$L_{q,p}$-Cohomology of Riemannian Manifolds with Negative Curvature

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Dedicated to the memory of Sergey L'vovich Sobolev

Abstract We consider the $L_{q,p}$-cohomology of a complete simply connected Riemannian manifold (M, g) with pinched negative curvature. The connection between the $L_{q,p}$-cohomology of (M, g) and Sobolev inequalities for differential forms on (M, g) was established by the authors in the previous publications.

1 Introduction

In [5], we established a connection between Sobolev inequalities for differential forms on a Riemannian manifold (M, g) and an invariant, called the $L_{q,p}$-*cohomology* $\left(H^k_{q,p}(M)\right)$ of the manifold (M, g). In this paper, we prove nonvanishing results for the $L_{q,p}$-cohomology of simply connected complete manifolds with negative curvature.

1.1 $L_{q,p}$-cohomology and Sobolev inequalities

To define the $L_{q,p}$-cohomology of a Riemannian manifold (M, g), we first recall the notion of the *weak exterior differential* of a locally integrable

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differential form. Denote by $C_c^\infty(M, \Lambda^k)$ the space of smooth differential forms of degree k with compact support on M.

Definition. A form $\theta \in L^1_{loc}(M, \Lambda^k)$ is the *weak exterior differential* of a form $\varphi \in L^1_{loc}(M, \Lambda^{k-1})$ and one writes $d\varphi = \theta$ if for every $\omega \in C_c^\infty(M, \Lambda^{n-k})$

$$\int_M \theta \wedge \omega = (-1)^k \int_M \varphi \wedge d\omega.$$

The Sobolev space $W^{1,p}(M, \Lambda^k)$ of differential k-forms is defined as the space of k-forms φ in $L^p(M)$ such that $d\varphi \in L^p(M)$ and $d(*\varphi) \in L^p(M)$, where $* : \Lambda^k \to \Lambda^{n-k}$ is the Hodge star homomorphism. In this paper, we are interested in a different "Sobolev type" space of differential forms, denoted by $\Omega^k_{q,p}(M)$; namely, the space of all k-forms φ in $L^q(M)$ such that $d\varphi \in L^p(M)$ $(1 \leqslant q, p \leqslant \infty)$. It is a Banach space relative to the graph norm

$$\|\omega\|_{\Omega^k_{q,p}} := \|\omega\|_{L^q} + \|d\omega\|_{L^p}\,. \tag{1.1}$$

When $k = 0$ and $q = p$, the space $\Omega^0_{p,p}(M)$ coincides with the classical Sobolev space $W^1_p(M)$ of functions in L^p with gradient in L^p. Note that a more general space $\Omega^0_{q,p}(M)$ was considered in [10] in the context of embedding theorems and Sobolev inequalities.

To define the $L_{q,p}$–cohomology of (M, g), we introduce the space of weakly closed forms

$$Z^k_p(M) = \{\omega \in L^p(M, \Lambda^k) \mid d\omega = 0\}$$

and the space of differential forms in $L^p(M)$ having a primitive in $L^q(M)$

$$B^k_{q,p}(M) = d(\Omega^{k-1}_{q,p}).$$

Note that $Z^k_p(M) \subset L^p(M, \Lambda^k)$ is always a closed subspace, but this is, in general, not the case of $B^k_{q,p}(M)$, and we denote by $\overline{B}^k_{q,p}(M)$ its closure in the L^p-topology. We also note that $\overline{B}^k_{q,p}(M) \subset Z^k_p(M)$ (by continuity and $d \circ d = 0$). Thus,

$$B^k_{q,p}(M) \subset \overline{B}^k_{q,p}(M) \subset Z^k_p(M) = \overline{Z}^k_p(M) \subset L^p(M, \Lambda^k).$$

Definition. The $L_{q,p}$-*cohomology* of (M, g) (where $1 \leqslant p, q \leqslant \infty$) is

$$H^k_{q,p}(M) := Z^k_p(M)/B^k_{q,p}(M)$$

and the *reduced* $L_{q,p}$-*cohomology* of (M, g) is

$$\overline{H}^k_{q,p}(M) := Z^k_p(M)/\overline{B}^k_{q,p}(M)\,.$$

The reduced cohomology is a Banach space, whereas the unreduced cohomology is a Banach space if and only if it coincides with the reduced one.

In [5, Theorem 6.1], we established the following connection between Sobolev inequalities for differential forms on a Riemannian manifold (M, g) and the $L_{q,p}$–cohomology of (M, g).

Theorem 1.1. $H^k_{q,p}(M, g) = 0$ *if and only if there exists a constant* $C < \infty$ *such that for any closed* p*-integrable differential form* ω *of degree* k *there exists a differential form* θ *of degree* $k-1$ *such that* $d\theta = \omega$ *and*

$$\|\theta\|_{L^q} \leqslant C \|\omega\|_{L^p} .$$

Suppose that $k = 1$. If M is simply connected (or, more generally, $H^1_{\text{deRham}}(M) = 0$), then any $\omega \in Z^1_p(M)$ has a primitive locally integrable function f, $df = \omega$. This means that for simply connected manifolds the space $Z^1_p(M)$ coincides with the seminormed Sobolev space $L^1_p(M)$, $\|f\|_{L^1_p(M)} := \|df\|_{L^p(M)}$. Then Theorem 1.1 reads as follows.

Corollary 1.2. *Suppose that* (M, g) *is a simply connected Riemannian manifold. Then* $H^1_{q,p}(M, g) = 0$ *if and only if there exists a constant* $C < \infty$ *depending only on* M*,* (q, p)*, and a constant* $a_f < \infty$ *depending also on* $f \in L^1_p(M, g)$ *such that*

$$\|f - a_f\|_{L^q} \leqslant C \|df\|_{L^p}$$

for any $f \in L^1_p(M, g)$*.*

In this paper, we prove nonvanishing results for the $L_{q,p}$-cohomology of simply connected complete manifolds with negative curvature, which concerns the nonexistence of Sobolev inequalities for such pairs (q, p).

1.2 Statement of the main result

The main goal of the present paper is to prove the following nonvanishing result for the $L_{q,p}$-cohomology of simply connected complete manifolds with negative curvature.

Theorem 1.3. *Let* (M, g) *be an* n*-dimensional Cartan–Hadamard manifold*[1] *with sectional curvature* $K \leqslant -1$ *and Ricci curvature* $\operatorname{Ric} \geqslant -(1+\varepsilon)^2(n-1)$*.*

(A) *Assume that*

$$\frac{1+\varepsilon}{p} < \frac{k}{n-1} \quad \text{and} \quad \frac{k-1}{n-1} + \varepsilon < \frac{1+\varepsilon}{q}.$$

[1] Recall that a *Cartan-Hadamard* manifold is a complete simply connected Riemannian manifold of nonpositive sectional curvature.

Then $H^k_{q,p}(M) \neq 0$.

(B) *If, in addition,*

$$\frac{1+\varepsilon}{p} < \frac{k}{n-1} \quad \textit{and} \quad \frac{k-1}{n-1} + \varepsilon < \min\left\{\frac{1+\varepsilon}{q}, \frac{1+\varepsilon}{p}\right\},$$

then $\overline{H}^k_{q,p}(M) \neq 0$.

Theorem 1.3, together with Theorem 1.1, has the following (negative) consequence regarding Sobolev inequalities for differential forms.

Corollary 1.4. *Let* (M, g) *be a Cartan–Hadamard manifold as above. If* q *and* p *satisfy assumption* (A) *of Theorem* 1.3*, then there is no finite constant* C *such that any smooth closed* k*-form* ω *on* M *admits a primitive* θ *such that* $d\theta = \omega$ *and*

$$\|\theta\|_{L^q(M)} \leqslant C\,\|\omega\|_{L^p(M)}.$$

The proof of Theorem 1.3 is based on the duality principle established in [5] and a comparison argument inspired by Chapt. 8 of the book [8] by Gromov. Now, we discuss some particular cases.

- If M is the hyperbolic plane $\mathbb{H}^2$ $(n = 2, \varepsilon = 0)$, Theorem 1.3 says that $\overline{H}^1_{q,p}(\mathbb{H}^2) \neq 0$ for any $q, p \in (1, \infty)$; and another proof can be found in [5, Theorem 10.1].

- For $q = p$ Theorem 1.3 says that $\overline{H}^k_{p,p}(M) \neq 0$ provided that

$$\frac{k-1}{n-1} + \varepsilon < \frac{1+\varepsilon}{p} < \frac{k}{n-1}; \tag{1.2}$$

  this result was already known [8, p. 244]. The inequalities (1.2) can also be written in terms of k as follows:

$$\frac{n-\tau}{p} < k < \frac{n-\tau}{p} + \tau$$

  with $\tau = 1 - \varepsilon(n-1)$.

- By contrast, Pansu [11, Theorem A] proved that $H^k_{p,p}(M) = 0$ if the sectional curvature satisfies

$$-(1+\varepsilon)^2 \leqslant K \leqslant -1 \quad \text{and} \quad (1+\varepsilon)\,p \leqslant \frac{n-1}{k} + \varepsilon.$$

- A Poincaré duality for the reduced L^p-cohomology was proved in [2], it says that for a complete Riemannian manifold

$$\overline{H}^k_{p,p}(M) = \overline{H}^{n-k}_{p',p'}(M)$$

with $p' = p/(p-1)$. This duality, together with the result of Pansu and some algebraic computations, implies that for a manifold M as in Theorem 1.3 we also have $\overline{H}^k_{p,p}(M) = 0$ if

$$p \geqslant \frac{(n-1) + \varepsilon(n-k)}{k-1}.$$

- Consider, for example, the hyperbolic space $\mathbb{H}^n$. This space is a Cartan–Hadamard manifold with constant sectional curvature $K \equiv -1$, and the reduced cohomology is known. Indeed, we have $\varepsilon = 0$ and, by the above three inequalities, $\overline{H}^k_{p,p}(\mathbb{H}^n) \neq 0$ if and only if $p \in (\frac{n-1}{k}, \frac{n-1}{k-1})$ (or, equivalently, $\frac{n-1}{p} < k < \frac{n-1}{p} + 1$). This assertion also follows from the computation of the L_p-cohomology of warped cylinders in [3, 4].

For $\varepsilon > 0$ there is still a gap between vanishing and nonvanishing results for the $L_{p,p}$-cohomology. When $\varepsilon \geqslant \frac{1}{n-1}$, the estimate (1.2) no longer gives any information about the $L_{p,p}$-cohomology. Note, by contrast, that Theorem 1.3 always produces some nonvanishing $L_{q,p}$-cohomology.

2 Manifolds with Contraction onto the Closed Unit Ball

Theorem 2.1 below, inspired by [8], can be regarded as an application of the concept of almost duality in [5]. It will be used in the proof of Theorem 1.3. Recall that, by the Rademacher theorem, a Lipshitz map $f : M \to N$ is differentiable for almost all $x \in M$ and its differential df_x defines a homomorphism

$$\Lambda^k f_x : \Lambda^k(T_{fx}N) \to \Lambda^k(T_x M).$$

Denote by $|\Lambda^k f_x|$ the norm of this homomorphism.

Theorem 2.1. *Let (M, g) be a complete Riemannian manifold, and let $f : M \to \overline{\mathbb{B}}^n$ be a Lipschitz map such that*

$$|\Lambda^k f| \in L^p(M) \quad \textit{and} \quad |\Lambda^{n-k} f| \in L^{q'}(M),$$

where $\overline{\mathbb{B}}^n$ is the closed unit ball in $\mathbb{R}^n$ and $q' = q/(q-1)$. Assume that

$$f^*\omega \in L^1(M) \quad \textit{and} \quad \int\limits_M f^*\omega \neq 0,$$

where $\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is the standard volume form on $\overline{\mathbb{B}}^n$. Then $H^k_{q,p}(M) \neq 0$.

Furthermore, if $|\Lambda^{n-k} f| \in L^{p'}(M)$ for $p' = \frac{p}{p-1}$, then $\overline{H}^k_{q,p}(M) \neq 0$.

The proof of this theorem uses the following "almost duality" result.

Proposition 2.2. *Let (M, g) be a complete Riemannian manifold, and let $\alpha \in Z_p^k(M)$. Assume that there exists a closed $(n-k)$-form $\gamma \in Z_{q'}^{n-k}(M)$ for $q' = \frac{q}{q-1}$ such that $\gamma \wedge \alpha \in L^1(M)$ and*

$$\int_M \gamma \wedge \alpha \neq 0.$$

Then $H_{q,p}^k(M) \neq 0$. Furthermore, if $\gamma \in Z_{p'}^{n-k}(M) \cap Z_{q'}^{n-k}(M)$ for $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$, then $\overline{H}_{q,p}^k(M) \neq 0$.

This assertion is contained in [5, Propositions 8.4 and 8.5].

We also need some facts about locally Lipschitz differential forms.

Lemma 2.3. *For any locally Lipschitz functions $g, h_1, \dots, h_k : M \to \mathbb{R}$*

$$d(g\, dh_1 \wedge dh_2 \wedge \dots \wedge dh_k) = dg \wedge dh_1 \wedge dh_2 \wedge \dots \wedge dh_k$$

in the weak sense.

Denote by $\mathrm{Lip}^*(M)$ the algebra generated by locally Lipschitz functions and the wedge product. By Lemma 2.3, $\mathrm{Lip}^*(M)$ is a graded differential algebra. Elements of $\mathrm{Lip}^*(M)$ are referred to as *locally Lipschitz forms.*

Proposition 2.4. *For any locally Lipschitz map $f : M \to N$ between two Riemannian manifolds the pullback $f^*(\omega)$ of any locally Lipschitz form ω is a locally Lipschitz form and $d(f^*(\omega)) = f^*(d\omega)$.*

A proof of Lemma 2.3 and Proposition 2.4 can be found in [1] (see also [6] for some related results).

Proof of Theorem 2.1. We set $\omega' = dx_1 \wedge dx_2 \wedge \dots \wedge dx_k$ and $\omega'' = dx_{k+1} \wedge dx_2 \wedge \dots \wedge dx_n$. Using the inequality

$$|(f^*\omega)_x| \leqslant |\Lambda^k f| \cdot \left|\omega_{f(x)}\right|,$$

we find that

$$\|f^*\omega'\|_{L^p(M,\Lambda^k)} = \left(\int_M |(f^*\omega')_x|^p \, dx\right)^{\frac{1}{p}} \leqslant \left(\int_M \left(|\Lambda^k f|^p \cdot \left|\omega'_{f(x)}\right|^p\right) dx\right)^{\frac{1}{p}}$$
$$\leqslant \left\|\Lambda^k f\right\|_{L^p(M)} \|\omega'\|_{L^\infty(M,\Lambda^k)} < \infty.$$

We set $\alpha = f^*\omega'$. Since f is a Lipschitz map, α is a Lipshitz form and, by Proposition 2.4, we have $d\alpha = f^*d\omega' = 0$. The previous inequality says that $\alpha \in L^p(M, \Lambda^k)$. Thus, $\alpha \in Z_p^k(M)$. The same argument shows that $\gamma =\in Z_{q'}^{n-k}(M)$, where $\gamma = f^*\omega''$.

By assumption, $\alpha \wedge \gamma = f^*(\omega' \wedge \omega'') = f^*(\omega) \in L^1(M)$ and

$$\int\limits_M \gamma \wedge \alpha = \int\limits_M f^*\omega \neq 0.$$

Hence, by Proposition 2.2, we have $H_{q,p}^k(M) \neq 0$.

If, additionally, we assume that $\Lambda^{n-k} f_x \in L^{p'}(M)$ for $p' = \frac{p}{p-1}$, then $\gamma \in Z_{p'}^{n-k}(M)$ and, by the second part of Proposition 2.2, $\overline{H}_{q,p}^k(M) \neq 0$. □

The paper [7] contains other results relating the $L_{q,p}$-cohomology and classes of mappings.

3 Proof of the Main Result

Let (M, g) be a complete simply connected manifold of negative sectional curvature of dimension n. Fix a base point $o \in M$ and identify T_oM with $\mathbb{R}^n$ by a linear isometry. Then the exponential map $\exp_o : \mathbb{R}^n = T_oM \to M$ is a diffeomorphism, and we define a map $f : M \to \overline{\mathbb{B}}^n$, where $\overline{\mathbb{B}}^n \subset \mathbb{R}^n$ is the closed Euclidean unit ball, by the formula

$$f(x) = \begin{cases} \exp_o^{-1}(x) & \text{if } |\exp_o^{-1}(x)| \leqslant 1, \\ \dfrac{\exp_o^{-1}(x)}{|\exp_o^{-1}(x)|} & \text{if } |\exp_o^{-1}(x)| \geqslant 1. \end{cases}$$

Using the polar coordinates (r, u) on M, i.e., writing a point $x \in M$ as $x = \exp_o(r \cdot u)$ with $u \in \mathbb{S}^{n-1}$ and $r \in [0, \infty)$, we can write this map as $f(r, u) = \min(r, 1) \cdot u$. Since the exponential map is expanding, the map $f : M \to \overline{\mathbb{B}}^n$ is contracting and, in particular, is a Lipschitz map.

Recall that $\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is the volume form on $\overline{\mathbb{B}}^n$. It can also be written as $r^{n-1} dr \wedge d\sigma_0$, where $d\sigma_0$ is the volume form of the standard sphere $\mathbb{S}^{n-1}$. It follows that $f^*\omega = 0$ on the set $\{x \in M \mid d(o, x) > 1\}$ and $f^*\omega$ has compact support and is integrable. Denote by $U_1 = \{x \in M \mid d(o, x) < 1\}$ the Riemannian open unit ball in M. The restriction of f to U_1 is a diffeomorphism onto $\mathbb{B}^n$.

Therefore,

$$\int\limits_M f^*\omega = \int\limits_{U_1} f^*\omega = \int\limits_{\mathbb{B}^n} \omega = \text{Vol}\,(\mathbb{B}^n) > 0.$$

The next lemma implies that $|\Lambda^k f| \in L^p(M)$ if

$$\frac{1+\varepsilon}{p} < \frac{k}{n-1}$$

and $|\Lambda^{n-k} f| \in L^{q'}(M)$ if

$$\frac{1+\varepsilon}{q'} < \frac{n-k}{n-1}.$$

Note that the inequality

$$\frac{1+\varepsilon}{q'} < \frac{n-k}{n-1}$$

is equivalent to the inequality

$$\frac{k-1}{n-1} + \varepsilon < \frac{1+\varepsilon}{q}$$

since $q' = q/(q-1)$. Likewise, $|\Lambda^{n-k} f| \in L^{p'}(M)$ if

$$\frac{k-1}{n-1} + \varepsilon < \frac{1+\varepsilon}{p}.$$

In conclusion, the map f satisfies all the assumptions of Theorem 2.1, as soon as assumption (A) or (B) of Theorem 1.3 is fulfilled. The proof of Theorem 1.3 is complete.

Lemma 3.1. *The map* $f : M \to \overline{\mathbb{B}}^n$ *satisfies* $|\Lambda^m f| \in L^s(M)$ *if*

$$\frac{1+\varepsilon}{s} < \frac{m}{n-1}.$$

Proof. By the Gauss lemma from Riemannian geometry, we know that, in the polar coordinates, $M \simeq [0, \infty) \times \mathbb{S}^{n-1}/(\{0\} \times \mathbb{S}^{n-1})$, the Riemannian metric can be written as

$$g = dr^2 + g_r,$$

where g_r is a Riemannian metric on the sphere $\mathbb{S}^{n-1}$. The Rauch comparison theorem tells us that if the sectional curvature of g satisfies $K \leqslant -1$, then

$$g_r \leqslant (\sinh(r))^2 \, g_0, \tag{3.1}$$

where g_0 is the standard metric on the sphere $\mathbb{S}^{n-1}$ (see any textbook on Riemannian geometry, for example, [12, Sect. 6.2, Corollary 2.4] or [9, Corollary 4.6.1]). Using the fact that the Euclidean metric on $\mathbb{R}^n = T_oM$ is written in the polar coordinates as $ds^2 = dr^2 + r^2 g_0$ and taking into account the inequality (3.1), we find

$$|f^*(\theta)| \leqslant \frac{r}{\sinh(r)} |\theta|$$

for any covector $\theta \in T^*_{(r,u)}M$, orthogonal to dr. Since $f^*(dr)$ has compact support, we conclude that

$$|f^*(\varphi)| \leqslant \text{const} \left(\frac{r}{\sinh(r)}\right)^m |\varphi|$$

for any m-form $\varphi \in \Lambda^m(T^*_{(r,u)}M)$. In other words, we have the pointwise estimate

$$|\Lambda^m f|_{(r,u)} \leqslant \text{const} \left(\frac{r}{\sinh(r)}\right)^m . \tag{3.2}$$

By the Ricci curvature comparison estimate, $\text{Ric} \geqslant -(1+\varepsilon)^2(n-1)$ implies that the volume form of (M, g) satisfies

$$d\text{vol} \leqslant \left(\frac{\sinh((1+\varepsilon)r)}{1+\varepsilon}\right)^{n-1} dr \wedge d\sigma_0, \tag{3.3}$$

where $d\sigma_0$ is the volume form of the standard sphere $\mathbb{S}^{n-1}$ (see, for example, [12, Sect. 9.1.1]). The above inequalities give us a control of the growth of $|\Lambda^m f|^s_{(r,u)} d\text{vol}$. To be precise, let us choose a number t such that

$$\frac{m(1+\varepsilon)}{n-1} < t < s.$$

Then (3.2) and (3.3) imply

$$|\Lambda^m f|^s_{(r,u)} d\text{vol} \leqslant \text{const } \mathrm{e}^{-ar} dr \wedge d\sigma_0$$

with $a = mt - (n-1)(1+\varepsilon) > 0$. The last inequality implies the integrability of $|\Lambda^m f|^s_{(r,u)}$:

$$\int_M |\Lambda^m f|^s_{(r,u)} d\text{vol} \leqslant \text{Vol } (\mathbb{S}^{n-1}) \int_0^\infty \mathrm{e}^{-ar} dr < \infty.$$

The lemma is proved. □

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