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The L_{pq}-Cohomology of SOL[(*)]

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RÉSUMÉ. — On prouve un résultat de non-annulation de la cohomologie non réduite du groupe de Lie SOL.

ABSTRACT. — We prove a non vanishing result for the unreduced L_{pq}-cohomology of the Lie group SOL.

1. Introduction

SOL is the three dimensional Lie group of 3×3 real matrices of the form

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} .$$

This is a solvable and unimodular group; it is diffeomorphic to $\mathbb{R}^3$ (with coordinates x, y, z). A left invariant Riemannian metric is $\mathrm{d}s^2 = e^{-2z}\,\mathrm{d}x^2 + e^{2z}\,\mathrm{d}y^2 + \mathrm{d}z^2$; its volume measure is given by $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and is bi-invariant. For more information about the geometry of this group, see [9].

Let us recall the definition of the unreduced L_{pq}-cohomology groups, let $(M, \mathrm{d}s^2)$ be a complete connected Riemannian manifold of dimension n. We

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note $L^p(M,\Lambda^k)$ the space of differential forms of degree k with measurable coefficients on M such that

$$\|\theta\|_p := \left(\int_M |\theta|^p\right)^{1/p} < \infty\,.$$

We denote by $Z_p^k(M)$ the set of differential forms in $L^p(M,\Lambda^k)$ which are closed in the sense of current and by $B_{pq}^k(M)$ the set of forms $\theta \in L^p(M,\Lambda^k)$ such that there exists a form $\phi \in L^q(M,\Lambda^{k-1})$ with $\mathrm{d}\phi = \theta$. The unreduced L_{pq}-cohomology of $(M, \mathrm{d}s^2)$ is by definition the quotient

$$H_{pq}^k(M) := Z_p^k(M)/B_{pq}^k(M)\,.$$

Other papers dealing with L_{pq} cohomology are [2], [3], [8] and [10]. The goal of this paper is to prove the following result about the unreduced L_{pq}-cohomology of SOL.

THEOREM 1.— *We have* $\dim\big(H_{pq}^2(\mathrm{SOL})\big) = \infty$ *for every* $1 < p,\ q < \infty$.

2. Auxiliary results

The main ingredient in the proof of Theorem 1 is the next proposition (which is a kind of duality argument in L_{pq}-cohomology).

PROPOSITION 2.1.— *Let* $\alpha \in Z_p^k(M)$, *and suppose that for every* $\epsilon > 0$, *there exists a form*

$$\gamma = \gamma_\epsilon \in L^{p'}(M,\Lambda^{n-k}) \cap L^{q'}(M,\Lambda^{n-k})$$

such that

$$\|\mathrm{d}\gamma\|_{q'} \le \epsilon \quad \textit{and} \quad \int_M \gamma \wedge \alpha \ge a$$

where $a > 0$ *is independent of* ϵ *(here* $1/q + 1/q' = 1/p + 1/p' = 1$*). Then* $\alpha \notin B_{pq}^k(M)$ *(in particular,* $H_{pq}^k(M) \neq 0$*).*

For the proof, we will need the following integration-by-part lemma (for differential forms of class C^1, this lemma is due to Gaffney [1]).

LEMMA 2.1.— *Let* M *be a complete Riemannian manifold. Let* $\beta \in L^q(M,\Lambda^{k-1})$ *be such that* $\mathrm{d}\beta \in L^p(M,\Lambda^k)$, *and* $\gamma \in L^{p'}(M,\Lambda^{n-k}) \cap$

$L^{q'}(M,\Lambda^{n-k})$ be such that $\mathrm{d}\gamma \in L^{q'}(M,\Lambda^{n-k+1})$ *where* $1/p + 1/p' = 1/q + 1/q' = 1$.

Then $\mathrm{d}\gamma \wedge \beta$ *and* $\gamma \wedge \mathrm{d}\beta$ *are integrable and*

$$\int_M \gamma \wedge \mathrm{d}\beta = (-1)^{n-k+1} \int_M \mathrm{d}\gamma \wedge \beta .$$

Proof.— By Hölder's inequality, the forms $\mathrm{d}\gamma \wedge \beta$, $\gamma \wedge \mathrm{d}\beta$ and $\gamma \wedge \beta$ all belong to L^1. For smooth forms β with compact support, the lemma is true by definition of the weak exterior differential (of γ).

Assume first that β is smooth with non compact support and satisfies the conditions of the lemma. On a complete Riemannian manifold M, we can constuct a sequence $\{\lambda_i\}$ of smooth functions with compact support such that $\lambda_i(x) \to 1$ uniformly on every compact subset, $0 \leq \lambda_i(x) \leq 1$ and $\left|\mathrm{d}\lambda_i\right|_x \leq 1$ for all $x \in M$. The forms $\lambda_i\,\beta$ have compact support, thus the lemma holds for each $\lambda_i\,\beta$. Since

$$\left|\gamma \wedge \mathrm{d}(\lambda_i\beta) + (-1)^{n-k}\,\mathrm{d}\gamma \wedge (\lambda_i\beta)\right| \leq \left|\mathrm{d}\gamma \wedge \beta\right| + \left|\gamma \wedge \mathrm{d}\beta\right| + \left|\gamma \wedge \beta\right| \in L^1 ,$$

we can apply Lebesgue's dominated convergence theorem. Thus we have

$$\int_M \left(\gamma \wedge \mathrm{d}\beta + (-1)^{n-k}\,\mathrm{d}\gamma \wedge \beta\right) =$$

$$= \lim_{i\to\infty} \int_M \left(\gamma \wedge\, \mathrm{d}(\lambda_i\beta) + (-1)^{n-k}\,\mathrm{d}\gamma \wedge (\lambda_i\beta)\right) = 0 .$$

Finally, for any $\beta \in L^q(M,\Lambda^{k-1})$ with $\mathrm{d}\beta \in L^p(M,\Lambda^k)$, we can construct a sequence β_j of smooth forms such that $\beta_j \to \beta$ in L^p-topology and $\mathrm{d}\beta_j \to \mathrm{d}\beta$ in L^p-topology (see Corollary 1 of [4]). Thus the same limiting process proves the lemma in all its generality. □

Proof of Proposition 2.1.— Suppose that $\alpha = \mathrm{d}\beta$ for some $\beta \in L^q(M,\Lambda^{k-1})$. We have by Lemma 1,

$$\int_M \gamma \wedge \alpha = \int_M \gamma \wedge\, \mathrm{d}\beta = (-1)^{n-k+1} \int_M \mathrm{d}\gamma \wedge \beta .$$

Using Hölder's inequality, we get

$$a \leq \int_M \gamma \wedge \alpha \leq \left|\int_M \mathrm{d}\gamma \wedge \beta\right| \leq \left\|\mathrm{d}\gamma\right\|_{q'} \left\|\beta\right\|_q \leq \epsilon \left\|\beta\right\|_q .$$

This is impossible since $\epsilon > 0$ is arbitrary. □

Proposition 2.1 can be completed in the following way.

LEMMA 2.2.— *Let $\alpha_1, \alpha_2, \ldots, \alpha_r \in Z_p^k(M)$ and suppose that we can find pairwise disjoint sets $S_i \subset M$ such that for every $\varepsilon > 0$ there exists $\gamma_i = \gamma_{i,\varepsilon} \in L^{p'}(M, \Lambda^{n-k}) \cap L^{q'}(M, \Lambda^{n-k})$ with $\operatorname{supp}(\alpha_i) \cup \operatorname{supp}(\gamma_i) \subset S_i$ and such that $\|\mathrm{d}\gamma_i\|_{q'} \leq \epsilon$ and $\int_M \gamma_i \wedge \alpha_i \geq a$ where $a > 0$ is independent of ϵ and i. Then $[\alpha_1], [\alpha_2], \ldots, [\alpha_r]$ are linearly independent elements of $H_{p,q}^k(M)$.*

Proof.— Choose $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, r$, and set $\alpha = \sum \lambda_i \alpha_i$ and $\gamma = \gamma_\varepsilon = \sum \lambda_i \gamma_i$. The assumption on the supports of these forms implies that

$$\int_M \alpha \wedge \gamma = \sum_i \lambda_i^2 \int_M \alpha_i \wedge \gamma_i \geq a \sum_i \lambda_i^2 .$$

This sum vanishes if and only if all $\lambda_i = 0$. Since $\gamma \in L^{p'} \cap L^{q'}$ and

$$\|\mathrm{d}\gamma\|_{q'} \leq \sum_i |\lambda_i| \, \|\mathrm{d}\gamma_i\|_{q'} \leq \varepsilon \sum_i |\lambda_i|$$

we can deduce from Proposition 1 that $\sum \lambda_i [\alpha_i] = [\alpha] \neq 0 \in H_{p,q}^k(M)$ unless all $\lambda_i = 0$. □

For all $x_0 \in \mathbb{R}$ the surface

$$\mathcal{H}_{x_0} := \{(x, y, z) \in \mathrm{SOL} \mid x = x_0\} \subset \mathrm{SOL}$$

is a totally geodesic surface isometric to the hyperbolic plane $\mathbb{H}^2$. In particular a function $f : \mathrm{SOL} \to \mathbb{R}$ which is invariant under all x-translations (i.e., $f = f(y, z)$) can be seen as a function on the hyperbolic plane.

LEMMA 2.3.— *There exists two non negative smooth functions f and g on $\mathbb{H}^2 \simeq \mathcal{H}_{x_0}$ such that:*

(1) $f(y, z) = g(y, z) = 0$ if $z \leq 0$ or $|y| \geq 1$;

(2) $\mathrm{d}f$ and $\mathrm{d}g \in L^r(\mathbb{H}^2, \Lambda^1)$ for any $1 < r \leq \infty$;

(3) the support of $\mathrm{d}f \wedge \mathrm{d}g$ *is contained in* $\{(y, z) \mid |y| \leq 1\,,\ 0 \leq z \leq 1\}$ *and* $\mathrm{d}f \wedge \mathrm{d}g \geq 0$;

(4) $\displaystyle\iint_{\mathbb{H}^2} \mathrm{d}f \wedge \mathrm{d}g = 1$;

(5) $\partial f/\partial y$ *and* $\partial g/\partial y \in L^\infty(\mathbb{H}^2)$, *and* $\partial f/\partial z$, $\partial g/\partial z$ *have compact support.*

Remark.— The forms $\mathrm{d}f$ and $\mathrm{d}g$ cannot have compact support, otherwise, by Stokes theorem, we would have

$$\int_{\mathbb{H}^2} \mathrm{d}f \wedge \mathrm{d}g = 0\,.$$

Proof.— Choose non negative smooth functions h_1, h_2 and $k : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $h_i(y) = 0$ if $|y| \geq 1$, $h_1'(y)h_2(y) \geq 0$ and $h_1(y)h_2'(y) \leq 0$ for all y;

(ii) the function $\big(h_1'(y)h_2(y) - h_1(y)h_2'(y)\big)$ has non empty support;

(iii) $k'(z) \geq 0$ for all z, furthermore

$$k(z) = \begin{cases} 1 & \text{if } z \geq 1 \\ 0 & \text{if } z \leq 0. \end{cases}$$

We set $f(y, z) := h_1(y)k(z)$ and $g(y, z) := h_2(y)k(z)$. Property (1) of the lemma is clear. We prove (3) (i.e., that $\mathrm{d}f \in L^r$ for any $1 < r \leq \infty$), we have

$$\mathrm{d}f = h_1(y)k'(z)\,\mathrm{d}z + k(z)h_1'(y)\,\mathrm{d}y\,.$$

The first term $h_1(y)k'(z)\,\mathrm{d}z$ has compact support, and the second term $k(z)h_1'(y)\,\mathrm{d}y$ has its support in the infinite rectangle $Q = \{|y| \leq 1\,, z \geq 0\}$.

Choose $D < \infty$ such that $\big|k(z)h_1'(y)\big| \leq D$ on Ω. We have

$$\big|k(z)h_1'(y)\,\mathrm{d}y\big| \leq D\big|\mathrm{d}y\big| = D\,e^{-z}\,,$$

thus, since the element of area of $\mathbb{H}^2$ is $\mathrm{d}A = e^z\,\mathrm{d}y\,\mathrm{d}z$, we have

$$\begin{aligned} \int_{\mathbb{H}^2} \big|k(z)h_1'(y)\,\mathrm{d}y\big|^r\,\mathrm{d}A &\leq D^r \int_Q e^{-rz}\,e^z\,\mathrm{d}y\,\mathrm{d}z \\ &\leq 2D^r \int_0^\infty e^{(1-r)z}\,\mathrm{d}z < \infty\,, \end{aligned}$$

from which one gets $\mathrm{d}f \in L^r$.

Now observe that

$$\mathrm{d}f \wedge \mathrm{d}g = \big(k(z)k'(z)\big)\big(h_1'(y)h_2(y) - h_1(y)h_2'(y)\big)\,\mathrm{d}y \wedge \mathrm{d}z\,,$$

hence the property (3) follows from the construction of h_1, h_2 and k.

Property (4) is only a normalisation, and property (5) is easy to check. □

The following is a vanishing result for some kind of "anisotropic weighted capacity".

LEMMA 2.4.— *Given any numbers δ and q' such that $1 < q' < \infty$ and $0 < \delta < (1/2)(q'-1)$, we can construct a family of Lipschitz functions $\psi_t = \psi_t(x,z)$, $t \geq 1$, on $\mathbb{R}^2$ such that:*

(i) $0 \leq \psi_t \leq 1$, $\operatorname{supp}\psi_t \subset \{(x,z) \mid x^2 + |z|^{2s} \leq 2t\}$, $\psi_t(x,z) = 1$ *if* $x^2 + |z|^{2s} \leq t$;

(ii) $\displaystyle\iint_{z>0}\left(\left|\frac{\partial\psi_t}{\partial x}\right|^{q'} + \left|\frac{\partial\psi_t}{\partial z}\,e^{-z}\right|^{q'}\right)\mathrm{d}x\,\mathrm{d}z \leq Ct^{-\delta}$,

where the constant $C = C(\delta)$ is independent of t.

Proof.— We first choose some number $s > 0$ so large that $(s+1-q's)/2s < -\delta$ and set $\rho(x,z) := x^2 + |z|^{2s}$. We now define $\psi_t : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\psi_t(x,z) = \begin{cases} 1 & \text{if } \rho(x,z) \leq t \\ \dfrac{\log(2t) - \log\big(\rho(x,z)\big)}{\log(2)} & \text{if } t \leq \rho(x,z) \leq 2t \\ 0 & \text{if } \rho(x,z) \geq 2t\,. \end{cases}$$

We will prove that

$$\iint_{z>0}\left(\left|\frac{\partial\psi_t}{\partial x}\right|^{q'} + \left|\frac{\partial\psi_t}{\partial z}\,e^{-z}\right|^{q'}\right)\mathrm{d}x\,\mathrm{d}z \leq Ct^{(s+1-q')s/2s}\,, \qquad (2.1)$$

where the constant C is independent of t.

It will be convenient to introduce new variables $X = x/\sqrt{t}$ and $Z = z^s/\sqrt{t}$ (we assume $z \geq 0$). We have

$$x = t^{1/2}X \quad \text{and} \quad z = t^{1/2s}Z^{1/s}$$

thus $\rho = t\,(X^2 + Z^2)$. Let us set $\Psi_t(X, Z) := \psi_t(x, z)$, then

$$\Psi_t(X, Z) = \begin{cases} 1 & \text{if } (X^2 + Z^2) \leq 1 \\ \dfrac{\log(2) - \log(X^2 + Z^2)}{\log(2)} & \text{if } 1 \leq (X^2 + Z^2) \leq 2 \\ 0 & \text{if } (X^2 + Z^2) \geq 2\,. \end{cases}$$

In particular, Ψ_t is independent of t (and will henceforth be written as Ψ) and its support is the annulus $A = \{(X, Z) \mid 1 \leq (X^2 + Z^2) \leq 2\}$.

The partial derivatives of ψ_t may be written as

$$\frac{\partial \psi_t}{\partial x} = t^{-1/2}\, \frac{\partial \Psi}{\partial X} \quad \text{and} \quad \frac{\partial \psi_t}{\partial z} = s t^{-1/2} z^{s-1}\, \frac{\partial \Psi}{\partial Z}\,. \tag{2.2}$$

The maximum of the function $z \rightarrow s z^{s-1} e^{-z}$ on $0 \leq z < \infty$ is achieved at $z = (s-1)$, hence

$$\left|z\right|^{s-1} e^{-z} \leq c_1 := s\big(s-1\big)^{s-1} e^{-s+1} \tag{2.3}$$

for all $z \geq 0$. From the second equation in (2.2) and (2.3), we conclude that

$$e^{-z} \left|\frac{\partial \psi_t}{\partial z}\right| \leq c_1\, t^{-1/2} \left|\frac{\partial \Psi}{\partial Z}\right|\,. \tag{2.4}$$

We see from the first equation in (2.2) and the inequality (2.4) that

$$\left(\left|\frac{\partial \psi_t}{\partial x}\right|^{q'} + \left|\frac{\partial \psi_t}{\partial z}\, e^{-z}\right|^{q'} \right) \leq t^{-q'/2} \left(\left|\frac{\partial \Psi}{\partial X}\right|^{q'} + \left|\frac{\partial \Psi}{\partial Z}\, c_1\right|^{q'} \right)\,.$$

Since

$$\mathrm{d}x\, \mathrm{d}z = \frac{1}{s}\, t^{(s+1)/2s}\, Z^{(s-1)/s}\, \mathrm{d}x\, \mathrm{d}z$$

we obtain (2.1) with

$$C = \iint_{A^+} \left(\left|\frac{\partial \Psi}{\partial X}\right|^{q'} + \left|\frac{\partial \Psi}{\partial Z}\, c_1\right|^{q'} \right) \frac{Z^{(s-1)/s}}{s}\, \mathrm{d}x\, \mathrm{d}z < \infty\,,$$

where the domain of integration is the half annulus $A^+ = \{(X, Z) \mid Z \geq 0 \text{ and } 1 \leq (X^2 + Z^2) \leq 2\}$. $\square$

3. Proof of the main theorem

The proof is technical and will be divided in five steps: we first fix some arbitrarily $\epsilon > 0$.

Step 1. We construct a closed 2-form $\alpha \in Z_p^2(\mathrm{SOL})$

We start by choosing a pair of functions $f = f(y,z)$ and $g = g(y,z)$ with the properties of Lemma 2.3. We then choose a smooth function $\lambda : \mathbb{R} \to \mathbb{R}$ such that

$$\lambda(u) = \begin{cases} 0 & \text{if } u \leq -1 \\ 1 & \text{if } u \geq 1 \\ 0 \leq \lambda'(u) \leq 1 & \text{for all } u \in \mathbb{R}. \end{cases}$$

Then we set $\varphi(x,z) = \lambda(e^{-z}x)$, and note that

$$\mathrm{d}\varphi = \left(\lambda'(e^{-z}\,x)\,e^{-z}\right)(\mathrm{d}x - x\,\mathrm{d}z)\,.$$

We finally define

$$\begin{aligned} \alpha &:= \mathrm{d}\varphi \wedge \mathrm{d}f = \mathrm{d}(\varphi\,\mathrm{d}f) \\ &= \left(\lambda'(e^{-z}x)\,e^{-z}\right)\left(\frac{\partial f}{\partial y}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}x \wedge \mathrm{d}z + x\frac{\partial f}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}z\right). \end{aligned}$$

Observe that $\mathrm{d}\alpha = 0$ and

- $\operatorname{supp}(\alpha) \subset \Omega := \left\{(x,y,z) \in \mathrm{SOL} \;\middle|\; |y| \leq 1\,,\ z > 0\,,\ |x| < e^{z}\right\}$;
- $\lambda'(e^{-z}\,x)\,\dfrac{\partial f}{\partial z}$ has compact support;
- $\left|\dfrac{\partial f}{\partial y}\right| \left|\mathrm{d}x \wedge \mathrm{d}y\right|$ is bounded (since $\left|\mathrm{d}x \wedge \mathrm{d}y\right| = 1$ and $\partial f/\partial y$ is bounded);
- $\left|\dfrac{\partial f}{\partial y}\right| |x| \left|\mathrm{d}y \wedge \mathrm{d}z\right|$ is bounded (since $\left|\mathrm{d}y \wedge \mathrm{d}z\right| = e^{z}$ and $|x| \leq e^{-z}$ on Ω).

From these estimates and $0 \leq \lambda' \leq 1$, we deduce easily that $|\alpha| \leq \mathrm{const}\,e^{-z}$ on Ω and

$$\int_\Omega |\alpha|^p \leq \mathrm{const} \int_0^\infty e^{(1-p)z}\,\mathrm{d}z < \infty$$

for all $1 < p \leq \infty$. It follows that $\alpha \in Z_p^2(\mathrm{SOL})$.

Step 2. We construct a family of almost closed forms $\gamma_t \in L^r(\mathrm{SOL}, \Lambda^1)$

Fix $0 < \delta < (1/2)(q' - 1)$ and choose a function $\psi_t = \psi_t(x, z)$ as in Lemma 2.4. Define $\gamma_t := \psi_t(x, z)\, \mathrm{d}g$. In order to show that $\gamma_t \in L^r(\mathrm{SOL}, \Lambda^1)$, observe that γ_t has its support contained in the box

$$Q_t := \left\{(x, y, z) \in \mathrm{SOL} \;\middle|\; |y| \leq 1\,,\ z \geq 0\,,\ |x| \leq \sqrt{2t}\right\}.$$

Recall that $0 \leq \psi_t(x, z) \leq 1$ and the volume form of SOL is $\mathrm{d(vol)} = \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$. We thus have

$$\begin{aligned}
\left\|\gamma_t\right\|_r^r &= \int_{x=-\sqrt{2t}}^{\sqrt{2t}} \int_{y=-1}^{1} \int_{z=0}^{\infty} \left|\psi_t(x, z)\right|^r \left|\mathrm{d}g\right|^r \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\
&\leq \int_{x=-\sqrt{2t}}^{\sqrt{2t}} \mathrm{d}x \int_{y=-1}^{1} \int_{z=0}^{\infty} \left|\mathrm{d}g\right|^r \mathrm{d}y\, \mathrm{d}z \\
&\leq \left(2\sqrt{2t}\right) \int_{y=-1}^{1} \int_{z=0}^{\infty} \left|\mathrm{d}g\right|^r e^z\, \mathrm{d}y\, \mathrm{d}z\,.
\end{aligned}$$

By Lemma 2.3, we know that

$$\int_{\mathbb{R}^2} \left|\mathrm{d}g\right|^r e^z\, \mathrm{d}y\, \mathrm{d}z < \infty \quad \text{for any } 1 < r < \infty\,,$$

from which one gets an estimates $\left\|\gamma_t\right\|_r \leq C_1(r)\, t^{1/2r}$. In particular

$$\gamma_t \in \bigcap_{1<r<\infty} L^r(\mathrm{SOL}, \Lambda^1)\,. \tag{3.1}$$

Step 3. We estimate $\left\|\mathrm{d}\gamma_t\right\|_{q'}$

We have

$$\mathrm{d}\gamma_t = \frac{\partial \psi_t}{\partial x} \frac{\partial g}{\partial y}\, \mathrm{d}x \wedge \mathrm{d}y + \frac{\partial \psi_t}{\partial x} \frac{\partial g}{\partial z}\, \mathrm{d}x \wedge \mathrm{d}z + \frac{\partial \psi_t}{\partial z} \frac{\partial g}{\partial y}\, \mathrm{d}z \wedge \mathrm{d}y$$

and

$$\left|\mathrm{d}x \wedge \mathrm{d}y\right| = 1\,, \quad \left|\mathrm{d}x \wedge \mathrm{d}z\right| = e^z\,, \quad \left|\mathrm{d}z \wedge \mathrm{d}y\right| = e^{-z}\,.$$

Recall that $\partial g/\partial y$ is bounded, $\partial g/\partial z$ has compact support and $\mathrm{d}\gamma_t$ has its support in the region Q_t. Thus

$$\left|\mathrm{d}\gamma_t\right| \leq C_2 \left(\left|\frac{\partial \psi_t}{\partial x}\right| + \left|\frac{\partial \psi_t}{\partial z}\right| e^{-z} \right).$$

Since $0 < \delta < (1/2)(q'-1)$, Lemma 2.4 implies

$$\left\|\mathrm{d}\gamma_t\right\|_{q'} \leq C_3\, t^{-\delta/q'} . \tag{3.2}$$

Step 4. We estimate the integral of $\alpha \wedge \gamma_t$

Let

$$A_t := \int_{\mathrm{SOL}} \alpha \wedge \gamma_t .$$

We have

$$\alpha \wedge \gamma_t = \psi_t(x,z)\,\mathrm{d}\varphi \wedge \mathrm{d}f \wedge \mathrm{d}g = \left(\lambda'\left(e^{-z}x\right)e^{-z}\psi_t(x,z)\right)\mathrm{d}x \wedge \mathrm{d}f \wedge \mathrm{d}g$$

(since $\mathrm{d}z \wedge \mathrm{d}f \wedge \mathrm{d}g = 0$). By Lemma 2.3, $\mathrm{d}f \wedge \mathrm{d}g \geq 0$, and since $\lambda'\left(e^{-z}x\right) \geq 0$ we see that $\alpha \wedge \gamma_t$ is a non negative 3-form. In particular $A_t \geq \int_\Delta \alpha \wedge \gamma_t$ for every measurable subset $\Delta \subset \mathrm{SOL}$.

We set

$$\Delta_t := \left\{(x,y,z) \in \mathrm{SOL} \;\middle|\; |y| \leq 1\,,\; 0 \leq z \leq 1\,,\; |x| \leq \sqrt{t}\right\}.$$

Recall that if $t \geq 1$, $0 \leq z \leq 1$ and $|x| \leq \sqrt{t}$, then $\psi_t(x,z) = 1$, we thus get

$$A_t \geq \int_{\Delta_t} \alpha \wedge \gamma_t = \int_{y=-1}^{+1}\int_{z=0}^{1}\int_{x=-\sqrt{t}}^{\sqrt{t}} \lambda'(\,e^{-z}x)\,e^{-z}\,\mathrm{d}x \wedge \mathrm{d}f \wedge \mathrm{d}g .$$

Now set $u = e^{-z}x$, $\mathrm{d}u = e^{-z}\,\mathrm{d}x$, $u_0 = -e^{-z}\sqrt{t}$ and $u_1 = e^{-z}\sqrt{t}$. We have

$$\int_{x=-\sqrt{t}}^{\sqrt{t}} \lambda'\left(e^{-z}x\right)e^{-z}\,\mathrm{d}x = \int_{u_0}^{u_1} \lambda'(u)\,\mathrm{d}u = 1$$

if t is large enough (i.e., $e^{-1}\sqrt{t} \geq 1$). Thus

$$A_t \geq C_4 := \int_{y=-1}^{1}\int_{z=0}^{1} \mathrm{d}f \wedge \mathrm{d}g > 0 . \tag{3.3}$$

Observe that the constant C_4 is positive and independent of t (in fact, using equation (4) of Lemma 2.3 and (i) of Lemma 2.4, we see that $A_t \to 1$ as $t \to \infty$).

Step 5. Recapitulation

Let us summarize the previous estimates (3.1), (3.2) and (3.3):

$$\|\gamma_t\|_{p'} + \|\gamma_t\|_{q'} < \infty\,, \quad \|d\gamma_t\|_{q'} \leq C_3\, t^{-\delta/q'} \quad \text{and} \quad \int_{\mathrm{SOL}} \alpha \wedge \gamma_t \geq C_4 > 0\,.$$

If we let $t \to \infty$ and apply Proposition 2.1, we obtain $\alpha \notin B^2_{pq}(M)$.

By the construction

$$S_t := \operatorname{supp}(\alpha) \cup \operatorname{supp}(\gamma_t) \subset Q := \big\{(x,y,z) \bigm| |y| \leq 1\,,\ 0 \leq z\big\}\,.$$

Using the group of isometries $T : (x,y,z) \to (x, y+2k, z)$, $k \in \mathbb{Z}$, we can produce an infinite family of forms $\alpha_i \in Z^k_p(\mathrm{SOL})$ satisfying the hypothesis of Lemma 2.2. Therefore

$$\dim H^2_{pq}(\mathrm{SOL}) = \infty$$

for all $1 < p,\ q < \infty$. The proof is complete □

4. Final remark

The above proof of Theorem 1 is only true for unreduced cohomology. In fact, the work of Jeff Cheeger and Mikhael Gromov gives us the following result in the reduced case (for $p = q = 2$).

THEOREM 2.— *The reduced L_2-cohomology of* SOL *is trivial.*

Proof.— The Lie group SOL admits uniform lattices (i.e., discrete cocompact subgroups), see [11] for explicit constructions. The result thus follows from [5], [6] and [7] since every lattice in SOL is amenable. □

Acknowledgments

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