

$$\frac{(1 - \mu)(1 - v)}{\lambda v} = \frac{\rho}{1 - \rho} = -\frac{\mu}{1 - \lambda}$$

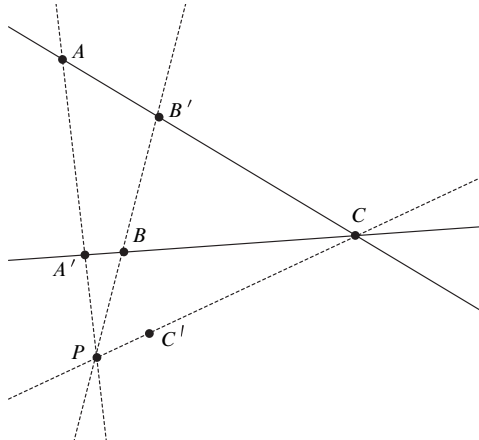
and we conclude that C' is aligned with A' and B' if and only if

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = \frac{(\lambda - 1)(\mu - 1)(v - 1)}{\lambda\mu v} = 1. \quad \square$$

Remark. Using similar arguments, we can also prove Ceva's Theorem. Both theorems are dual to each other. Let us recall the statement:

Proposition A.2 (Ceva's Theorem). *Let ABC be a non-degenerate Euclidean triangle and let A', B', C' be three arbitrary points on the lines containing the sides BC, AC, AB such that $A' \neq C, B' \neq A$ and $C' \neq B$. Then, the lines AA', BB' and CC' are concurrent or parallel if and only if we have*

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1.$$



Proof. Let us give the main step of the proof of Ceva's Theorem. Suppose that the lines AA' , BB' and CC' meet at a point P . Then we can find r, s and t in \mathbb{R} such that

$$P = tA + (1-t)A' = sB + (1-s)B' = rC + (1-r)C'.$$

We also have as before $A' = \lambda B + (1-\lambda)C$, $B' = \mu C + (1-\mu)A$ and $C' = \nu A + (1-\nu)B$. This implies

$$\begin{aligned} P &= tA + (1-t)\lambda B + (1-t)(1-\lambda)C \\ &= (1-s)(1-\mu)A + sB + (1-s)\mu C \\ &= (1-r)\nu A + (1-r)(1-\nu)B + rC. \end{aligned}$$

By uniqueness of the barycentric coordinates with respect to the triangle ABC , we have

$$\begin{aligned} t &= (1-s)(1-\mu) = (1-r)\nu, \\ s &= (1-t)\lambda = (1-r)(1-\nu), \\ r &= (1-t)(1-\lambda) = (1-s)\mu. \end{aligned}$$

Therefore, we have

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = \frac{(\lambda-1)(\mu-1)(\nu-1)}{\lambda\mu\nu} = -\frac{r}{s} \cdot \frac{t}{r} \cdot \frac{s}{t} = -1.$$

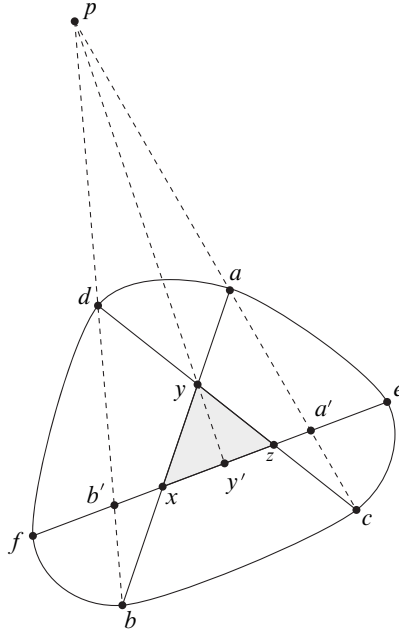
We leave it to the reader to discuss the case where the lines AA' , BB' and CC' are parallel and to prove the converse direction. \square

B The classical proof of the triangle inequality for the Funk metric

The triangle inequality is proved in Section 7. Note that it also easily follows from Corollary 2.6 (see also [27]), and it is also a consequence of the Finslerian description of the Funk metric (see Chapter 3 ([24]) in this volume).

In this appendix, we present the classical proof of the triangle inequality following Zaustinsky [28]. This proof is similar to the original proof of the triangle inequality for the Hilbert distance, as given by D. Hilbert in [11], although the proof in the case of the Hilbert distance is a bit simpler and does not use the Menelaus theorem.

We now prove the triangle inequality for the Funk metric following [28] (see p. 85). Let x, y, z be three points in Ω . In view of Property (b) in Proposition 2.2, we may assume that they are not collinear. Let a, b, c, d, e, f be the intersections with $\partial\Omega$ of the lines xz , yx and zy , using the notation of the figure concerning the order of intersections.



From the invariance of the cross ratio from the perspective at p , we have

$$\frac{|x - a|}{|y - a|} \cdot \frac{|b - y|}{|b - x|} = \frac{|x - a'|}{|y' - a'|} \cdot \frac{|b' - y'|}{|b' - x|}$$

and

$$\frac{|y - c|}{|z - c|} \cdot \frac{|d - z|}{|d - y|} = \frac{|y' - a'|}{|z - a'|} \cdot \frac{|b' - z|}{|b' - y'|}.$$

Multiplying both sides of these two equations, we get

$$\frac{|x - a|}{|y - a|} \cdot \frac{|y - c|}{|z - c|} = \frac{|x - a'|}{|z - a'|} \cdot \frac{|b' - z|}{|b' - x|} \cdot \frac{|b - x|}{|b - y|} \cdot \frac{|d - y|}{|d - z|}.$$

The three points b , b' and d lie on the sides of the triangle xyz and are aligned, therefore we have by Menelaus' Theorem (Theorem A.1):

$$\frac{|b' - z|}{|b' - x|} \cdot \frac{|b - x|}{|b - y|} \cdot \frac{|d - y|}{|d - z|} = 1$$

and

$$\frac{|d - x|}{|d - y|} \cdot \frac{|f - y|}{|f - z|} = \frac{|a' - x|}{|a' - z|}.$$

This gives

$$\frac{|x - a|}{|a - c|} \cdot \frac{|y - c|}{|z - c|} = \frac{|x - a'|}{|z - a'|} \geq \frac{|x - a|}{|z - a|},$$

and the inequality is strict unless $a = a'$. This inequality is equivalent to the triangle inequality for the Funk metric³. \square

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³Observe that the argument shows that the inequality is strict for all x, y, z unless $\partial\Omega$ contains a Euclidean segment; compare with Theorem 7.1.

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