manuscripta math. 72, 251 - 256 (1991)

manuscripta mathematica

THE SCHWARZ LEMMA FOR NONPOSITIVELY CURVED RIEMANNIAN SURFACES

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In this paper, we prove that if f is a conformal map between two Riemannian surfaces, and if the curvature of the target is nonpositive and less than or equal to the curvature of the source, then the map is contracting.

The well known Schwarz lemma of complex analysis has been generalized for Kähler manifolds by Yau in [7]. In the special case of Riemann surfaces, his result can be stated in the following way :

Theorem [7]. *Let (S_1, g_1) and (S_2, g_2) be two smooth Riemannian surfaces without boundary. Let K_i denote the curvature of (S_i, g_i) and assume the following :*

- *a)* (S_1, g_1) *is complete ;*
- *b)* $K_1 \geq -a_1$ *for some number* $a_1 \geq 0$ *;*
- *c)* $K_2 \leq -a_2 < 0$.

Then any conformal map $f : S_1 \to S_2$ satisfies :

$$f^*(g_2) \leq (\frac{a_1}{a_2}) g_1 .$$

In particular, if $a_1 = 0$, then f is locally constant.

When (S_1, g_1) is the hyperbolic plane, this result has already been proved by L.V. Ahlfors in 1938 (see [1]).

In Yau's theorem, we have to assume the target surface to have curvature bounded above by a negative constant. Our aim in this paper is to prove a Schwarz lemma for surfaces of non positive curvature.

Theorem (The Schwarz Lemma). *Let (S_1, g_1) be a smooth complete connected Riemannian surface with curvature K_1 bounded below and (S_2, g_2) be any smooth Riemannian surface with curvature K_2.*

Let $f : (S_1, g_1) \to (S_2, g_2)$ be a conformal mapping such that

$$\begin{cases} K_2 \circ f & \leq \min\{0, K_1\}, \quad K_2 \circ f \not\equiv 0 \quad \text{and} \\ K_2 \circ f & < -a < 0 \quad \text{on the complement of some compact subset of } S_1 . \end{cases}$$

Then f is a contracting map.

The proof of this theorem will rely on the generalized maximum principle of Yau. But let us show first how we may obtain simple proofs of some results in the geometry of surfaces using the Schwarz lemma.

Theorem (Liouville's Theorem). *Let (S_1, g_1) be a smooth complete connected Riemannian surface with curvature $K_1 \geq 0$, and (S_2, g_2) be any smooth Riemannian surface with curvature K_2.*

Then any conformal map $f : S_1 \to S_2$ such that

$$\begin{cases} K_2 \circ f & \leq 0, \quad K_2 \circ f \not\equiv 0 \quad \text{and} \\ K_2 \circ f & < -a < 0 \quad \text{on the complement of some compact subset of } S_1 . \end{cases}$$

is constant.

Proof. The map $f : (S_1, cg_1) \to (S_2, g_2)$ has to be contracting for any positive constant c. □

Let us now discuss some applications to Riemann surface theory. Recall first that a *Riemann surface* is a one dimensional complex manifold and that a *Riemannian surface* is 2-dimensional manifold with a Riemannian metric on it. A metric g on a Riemann surface S is said to be *conformal* if, on any domain $U \subset S$ of a complex coordinate z, the metric takes the form $g = \rho(z)|dz|^2$ (for some positive function ρ). Conversely, given a Riemannian metric g on an oriented surface S, there exists a unique complex structure on S for which this metric is conformal (Korn-Lichtenchtein theorem). Furthermore, a mapping $f : (S_1, g_1) \to (S_2, g_2)$ between oriented Riemannian surfaces is conformal if and only if it is holomorphic for the associated complex structure.

The complex line is denoted by $\mathbf{C}$, and the once punctured complex line by $\mathbf{C}^*$.

Corollary 1. *(Compare to [5].) There is no conformal metric g on $\mathbf{C}$ or $\mathbf{C}^*$ (whether complete or not) such that $K_g \leq 0$ and $K_g \leq -a < 0$ outside a compact set.*

Proof. Suppose that such a metric g exists on $S = \mathbf{C}$ or $\mathbf{C}^*$. Let g_0 be the canonical (conformal, complete and euclidean) metric on S. then the identity map $id : (S, g_0) \to (S, g)$ is constant, which is absurd. □

Observe that this result is false on any other open Riemann surface of finite topological type (see [3], th. A.1).

Corollary 2 (Picard's Theorem). *Let f be a non constant entire function (holomorphic function on $\mathbf{C}$). Then $\mathbf{C} \setminus f(\mathbf{C})$ contains at most one point.*

Proof. Let S_2 be $\mathbf{C}$ minus two points. Then S_2 carries a conformal metric g_2 of curvature -1. Suppose that f is an entire function whose image lies in S_2. Then the function f is a conformal map from $\mathbf{C} \to S_2$, hence a constant. □

Our next application of the Schwarz lemma says that a complete conformal metric on a Riemann surface is determined by its curvature (if the curvature is negative).

Theorem. *(Uniqueness of the solution to the Berger-Nirenberg Problem.)*

Let S be an open Riemann surface (whithout boundary and of finite topological type) and $K : S \to \mathbf{R}$ be a smooth non positive function such that

$$-b \leq K \leq -a < 0$$

on the complement of some compact set. Assume that g and h are two complete conformal metrics on S with curvature K. Then $g = h$.

Proof. Apply the Schwarz lemma to the identity map $(S, h) \to (S, g)$ and its inverse. □

The existence of metric with prescribed curvature on a given open Riemann surface (of finite type) is dicussed in [2], [3].

For an application of the higher dimensional Schwarz lemma, see [4].

In the proof of the Schwarz lemma, we will need the following definition; let β be some real number such that $\beta > -1$.

Definition : The conformal metric g on the Riemann surface S has a *conical singularity* of *order* β at $p \in S$ if there is a local complex coordinate z centered at p such that we have the local expression

$$g = e^{2v(z)}|z|^{2\beta}|dz|^2 ,$$

where v is a continuous function (this notion is independent of the choice of the coordinate z).

We will always assume our metrics to be smooth except perhaps for isolated conical singularities. In particular, the curvature K_g of such a metric g is a well defined continuous function off the singular set. Inequalities such as $K_g \leq 0$ will always be interpreted on the complement of the singular set.

Example. If S_1 and S_2 are two Riemann surfaces, g is a smooth conformal metric on S_2 and $f : S_1 \to S_2$ is a holomorphic map having a zero of order m at $p \in S_1$. Then f^*g obviously has a conical singularity of order $m - 1$ at p.

Proof of the Schwarz Lemma.

If f is constant, then there is nothing to prove, we may therefore assume that f is a non constant holomorphic map with isolated singularities.

Let $p_1, p_2, \cdots \in S_1$ be the list of all singular points of f and let $S_1' := S_1 \setminus \{p_1, p_2, \dots\}$. Then we may define a function u on S_1' by $f^*(g_2) = e^{2u} g_1$. We want to prove that $u \leq 0$.

In the neighbourhood of p_i, we have in some local coordinate z (with $z(p_i) = 0$)

$$f(z) = z^{m_i} ,$$

where $m_i > 1$ is some positive integer, therefore, $u(z) = v_i(z) + (m_i - 1)\log(z)$ for some locally defined continuous function v_i (in other words, p_i is a conical singularity of order $m_i - 1$ for the metric $f^*(g_2)$). We can thus find small neighbourhoods $\mathcal{U}_i$ of the p_i's and a smooth function $\tilde{u}$ on S_1 such that

$$\begin{cases} u \leq \tilde{u} \leq -1 & \text{on } \mathcal{U} \text{ for all } i , \\ u = \tilde{u} & \text{on } S_1 \setminus \mathcal{U} , \end{cases}$$

where $\mathcal{U} := \cup_i \mathcal{U}_i$.

On the complement of the set $\mathcal{U}$, the function u is smooth and we have

$$\begin{cases} \Delta u = (K_2 \circ f)e^{2u} - K_1 & \text{on } S \setminus \mathcal{U} , \\ u \leq -1 & \text{on } \partial(S \setminus \mathcal{U}) , \end{cases}$$

where Δ is the Laplacian for the metric g_1 (with the sign convention that a subharmonic function φ satisfies $\Delta\varphi \leq 0$).

Consider now the set $\mathcal{P} = \{x \in S : u(x) \geq 0\}$. It is clear that $\mathcal{P} \cap \mathcal{U} = \emptyset$, in particular, $u = \tilde{u}$ on $\mathcal{P}$. Observe also that u is subharmonic on $\mathcal{P}$ since

$$\begin{aligned} \Delta u = (K_2 \circ f)e^{2u} - K_1 &= (K_2 \circ f)(e^{2u} - 1) + ((K_2 \circ f) - K_1) \\ &\leq (K_2 \circ f)(e^{2u} - 1) \leq 0 . \end{aligned}$$

Case 1 : u achieves its maximum and $\partial\mathcal{P} \neq \emptyset$.

The maximum principle tells us that $\max u$ is attained on $\partial\mathcal{P}$. But it follows from the definitions of $\mathcal{P}$ that $u \leq 0$ on $\partial\mathcal{P}$. Hence $\max u \leq 0$ on S.

Case 2 : u achieves its maximum and $\partial\mathcal{P} = \emptyset$.

Then either $\mathcal{P} = \emptyset$ or $\mathcal{P} = S$. In the latter case, u is subharmonic on S and hence is a constant. It is easy to see that the hypothesis on K_1 and $(K_2 \circ f)$ imply that this constant is non positive.

Case 3 : u does not achieve its maximum. Then either $\mathcal{P} = \emptyset$, or we can find $\eta > 0$ and $x \in S$ such that

$$u(x) > \eta.$$

We shall use the generalized maximum principle of Yau (see theorem 1, p. 200 in [7], or theorem 1, p. 206 in [Y1]). It says that if v is a C^2 function on a complete manifold (S_1, g_1) with (Ricci) curvature bounded below, and if $\sup(v) < \infty$ is not attained, then for any $\epsilon, \delta > 0$, the set

$$(*) \quad \Omega = \{x \in S \setminus N : \Delta v(x) > -\epsilon\,,\ |\nabla v(x)|^2 < \epsilon \text{ and } v(x) > \sup(v) - \delta\}$$

is not empty where $N \subset S$ is any given compact set.

We will apply this principle to the bounded function $v : S_1 \to \mathbf{R}$ defined by

$$v(x) = \frac{1}{1 + e^{-\tilde{u}(x)}}.$$

Choose $\delta > 0$ small enough so that $u(x) > \eta > 0$ for $v(x) > \sup(v) - \delta$ (i.e. $\delta < \sup(v) - (1 + e^{-\eta})^{-1}$). Choose $N \subset S_1$ to be the (compact) set of points in S_1 such that $K_2 \circ f \geq -a$. Finally, choose ϵ so that

$$4\epsilon \leq \frac{a\sinh(\eta)}{1 + 2\sinh(\eta)}.$$

And let Ω be the set defined by $(*)$ with these choices of v,ϵ, δ and N.

A computation gives us

$$\frac{e^{-u}}{(1 + e^{-u})^2}\,\Delta u = \Delta v - 2\sinh\,(u)|\nabla v|^2\,.$$

Since $u > 0$ on Ω, we have $(1 + e^{-u})^2 \leq 4$ on Ω, using also the estimates on Δv and $|\nabla v|^2$, we see that

$$e^{-u}\Delta u \geq -4\epsilon(1 + 2\sinh(u))$$

on Ω.

From $\Delta u = (K_2 \circ f)e^{2u} - K_1$, we get

$$(K_2 \circ f)e^{u} - K_1 e^{-u} \geq -4\epsilon(1 + 2\sinh(u))\,.$$

Dividing by $(K_2 \circ f)$ and using $K_2 \circ f \leq K_1$ and $K_2 \circ f < -a$, we obtain

$$e^{u} - e^{-u} \leq e^{u} - \frac{K_1}{K_2 \circ f}e^{-u} \leq -\frac{4\epsilon(1 + 2\sinh(u))}{K_2 \circ f} \leq \frac{4\epsilon(1 + 2\sinh(u))}{a}\,.$$

Whence

$$4\epsilon \geq 2\left(\frac{a\sinh(u)}{1 + 2\sinh(u)}\right) \geq 2\left(\frac{a\sinh(\eta)}{1 + 2\sinh(\eta)}\right)\,.$$

This last inequality contradicts our choice of ϵ. This contradiction implies either that $\mathcal{P}$ is empty or that u attains its maximum.

In all cases, we have proved that $u \leq 0$ on S. □

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(Received December 18, 1990;
in revised form May 7, 1991)