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SOLVING THE p-LAPLACIAN ON MANIFOLDS

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ABSTRACT. We prove in this paper that the equation $\Delta_p u + h = 0$ on a p-hyperbolic manifold M has a solution with p-integrable gradient for any bounded measurable function $h : M \to \mathbb{R}$ with compact support.

1. INTRODUCTION

The p-Laplacian of a function f on a connected oriented Riemannian manifold without boundary M is defined by $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f)$; it is the Euler-Lagrange operator associated with the functional $\int_M |\nabla f|^p$.

A function $u \in W^{1,p}_{loc}(M)$ is said to be a weak solution to the equation

$$\Delta_p u + h = 0 \tag{1}$$

if for all $\psi \in C^1_0(M)$ one has

$$\int_M \left\langle |\nabla u|^{p-2}\, \nabla u, \nabla\psi \right\rangle = \int_M h\, \psi\, .$$

We introduce the p-Dirichlet space $\mathcal{L}^{1,p}(M)$ of functions $u \in W^{1,p}_{loc}(M)$ admitting a weak gradient such that $\int_M \|\nabla u\|^p < \infty$.

In [2], the following result has been proved:

Theorem 1. *Suppose that M is p-parabolic, and let $h \in L^1(M)$ be a function such that $\int_M h \neq 0$. Then (1) has no weak solution $u \in \mathcal{L}^{1,p}(M)$.*

The goal of this paper is to prove the following result in the converse direction.

Theorem 2. *Suppose that M is a p-hyperbolic manifold ($1 < p < \infty$) and that $h \in L^\infty(M)$ has compact support. Then (1) has a weak solution $u \in \mathcal{L}^{1,p}(M)$. Moreover u is of class $C^{1,\alpha}$ on each compact set (where $\alpha \in (0,1)$ may depend on the compact set).*

The notion of p-hyperbolic and p-parabolic manifolds will be recalled below (see also [6]). As an example, the euclidean space $\mathbb{R}^n$ is p-hyperbolic if and only if $p < n$.

Remark. If $M = \mathbb{R}^n$ with $1 < p < n$ and $h \geq 0$, then equation (1) (and in fact a more general eigenvalue problem) is solved in [1].

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2. Preliminaries on p-hyperbolicity

Definition. Let (M, g) be a connected Riemannian manifold, and $K \subset M$ a compact set. For $1 < p < \infty$, the p-capacity of K is defined by

$$\mathrm{Cap}_p(K) := \inf \left\{ \int_M |\nabla u|^p : u \in C_0^1(M),\, u \geq 1 \text{ on } K \right\}.$$

The manifold M is said to be p-parabolic if $\mathrm{Cap}_p(K) = 0$ for all compact subsets $K \subset M$ and p-hyperbolic otherwise. It is a well known fact that, in a p-hyperbolic manifold, the p-capacity of any compact set with non empty interior is always positive (see e.g. [6]).

Let $D \subset M$ be a non empty bounded domain. We introduce the Banach space $E^p = E^p(D, M)$ of functions $u \in W^{1,p}_{loc}(M)$ such that

$$\|u\|_E^p := \int_D |u|^p dx + \int_M |\nabla u|^p dx < \infty \ .$$

We denote by E_0^p the closure of $C_0^1(M)$ in E^p.

Lemma 1. *If M is p-parabolic, then $1 \in E_0^p$.*

Proof. By hypothesis $\mathrm{Cap}_p(\overline{D}) = 0$; hence for all $\epsilon > 0$, there exists a function $u \in C_0^1(M)$ such that $u \equiv 1$ on D and $\int_M |\nabla u|^p dx < \epsilon$. Thus we have

$$\|1 - u\|_E^p := \int_D |1 - u|^p dx + \int_M |\nabla u|^p dx = \int_M |\nabla u|^p dx \leq \epsilon \ .$$

It follows that $1 \in E_0^p$. □

The next lemma is the well known Poincaré inequality.

Lemma 2. *Let D be any bounded regular domain in a Riemannian maniflold M and $1 \leq p < \infty$. Then there exists a constant A such that*

$$\left(\int_D |u - u_D|^p dx \right)^{1/p} \leq A \left(\int_D |\nabla u|^p dx \right)^{1/p}$$

for all $u \in W^{1,p}_{loc}(M)$, where $u_D = \frac{1}{vol(D)} \int_D u dx$ is the mean value of u on D.

A reference is [3, Lemma 3.8]. □

Combining this lemma with Hölder's (or Jensen's) inequality, we obtain

Corollary 1. *There exists a constant $c = c_D$ such that*

$$\int_D |u - u_D| dx \leq c_D \left(\int_M |\nabla u|^p dx \right)^{1/p} \tag{2}$$

for all $u \in W^{1,p}_{loc}(M)$. □

Proposition 1. *Suppose that M is p-hyperbolic and let $D \subset M$ be as in Lemma 2. Then there exists a constant C_1 such that for all $u \in E_0^p$*

$$\int_D |u| \, dx \leq C_1 \left(\int_M |\nabla u|^p dx \right)^{1/p} \ .$$

Proof. Suppose that such a constant does not exist. Then for all $\varepsilon > 0$ it is possible to find a function $u \in E_0^p$ such that

$$\int_D |u| dx = \mathrm{vol}(D) \qquad \text{and} \qquad \|\nabla u\|_{L^p(M)} \leq \varepsilon \ .$$

We may also assume $u \geq 0$ (else replace u by $|u|$). From Corollary 1 one gets

$$\int_D |u-1| dx \leq c_D \, \varepsilon \ . \tag{3}$$

Let us now choose a ball $B \subset\subset D$ and a function $\psi \in C_0^1(M)$ such that $0 \leq \psi \leq \frac{1}{2}$, $\mathrm{supp}(\psi) \subset D$ and $\psi \equiv \frac{1}{2}$ on B, and define the function $v \in E_0^p$ by $v = 2\max\{u; \psi\}$.

Observe first that $v \geq 1$ on B, and define the sets

$$A := \{ x \in D |\, \psi(x) \geq u(x)\} \qquad \text{and} \qquad A' := \left\{ x \in D |\, |u(x) - 1| \geq \ \frac{1}{2} \right\} \ .$$

We have $A \subset A'$ and by (3) we have $\frac{1}{2}\mathrm{vol}(A') \leq c_D\varepsilon$; thus

$$\mathrm{vol}(A) \leq 2c_D\varepsilon \ . \tag{4}$$

Now we have almost everywhere

$$\nabla v = \begin{cases} 2\nabla u & \text{on} \quad M \setminus A, \\ 2\nabla\psi & \text{on} \quad A; \end{cases}$$

in particular

$$|\nabla v| \leq 2|\nabla u| + 2\chi_A |\nabla \psi| \qquad \text{a.e.}$$

from which one deduces

$$\|\nabla v\|_{L^p(M)} \leq 2 \, \|\nabla u\|_{L^p(M)} + 2 \sup |\nabla \psi| \ \left(\mathrm{vol}(A)\right)^{1/p} \ . \tag{5}$$

From (4) and (5) one obtains

$$\|\nabla v\|_{L^p(M)} \leq \left(2\varepsilon + 2\sup |\nabla\psi| \ \left(2c_D\varepsilon\right)^{1/p} \right) \ .$$

Since $v \geq 1$ on B and ε is arbitrary, one deduces that $\mathrm{Cap}_p(B) = 0$, which contradicts the fact that M is p-hyperbolic. □

We may sum up our results so far in

Theorem 3. *The following conditions are equivalent:*

(a) *M is p-hyperbolic;*
(b) *There exists a constant C_2 such that for all $u \in E_0^p$ one has*

$$\|u\|_{L^p(D)} \leq C_2 \cdot \|\nabla u\|_{L^p(M)} \ ;$$

(c) $1 \notin E_0^p$.

Proof. The implication (b) $\Rightarrow$ (c) is obvious and (c) $\Rightarrow$ (a) is Lemma 1.

Let us write u as $u = (u - u_D) + u_D$; using Proposition 1 and Lemma 2, we see that

$$\begin{aligned}
\|u\|_{L^p(D)} &\leq \|u - u_D\|_{L^p(D)} + \|u_D\|_{L^p(D)} \\
&\leq A\left(\int_D |\nabla u|^p dx\right)^{1/p} + (\mathrm{Vol}(D))^{1/p}\,|u_D| \\
&\leq A\left(\int_D |\nabla u|^p dx\right)^{1/p} + (\mathrm{Vol}(D))^{(1-p)/p}\int_D |u| dx \\
&\leq A\left(\int_D |\nabla u|^p dx\right)^{1/p} + (\mathrm{Vol}(D))^{(1-p)/p}\,C_1\left(\int_M |\nabla u|^p dx\right)^{1/p} \\
&\leq C_2\left(\int_M |\nabla u|^p dx\right)^{1/p}.
\end{aligned}$$

This proves (a) $\Rightarrow$ (b). □

3. Proof of Theorem 2

We first choose a regular bounded domain $D \subset M$ such that $\mathrm{supp}(h) \subset D$. We then define a functional $\mathcal{J} : E_0^p \to \mathbb{R}$ by

$$\mathcal{J}(u) = \frac{1}{p}\left(\int_M |\nabla u|^p dx\right) - \int_M hu\; dx\,.$$

The manifold M beeing p-hyperbolic, we have

$$\begin{aligned}
\mathcal{J}(u) &\geq \frac{1}{p}\,\|\nabla u\|^p_{L^p(M)} - \left|\int_M hu\; dx\right| \\
&\geq \frac{1}{p}\,\|\nabla u\|^p_{L^p(M)} - \|h\|_{L^\infty}\cdot\|u\|_{L^1(D)} \\
&\geq \frac{1}{p}\,\|\nabla u\|^p_{L^p(M)} - C_1\,\|h\|_{L^\infty}\cdot\|\nabla u\|_{L^p(M)}\,,
\end{aligned}$$

where C_1 is the constant of Proposition 1. Since the function $g(x) = |x|^p - ax$ of the real variable x is bounded below, we conclude that the functional $\mathcal{J}$ is bounded below on the space E_0^p.

Set $m := \inf\{\mathcal{J}(u) | u \in E_0^p\}$, and let $\{u_i\} \subset E_0^p$ be a minimizing sequence for $\mathcal{J}$ (i.e. $\mathcal{J}(u_i) \to m$). Then from the inequality above, one deduces that $\{u_i\}$ is a bounded sequence in E_0^p. Since E_0^p is a reflexive Banach space, this sequence contains a weakly convergent subsequence (still denoted by $\{u_i\}$). Let us denote by u^* the weak limit of $\{u_i\}$. By the compactness of the embedding $E_0^p \subset L^1(D)$, we may assume that $\{u_i\}$ converges strongly in $L^1(D)$, in particular

$$\int_D hu_i \to \int_D hu^*\,. \tag{6}$$

By Theorem 3, $\|\nabla u\|_{L^p(M)}$ is an equivalent norm on E_0^p; hence by the weak lower semi-continuity of the norm on E_0^p we have

$$\|\nabla u^*\|_{L^p(M)} \leq \lim_{i\to\infty}\inf\|\nabla u_i\|_{L^p(M)}\,. \tag{7}$$

From (6) and (7) one deduces that $\mathcal{J}(u^*) \le \lim_{i\to\infty} \inf \mathcal{J}(u_i) = m$; hence $\mathcal{J}(u^*) = m$. By the usual arguments from variational calculus, one deduces that u^* is a weak solution to (1).

The $C^{1,\alpha}$ regularity follows from Theorem 1 in [5]. □

Remark. We have in fact solved (1) in the space $E_0^p \subset \mathcal{L}^{1,p}(M)$.

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