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Steiner's invariants and minimal connections

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Abstract. The aim of this note is to prove that any compact metric space can be made connected at a minimal cost, where the cost is taken to be the one-dimensional Hausdorff measure.

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1. Introduction

Recall that a *continuum* is a compact connected metric space. We denote by $\mathrm{Cont}(X)$ the set of all continua $C \subset X$ in an arbitrary metric space (X, d).

Definition. (A) Given a metric space (X, d), and a compact subset $S \subset X$, the *relative Steiner invariant of S in X* is defined as

$$\mathrm{St}(S, X) = \inf\{\mathscr{H}^1(C) \mid C \text{ and } C \cup S \in \mathrm{Cont}(X)\},$$

here, $\mathscr{H}^1$ is the one-dimensional Hausdorff measure.

(B) If (S, d) is a compact metric space, its *absolute Steiner invariant* is defined as

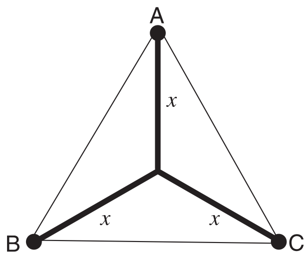
$$\mathrm{St}(S) = \inf\big\{\mathrm{St}\big(\imath(S), X\big) \mid X \text{ is an arbitrary metric space and } \imath : S \hookrightarrow X \text{ is an isometric embedding}\big\}.$$

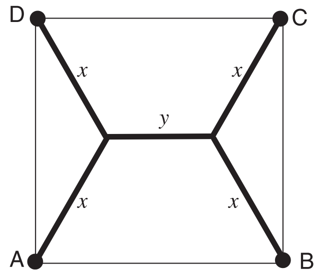
Our goal is to prove the following result.

Theorem 1 (The main theorem). *Let S be a compact metric space such that* $\mathrm{St}(S) < \infty$. *Then its Steiner invariant is realized, i.e., there exist a compact metric space Z and an isometric embedding* $\imath : S \hookrightarrow Z$ *such that* $\mathrm{St}\big(\imath(S), Z\big) = \mathrm{St}(S)$. *Furthermore, there exists* $C \in \mathrm{Cont}(Z)$ *such that* $C \cup \imath(S)$ *is also a continuum and* $\mathscr{H}^1(C) = \mathrm{St}(S)$.

The set $C \cup \imath(S)$ described in this theorem is thus a "minimal connection" of S, i.e., it is a shortest possible set that can be added to S to make it a continuum.

It should be noted that the "Steiner Problem", i.e., the problem of algorithmically finding a minimal connection of a given (generally finite) set in the Euclidean space or other metric spaces such as graphs, has attracted a considerable interest in the literature on optimization and combinatorial geometry, see e.g. [3], [4], [6]. Let us shortly discuss two elementary examples.





Let $X = \mathbb{R}^2$ be the Euclidean space and $S = \{A, B, C\} \subset \mathbb{R}^2$ be the three vertices of an equilateral triangle with unit side. Connecting A, B and C to the center of the triangle gives us a trivalent graph whose total length is $3x$, where $x = 1/\big(2\cos(\pi/6)\big) = \sqrt{3}/3$. It is known that this graph is the shortest network connecting A, B and C and thus $\mathrm{St}(S, \mathbb{R}^2) = 3x = \sqrt{3}$.

However, the absolute Steiner invariant of that set is smaller. Indeed, one can consider the graph as an abstract metric space. Since $d(A, B) = d(A, C) = d(B, C) = 1$, all edges must have length $x' = \frac{1}{2}$ and the absolute Steiner invariant of our set is $\mathrm{St}(S) = 3x' = \frac{3}{2} < \sqrt{3}$.

As a second example, consider the set $T = \{A, B, C, D\} \subset \mathbb{R}^2$ given by the four vertices of a square with unit side. It is known that the shortest network connecting A, D and B, C is a trivalent tree with rectilinear edges forming 120^0 angles. From this sole information, one can deduce the shape of that network and compute the lengths. Since $d(A, D) = 1$, we have as before $x = \sqrt{3}/3$, we then compute that $y = 1 - \sqrt{3}/3$ and we obtain $\mathrm{St}(T, \mathbb{R}^2) = 4x + y = 1 + \sqrt{3}$.

Again, one may consider the network as an abstract metric space. Since $d(A, D) = 2x' = 1$ the exterior edges have length $x' = \frac{1}{2}$, and since $d(A, C) = 2x' + y' = \sqrt{2}$, the interior edge has length $y' = \sqrt{2} - 1$. The absolute Steiner invariant of that set is then $\mathrm{St}(T) = 4x' + y' = 1 + \sqrt{2} < 1 + \sqrt{3}$.

2. Useful results

It is known that the relative Steiner invariant is always realized in a proper metric space (recall that a metric space is *proper* if every closed ball in it is compact):

Theorem 2. *Let (X, d) be a proper metric space and $S \subset X$ a compact subset such that* $\mathrm{St}(S, X) < \infty$. *Then* $\mathrm{St}(S, X)$ *is realized, i.e., there exists* $C \in \mathrm{Cont}(X)$ *such that* $C \cup S \in \mathrm{Cont}(X)$ *and* $\mathscr{H}^1(C) = \mathrm{St}(S, X)$.

If S contains only two points $\{x, y\}$, then this theorem simply says that the two points can be joined by a shortest curve. This is the Hopf–Rinow theorem for proper metric spaces.

This result can be found in [1], Theorem 4.4.20, (see also [4], chapter 2, for the special case of a finite set in a complete Riemannian manifold). The proof is essentially based on the Blaschke compactness theorem for the Hausdorff distance and a semi-continuity property of the Hausdorff measure due to Golab. Let us recall these results.

Proposition 3 (Blaschke). *Let (X, d) be an arbitrary metric space. We denote by $\mathscr{K}(X)$ the family of all non empty compact subsets of X. This is a metric space for the Hausdorff distance d_H. We then have:*

a) *If (X, d) is compact, then so is $(\mathscr{K}(X), d_H)$.*

b) *If (X, d) is proper, then so is $(\mathscr{K}(X), d_H)$.*

This theorem has been originally proved by Blaschke in the context of convex bodies in Euclidean space. We refer to [1], Theorem 4.4.15, or [2], Theorem 7.3.8, for a modern proof. □

It is not difficult to check that $\mathrm{Cont}(X) \subset \mathscr{K}(X)$ is a closed subset for the topology induced by the Hausdorff distance. Furthermore:

Proposition 4 (Golab). *Let (X, d) be a complete metric space and let $\{C_n\} \subset$* $\mathrm{Cont}(X)$ *be a sequence of continua such that $C_n \to C$ for the Hausdorff distance. Then* $C \in \mathrm{Cont}(X)$ *and*

$$\mathscr{H}^1(C) \leq \liminf_{n\to\infty} \mathscr{H}^1(C_n).$$

See [1], Theorem 4.4.17, for a proof. □

Our main theorem is an extension of Theorem 2. In its proof we will need to replace the Hausdorff distance by the Gromov–Hausdorff distance and the Blaschke theorem will be replaced by the Gromov compactness criterion. To recall this criterion, remember that the *packing number* of the metric space X at mesh $\varepsilon > 0$ is the number

$$P(X, \varepsilon) = \min\{n \mid \text{there exists } x_1, \ldots, x_n \in X \text{ such that}$$
$$\text{if } i \neq j \text{ then } B(x_i, \varepsilon) \cap B(x_j, \varepsilon) = \emptyset\}.$$

Recall that metric space X is totally bounded if $P(X, \varepsilon) < \infty$ for every $\varepsilon > 0$. The Gromov compactness criterion says that a family of isometry classes of compact metric spaces is totally bounded for the Gromov–Hausdorff distance if and only if it is *uniformly totally bounded*:

Theorem 5 (Gromov). *Let $\mathscr{M}$ be a family of isometry classes of compact metric spaces. Then the following conditions are equivalent:*

i) *$\mathscr{M}$ is totally bounded for the Gromov–Hausdorff distance.*

ii) $\sup_{X \in \mathscr{M}} P(X, \varepsilon) < \infty$ *for any* $\varepsilon > 0$.

See [2], Theorem 7.4.15. □

Another useful result on the Gromov–Hausdorff distance says that any sequence of compact metric spaces, which is Cauchy for the Gromov–Hausdorff distance, contains a subsequence which can be realized as a sequence of subsets of a single compact metric space:

Proposition 6. *Let $\{X_n\}$ be a sequence of compact metric spaces which is a Cauchy sequence in the Gromov–Hausdorff sense. Then there exist a subsequence $\{X_{n'}\}$, a compact metric space Z and isometric embeddings $X_{n'} \hookrightarrow Y_{n'} \subset Z$ and $X \hookrightarrow Y \subset Z$ such that $Y_{n'} \to Y$ for the Hausdorff distance in Z.*

This result is Theorem 4.5.7 in [1]. □

3. Proof of the main theorem

We first need a lemma:

Lemma 7. *Let (X, d) be a compact metric space, and $C \in \mathrm{Cont}(X)$. Then for any $a \in C$ and $0 < \varepsilon < \mathrm{diam}(C)/2$, we have*

$$\mathscr{H}^1\big(C \cap B(a, \varepsilon)\big) \geq \varepsilon;$$

in particular, we have

$$P(C, \varepsilon) \leq \frac{1}{\varepsilon} \mathscr{H}^1(C).$$

The proof of this result can be found for instance in [1], Lemma 4.4.5. □

We then need the following generalization of Golab's semi-continuity result:

Proposition 8. *Let $\{X_n\}$ be a sequence of compact metric spaces such that $X_n \to X$ in the Gromov–Hausdorff sense. Suppose that X_n is connected for each n. Then X is compact and connected, and moreover,*

$$\mathscr{H}^1(X) \leq \liminf_{n\to\infty} \mathscr{H}^1(X_n).$$

Proof. From Proposition 6, we know that, choosing a subsequence if necessary, there exist a compact metric space Z and isometric copies of X_n and X embedded in Z, say Y_n, Y, such that $Y_n \to Y$ in the Hausdorff sense. Since X_n and Y_n are isometric and each Y_n is compact and connected, we deduce from Proposition 4 that $Y \in \mathrm{Cont}(Z)$ and

$$\mathscr{H}^1(Y) \leq \liminf_{n\to\infty} \mathscr{H}^1(Y_n).$$

Now since Y is isometric to X, we conclude that X is compact and connected as well, and that

$$\mathscr{H}^1(X) \leq \liminf_{n\to\infty} \mathscr{H}^1(X_n),$$

because $\mathscr{H}^1(X_n) = \mathscr{H}^1(Y_n)$ and $\mathscr{H}^1(X) = \mathscr{H}^1(Y)$. □

Proof of Theorem 1. Let S be a compact metric space, and let $\{(S_k, X_k)\}$ be a minimizing sequence for the absolute Steiner invariant of S, that is:

i) X_k is a compact metric space, and $S_k \subset X_k$ is an isometric copy of S;

ii) $\mathrm{St}(S_k, X_k) \to \mathrm{St}(S)$.

We first prove that the sequence $\{X_k\}$ can be assumed to be uniformly totally bounded: indeed, if this where not the case, we could replace X_k by $S_k \cup C_k$, where $C_k \in \mathrm{Cont}(X_k)$ realizes $\mathrm{St}(S_k, X_k)$ (such a set exists by Theorem 2). From the compactness of S, we know that $P(S_k, \varepsilon) = P(S, \varepsilon) < \infty$ for all k and all ε. Moreover, $\mathscr{H}^1(C_k) = \mathrm{St}(S_k, X_k)$, therefore $\sup_k \mathscr{H}^1(C_k) < \infty$ and the family $\{C_k\}$ is uniformly totally bounded by Lemma 7. The families $\{S_k\}$ and $\{C_k\}$ being uniformly totally bounded, so is $\{S_k \cup C_k\}$.

We henceforth assume $\{X_k\}$ to be uniformly totally bounded. By the Gromov compactness criterion, Theorem 5, we know that $\{X_k\}$ contains a subsequence which is Cauchy in the Gromov–Hausdorff distance. From Proposition 6, we can further take a subsequence which can globally be embedded in a compact metric space Z. Finally, using the Blaschke compactness theorem, we can take one more subsequence, which converges for the Hausdorff distance in Z.

To sum up, there exist a subsequence $\{X_{k'}\}$, a compact metric space Z and isometric embeddings $\iota_{k'} : X_{k'} \hookrightarrow Y_{k'} \subset Z$ and a subset $Y \subset Z$ such that $Y_{k'} \to Y$ for the Hausdorff distance in Z.

Let $T_{k'} = \iota_{k'}(S_{k'}) \subset Y_{k'} \subset Z$. By Theorem 2, we can find $C_{k'} \in \mathrm{Cont}(Y_{k'})$ such that $C_{k'} \cup T_{k'} \in \mathrm{Cont}(Y_{k'})$ and $\mathscr{H}^1(C_{k'}) = \mathrm{St}(T_{k'}, Y_{k'}) = \mathrm{St}(S_{k'}, X_{k'})$.

By Blaschke's theorem again, we may assume (taking once more a subsequence if needed) that $\{C_{k'}\}$ converges to a subset $C \subset Z$ for the Hausdorff distance in Z. Likewise, we may assume that $\{T_{k'}\}$ converges to a subset $T \subset Z$ (since $C_{k'} \cup T_{k'} \subset Y_{k'}$ and $Y_{k'} \to Y$, we have in fact $C \cup T \subset Y$).

Furthermore, we know from Proposition 4 that C and $C \cup T$ are continua and that $\mathscr{H}^1(C) \le \liminf_{k' \to \infty} \mathscr{H}^1(C_{k'})$. But we have $\mathscr{H}^1(C_{k'}) = \mathrm{St}(T_{k'}, Y_{k'}) = \mathrm{St}(S_{k'}, X_{k'})$, which converges to $\mathrm{St}(S)$. Thus

$$\mathscr{H}^1(C) \le \mathrm{St}(S).$$

On the other hand C and $C \cup T$ are continua, hence $\mathscr{H}^1(C) \ge \mathrm{St}(T) = \mathrm{St}(S)$ by definition of the Steiner invariant. We therefore have equality.

To sum up, we have found a pair of subsets $C, T \subset Z$ such that C and $C \cup T$ are continua, T is isometric to S and $\mathscr{H}^1(C) = \mathrm{St}(S)$. The proof is complete. □

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