ELEMENTARY GAGA

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to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Sir Isaac Newton

1. Introduction

The study of algebraic geometry may be traced back in history almost arbitrarily far. For example, the old Greek, Arab and Persian mathematicians already used geometric methods to solve cubic equations, although one could argue about whether this is considered real algebraic geometry (no pun intended). The next vital step towards algebraic geometry was certainly the introduction of algebra into geometry, i.e. the introduction of Cartesian coordinates in the 17th century. Again one can argue whether this should be considered algebraic geometry or not. However, one of the first definite guises of algebraic geometry can be found in Newton’s classification of cubic curves (as published in John Harris’s *Lexicon Technicum*, vol. 2 of 1710).

This list goes on via the advent of projective and Kleinian geometry to Bernhard Riemann’s development of Riemann surfaces and the Italian school of algebraic geometry.

Obviously neither Descartes, nor Newton, nor Riemann knew what a “Zariski topology” was (or a “topology” for that matter) let alone a (maximal) spectrum of a ring or a structure sheaf. If there was some form of topology lurking in the background then it was certainly the analytic one induced by the standard Euclidean metric. It was not until the middle of the 20th century that the Zariski topology and later on the étale topology (although not a topology in the usual sense) were introduced and became the standard way of studying varieties and schemes.

The interplay of the Zariski topology on an (affine) algebraic variety and the classical one induced by the Euclidean norm of an ambient complex affine space has always been an intensely studied question. However, the theory of (complex analytic) manifolds turns out to be insufficient for this because an algebraic variety may have singularities. This obstacle was quickly removed in 1956 only shortly after the introduction of the Zariski topology when Jean-Pierre Serre published his influential paper [GAGA], introducing the notion of analytic spaces. To this date, Serre’s original paper remains a primary source of information about the above mentioned interplay between algebraic and analytic geometry and its importance is perhaps best reflected in the fact that the study of this is nowadays colloquially referred to simply as “GAGA”.

However, Serre’s paper is mainly concerned with his newly developed sheaf cohomology and the relations between coherent algebraic and analytic sheaves, while only addressing the necessary topological questions. The standard reference containing a lot more comparison theorems between the topological structures is [SGA 1, Exposé XII] which has the obvious drawback of being highly technical because everything is considered in the more general context of schemes of finite type over $\mathbb{C}$, which is usually not necessary in classical algebraic geometry. There are however more elementary accounts like [Mumford, 1999] and [Neeman, 2007] but the former is only concerned with the analytification of complete varieties and the latter uses GAGA as a mere motivational device to develop scheme theory from scratch. For this reasons it would be desirable to have an elementary account at least for some of the more important topological aspects of GAGA and this is exactly the goal of this thesis. The topological aspects that were picked out are the following three theorems:

1.1 Theorem. If $\varphi : X \to Y$ is a finite morphism between algebraic varieties, then the analytification $\varphi_{an} : X_{an} \to Y_{an}$ is a closed map.

1.2 Theorem. If $\varphi : X \to Y$ is a flat morphism between algebraic varieties, then the analytification $\varphi_{an} : X_{an} \to Y_{an}$ is an open map.
(1.3) **Theorem.** If \( X \) is an irreducible algebraic variety, then \( X_{\text{an}} \) is connected.

The first theorem was settled in a highly satisfactory manner and relies on a short and elementary proof of Chow’s lemma for quasi-projective varieties, whose presence in the literature the author is not aware of. Unfortunately, we were not able to address the second question in time and it turns out to be very complicated. However, we included a proof for the étale case (rather than the flat one), where the analytification is even a local homeomorphism although the proof can hardly be considered elementary because it relies on the fact that any étale morphism looks locally like a standard étale one, which in turn relies on Zariski’s main theorem. As for the last theorem, we were able to give two proofs. The first one has the advantage of being purely algebraic but with the drawback of needing a weak form of the famous Riemann-Roch theorem and admittedly proving the theorem in a roundabout way. It establishes the theorem for affine algebraic curves, enabling us to generalise it to arbitrary dimensions because on an irreducible affine variety, any two points are always contained in an irreducible one-dimensional affine closed subvariety, as shown in [De Capitani, 2006]. The second proof doesn’t take a detour via algebraic curves and is more elementary but uses tools from complex analysis.

A quick word on the notations and names used in the text: We are always working over the ground field \( \mathbb{C} \), the words “algebraic variety” and “variety” are used interchangeably and the word “curve” is used as a synonym for a one-dimensional variety. To distinguish the two topologies on the complex numbers that we are concerned with, we use \( \mathbb{A}^1 \) for the Zariski topology and \( \mathbb{C} \) for the one induced by the Euclidean norm (or any other norm for that matter). Unfortunately, there is no such notation for projective space. By a “morphism” we always mean a morphism of algebraic varieties and a morphism to \( \mathbb{A}^1 \) is also called a “regular function”. The structure sheaf of an algebraic variety \( X \) is denoted by \( \mathcal{O}_X \) and its stalk at a point \( x \in X \) by \( \mathcal{O}_{X,x} \). However, we will usually only consider affine varieties, so that only its global sections (a.k.a. its coordinate ring) \( \mathcal{O}_X(X) \) are important. Finally, if \( X \) is an irreducible affine variety its “rational function field” is the field of fractions \( \text{Frac} \mathcal{O}_X(X) \) and is denoted by \( \mathbb{C}(X) \).

I’d like to finish this introduction with the most important thing, namely to thank all the people who supported me, the above mentioned little boy on the sea-shore. I especially want to emphasise the importance of the role my adviser Hanspeter Kraft played in the creation of this work, without whose excellent supervision I would have failed desperately. Not a bit less important was the help of his PhD-student Immanuel Stampfli, who provided me with all kinds of mathematical insights during our very frequent and lively discussions. Last but not least, all the mathematical insight in the world won’t suffice if there isn’t a single person providing moral support. For that reason I’d like to thank all my friends who accompanied me during this important and hard but at the same time very exciting time of my life.

## 2. Topological Preliminaries

For readability’s sake, we gather some topological results in this section that we are going to use at some point(s) in the text. The reader can skip this section without further ado and return to it later, when a result stated here is needed.

(2.1) **Proposition.** Let \( X, Y \) be topological spaces with \( Y \) Hausdorff, \( D \subseteq X \) a dense subset and \( f, g : X \to Y \) continuous maps that agree on \( D \) (i.e. \( f|_D = g|_D \)). Then even \( f = g \).

**Proof.** Because \( Y \) is Hausdorff, the diagonal \( \Delta_Y := \{ (y, y) \mid y \in Y \} \subseteq Y \times Y \) is closed and

\[
(f, g)^{-1}\Delta_Y = \{ x \in X \mid fx = gx \} \supseteq D.
\]

But then the continuity of \( f \) and \( g \) implies that \( \overline{D} = X \subseteq (f, g)^{-1}\Delta_Y \) and the claim follows. \( \square \)
(2.2) **Definition.** Recall that a continuous map \( \varphi: X \to Y \) is called *proper* iff for every other topological space \( Z \), the pullback
\[
\varphi \times 1_Z: X \times Z \to Y \times Z
\]
along the projection \( Y \times Z \to Y \) is closed. Equivalently, \( \varphi \) is proper iff it is closed and all its fibres are quasi-compact. This implies that for \( \varphi \) proper, the preimages of quasi-compact subsets are quasi-compact. The converse of this implication is true for \( X \) Hausdorff and \( Y \) locally compact Hausdorff. Recall furthermore, that a continuous map \( \varphi: X \to Y \) is called *finite* iff it is closed and all its fibres are finite. In particular, a finite map is proper.

(2.3) **Proposition.** Let \( \varphi: X \to Y \) a continuous map such that \( Y \) has an open cover \( (Y_i)_{i \in I} \), satisfying that for each \( i \in I \) the restriction \( \varphi^{-1}Y_i \to Y_i \) of \( \varphi \) is closed/proper/finite. Then \( \varphi \) is closed/proper/finite.

**Proof.** We put \( X_i := \varphi^{-1}Y_i \) and first check the claim for closedness. If \( A \subseteq X \) is closed and \( B := \varphi A \subseteq Y \), we write
\[
B_i := B \cap Y_i = \varphi A \cap Y_i = \varphi(A \cap X_i),
\]
where the last equality holds because \( X_i = \varphi^{-1}Y_i \). Therefore, by hypothesis, all the \( B_i \subseteq Y_i \) are closed. Now if \( y \in Y \setminus B \) there is some \( i \in I \), such that \( y \in Y_i \) and because \( B_i \subseteq Y_i \) is closed, there is some open neighborhood \( V \subseteq Y_i \) of \( y \) such that \( V_i \cap B_i = \emptyset \). But \( V \subseteq Y_i \) and \( B_i = Y_i \cap B \), thence \( V \cap B = V \cap Y_i \cap B = V \cap B_i = \emptyset \), so that \( B = \varphi A \subseteq Y \) is indeed closed. As for properness (resp. finiteness), we only need to check that all fibres are quasi-compact (resp. finite), which is obvious by hypothesis. \( \square \)

(2.4) **Definition.** A continuous map \( p: E \to X \) between two topological spaces is called a *local homeomorphism* (or *étale*) iff for every \( e \in E \), there is some open neighborhood \( U \subseteq E \) of \( e \), such that \( pU \subseteq X \) is open and \( p \) restricts to a homeomorphism \( U \cong pU \). Furthermore, we say that a continuous map \( p: E \to X \) has the (unique) path-lifting property iff for every path \( \gamma: [0, 1] \to X \) with starting point \( x_0 := \gamma(0) \) and every \( e_0 \in p^{-1}x_0 \) there is a (unique) path \( \tilde{\gamma}: [0, 1] \to E \) that starts at \( e_0 \) and satisfies \( p \circ \tilde{\gamma} = \gamma \).

(2.5) **Remark.** It follows immediately from the definition, that local homeomorphisms have discrete fibres and are open.

(2.6) **Example.** It is well-known and probably part of every introduction to the homotopy theory of topological spaces, that covering spaces (which are instances of local homeomorphisms) have the unique path-lifting property (for a proof, see e.g. or [Hatcher, 2002, p. 60]). However, proving that a continuous map is actually a covering is sometimes a bit of a hassle. But we do have the following useful criterion from [Forster, 1981, p. 29].

(2.7) **Proposition.** Any étale and proper continuous map \( p: E \to X \) with \( E \) Hausdorff is even a covering map.
Proof. First, we note that \( p \) has finite fibres (they are discrete because \( p \) is étale and quasi-compact by properness), so that for \( x \in X \), we have \( p^{-1}x = \{ e_1, \ldots, e_n \} \subseteq E \) for some pairwise distinct \( e_1, \ldots, e_n \in E \). As \( p \) is étale, we find for every \( i \in \{ 1, \ldots, n \} \) some open neighborhood \( U_i \subseteq E \) of \( e_i \), which is mapped homeomorphically onto the open set \( pU_i \subseteq X \) via \( p \). Moreover, since \( E \) is Hausdorff, we can assume without loss of generality, that \( U_1, \ldots, U_n \) are pairwise disjoint. By the lemma below, there is some open neighborhood \( V \subseteq pU_1 \cap \ldots \cap pU_n \) of \( x \), such that \( p^{-1}V \subseteq U_1 \cup \ldots \cup U_n \). Putting \( V_i := U_i \cap p^{-1}V \) for \( i \in \{ 1, \ldots, n \} \), one readily checks that

\[
p^{-1}V = V_1 \cup \ldots \cup V_n
\]

is a disjoint union of open subsets of \( E \), with \( p \) restricting to a homeomorphism \( V_i \cong V \) for each \( i \in \{ 1, \ldots, n \} \). As \( x \in X \) was an arbitrary point, \( p \) is indeed a covering map. \( \Box \)

(2.8) **Lemma.** Let \( p: E \rightarrow X \) be a closed continuous map, \( x \in X \) and \( U \subseteq E \) open with \( p^{-1}x \subseteq U \). Then there is some open neighborhood \( V \subseteq X \) of \( x \) such that \( p^{-1}V \subseteq U \).

**Proof.** \( A := p(E \setminus U) \subseteq X \) is closed and \( V := X \setminus A \) is the sought for open neighborhood. \( \Box \)

A direct consequence of this lemma, which we shall find useful to relate the closures of a constructible set in the Zariski- and the strong topology, is the following.

(2.9) **Lemma.** Let \( \varphi: X \rightarrow Y \) be a closed continuous surjection, \( x \in X \) with \( \varphi^{-1}\varphi x = \{ x \} \) and \( S \subseteq X \). If \( \varphi x \) is not an interior point of \( \varphi S \), then \( x \) is not an interior point of \( S \) either.

**Proof.** Assume there is some \( U \subseteq X \) open with \( x \in U \) and \( U \subseteq S \). Then by the previous lemma, there is some open neighborhood \( V \subseteq Y \) of \( \varphi x \) such that \( \varphi^{-1}V \subseteq U \subseteq S \), implying that \( V = \varphi \varphi^{-1}V \subseteq \varphi S \), which is a contradiction. \( \Box \)

Finally, we need a result concerning the interplay between a covering and the connected components of its total space, following from the above mentioned path-lifting property.

(2.10) **Proposition.** Let \( p: E \rightarrow X \) be a surjective covering map with \( X \) pathwise connected and let \( (E_i)_{i \in I} \) be the connected components (or alternatively the path-components) of \( E \). Then every restriction \( p|_{E_i}: E_i \rightarrow X \) with \( i \in I \) is again surjective. \( \Box \)

### 3. Algebraic Preliminaries

As mentioned in the introduction, we didn’t fully succeed in giving elementary proofs for all desired facts. Obviously, the notion of being elementary is highly subjective and depends heavily on the mathematical history and talents of the “definer” (that is the person defining “elementary”). However, while our proof that étale morphisms are local homeomorphisms in the strong topology is a lost cause for being classified as elementary by someone not at ease with the theory of schemes, the strong connectedness of algebraic curves might be less so.

Still, we will need some very weak form of the Riemann-Roch theorem in section 7 and this is usually not considered to be elementary although most people with some background in algebraic geometry will probably have encountered it at some point during their studies. We shall state the theorem needed in this preliminary section rather than the section it is actually used. The reason for this is that we want the structure of the text to reflect the mental disjunction between this non-elementary part of the proof and the more elementary part found in the above mentioned section.

(3.1) **Theorem.** Let \( K \) be a field containing \( \mathbb{C} \) with \( \text{trdeg}_\mathbb{C} K = 1 \). Then there is some \( f \in K \setminus \mathbb{C} \) such that for every affine algebraic curve \( C \) together with a chosen isomorphism of \( \mathbb{C} \)-algebras \( \mathbb{C}(C) \cong K \) (thus enabling us to view \( f \) as a rational function on \( C \)), \( f \) has at most one zero on \( C \). \( \Box \)
By definition, any affine algebraic variety (over the complex numbers) can be embedded as a closed subvariety into some affine space $\mathbb{A}^n$. Moreover, any algebraic variety is “patched together” by (finitely many) affine varieties. More precisely, any algebraic variety $X$ has a finite open cover $(X_1, \ldots, X_n)$ by affine varieties, meaning that for each $i \in \{1, \ldots, n\}$ the locally ringed space $(X_i, \mathcal{O}_{X_i})$ is an affine variety (where $\mathcal{O}_{X_i}$ is the restriction of $\mathcal{O}_X$ to the open subspace $X_i$).

(4.1) **Theorem. (Analytification)** There is a functor called analytification $-_{\text{an}} : \text{AlgVar}_C \to \text{Top}$ from the category of algebraic varieties over the complex numbers to the category of topological spaces, which is characterised by the following properties:

(a) if $\text{Sp} : \text{AlgVar}_C \to \text{Top}$ is the functor, which sends each algebraic variety to its underlying topological space and $U : \text{Top} \to \text{Sets}$ the forgetful functor, then $U \circ -_{\text{an}} = U \circ \text{Sp}$ and we have a natural transformation $-_{\text{an}} \Rightarrow \text{Sp}$, whose components are the identity maps;

(b) $-_{\text{an}}$ sends open (resp. closed) immersions to open (resp. closed) embeddings;

(c) $-_{\text{an}}$ preserves products;

(d) $\mathbb{A}^1_{\text{an}} = \mathbb{C}$ with the usual topology induced by any norm on $\mathbb{C}$.

(4.2) **Remark.** Property (a) in the theorem just says that $X = X_{\text{an}}$ as sets and the topology of $X_{\text{an}}$ is stronger (i.e. finer) than the Zariski topology on $X$. Because of this, we speak of the “strong” (or “analytic”) topology on $X$. Moreover, a morphism $\varphi : X \to Y$ is sent to the same map between sets and turns out to be continuous for the strong topologies on $X$ and $Y$.

**Proof.** It is immediate that for $n \in \mathbb{N}$, we must have $\mathbb{A}^n_{\text{an}} = \mathbb{C}^n$ and moreover, if $\varphi : \mathbb{A}^m \to \mathbb{A}^n$ is a morphism of affine spaces, that the analytification $\varphi_{\text{an}} : \mathbb{C}^m \to \mathbb{C}^n$ (which by definition is the same as $\varphi$ if viewed as a mere map between sets) is continuous.

More generally, for $X$ any affine algebraic variety, we have $X = X_{\text{an}}$ as a set and as there is some closed immersion $X \hookrightarrow \mathbb{A}^n$ into an affine space, $X_{\text{an}}$ must carry the subspace topology induced by $X_{\text{an}} \hookrightarrow \mathbb{C}^n$. This is independent of the closed immersion chosen, for if $\varphi : X \hookrightarrow \mathbb{A}^m$ and $\psi : X \hookrightarrow \mathbb{A}^n$ are two closed immersions, and $X'_{\text{an}}$, $X''_{\text{an}}$ carry the initial topology with respect to the inclusions $\varphi_{\text{an}} : X'_{\text{an}} \hookrightarrow \mathbb{C}^m$ and $\psi_{\text{an}} : X''_{\text{an}} \hookrightarrow \mathbb{C}^n$ respectively, we can lift the identity morphism $\text{id}_X : X \to X$ to a morphism $\chi : \mathbb{A}^m \to \mathbb{A}^n$, which then gives us a commutative square

$$
\begin{array}{ccc}
  X'_{\text{an}} & \xrightarrow{1_{X_{\text{an}}}} & X''_{\text{an}} \\
  \varphi_{\text{an}} \downarrow & & \downarrow \psi_{\text{an}} \\
  \mathbb{C}^m & \xrightarrow{\chi_{\text{an}}} & \mathbb{C}^n
\end{array}
$$

where the upper morphism must be continuous because it is a restriction of $\chi_{\text{an}}$. By reversing the roles of $X'_{\text{an}}$ and $X''_{\text{an}}$ we also see that $1_{X_{\text{an}}} : X''_{\text{an}} \to X'_{\text{an}}$ is continuous, so that indeed $X'_{\text{an}} = X''_{\text{an}}$ as topological spaces.

For a general algebraic variety $X$ and $U \subseteq X$ an affine open subvariety, the inclusion $U_{\text{an}} \hookrightarrow X_{\text{an}}$ must be an open embedding. Hence, the topology on $X_{\text{an}}$ is coarser than the final
topology with respect to all inclusions $U_{an} \hookrightarrow X_{an}$ with $U \subseteq X$ any affine open subvariety of $X$. But it must be finer, too, because the set

$$U := \{ U \subseteq X \mid U \text{ affine and open} \}$$

provides an open cover of $X$ and if we write $i_{U_{an}} : U_{an} \rightarrow X_{an}$ for the standard inclusions, each of these must be open and continuous. Thence, if $V \subseteq X_{an}$ is such that $U_{an} \cap V$ is open in $U_{an}$ for every $U \in U$, then we have

$$V = \bigcup_{U \in U} i_{U_{an}}(U_{an} \cap V),$$

which must be open in $X_{an}$ by openness of the $i_{U_{an}}$.

One should observe that for proving the fact that for a general variety $X$, the topology on the analytification $X_{an}$ must be the final topology with respect to all inclusions $U_{an} \hookrightarrow X_{an}$ for $U \subseteq X$ affine and open, we only needed that these subvarieties cover all of $X$ and in fact, by the same argument, any cover by affine open subvarieties will do.

(4.3) **Proposition.** If $X$ is an algebraic variety and $X = X_1 \cup \ldots \cup X_n$ a cover by affine open subvarieties, then $X_{an}$ carries the final topology with respect to all inclusions $(X_i)_{an} \hookrightarrow X_{an}$ for $i \in \{1, \ldots, n\}$.

Our approach to equip the set $X_{an} = X$ with a topology is the one taken in probably most of the more elementary expository texts, e.g. [Mumford, 1999] and [Neeman, 2007]. An alternative and quicker definition of the analyticity that is mentioned in the original paper [GAGA] by Serre is the following: For an affine variety $X$, we put $X_{an} := X$ as a set and equip it with the initial topology with respect to all regular functions $X \rightarrow \mathbb{C}$ in $\mathcal{O}(X)$ but with the codomain $\mathbb{C}$ carrying the strong topology. Another alternative in a more abstract setting is provided by [SGA 1], which defines the analytification of a scheme of finite type over $\mathbb{C}$ as a representing object for a certain functor. Even just formulating this would require a vast amount of definitions (e.g. sheaves, locally ringed spaces and complex analytic spaces) but in principle, it boils down to the same characterisation as in the above theorem.

(4.4) **Remark.** As already implicitly mentioned in the last paragraph, analytification is not just a mere functor to $\text{Top}$ but $X_{an}$ carries the structure of a so-called analytic space (whence the name “analytification”), which can be viewed as a generalisation of a complex analytic manifold to include the possibility of it having singularities. A special case of this is the following well-known theorem.

(4.5) **Theorem.** If we write $\text{SmAlgVar}_\mathbb{C}$ for the full subcategory of $\text{AlgVar}_\mathbb{C}$ with objects the smooth algebraic varieties and $\text{AnMan}_\mathbb{C}$ for the the category of complex analytic manifolds then we have a commutative square of functors

$$\begin{array}{ccc}
\text{SmAlgVar}_\mathbb{C} & \xrightarrow{-_{an}} & \text{AnMan}_\mathbb{C} \\
\downarrow & & \downarrow U \\
\text{AlgVar}_\mathbb{C} & \xrightarrow{-_{an}} & \text{Top}
\end{array},$$

where $U$ is the forgetful functor. The functor $-_{an} : \text{SmAlgVar}_\mathbb{C} \rightarrow \text{AnMan}_\mathbb{C}$ is characterised by conditions analogous to (a) – (d) in (4.1) with (d) in this context reading as “$A^1_{an} = \mathbb{C}$ with the usual complex analytic structure”.


and just as in the proof of (4.1), one checks that the complex analytic structure of \( X \) depend on the chosen closed immersion of \( X \) cover their analytification. For instance, a subset by replacing each algebraic variety and each morphism of algebraic varieties occurring in (4.6) Nomenclature.

\( \phi \) the transition functions are all polynomial. By a similar argument, one easily checks that if \( \partial f_i / \partial x_j \big|_{x} \) is invertible. Hence, by the implicit function theorem, there are \( U \subseteq \mathbb{C}^d \), \( V \subseteq \mathbb{C}^{n-d} \) open and \( g : U \to V \) holomorphic such that \( x \in U \times V \) and

\[
X \cap (U \times V) = \Gamma_g = \{(a, ga) \mid a \in U\}.
\]

Put differently, the standard projection \( \mathbb{C}^n = \mathbb{C}^d \times \mathbb{C}^{n-d} \to \mathbb{C}^d \) restricts to a biholomorphic map \( X \cap (U \times V) \to U \) with inverse \( a \mapsto (a, ga) \). This proves that \( X_{an} \subseteq \mathbb{C}^n \) is a closed submanifold and just as in the proof of (4.1), one checks that the complex analytic structure of \( X_{an} \) does not depend on the chosen closed immersion of \( X \) into affine space.

If \( X \) is a general smooth variety then we can assume that it is irreducible because its irreducible components are disjoint (i.e. they are even the connected components). Choosing any cover \( (X_1, \ldots, X_n) \) of \( X \) by irreducible affine open subvarieties yields an atlas for \( X_{an} \) because the transition functions are all polynomial. By a similar argument, one easily checks that if \( \phi : X \to Y \) is a morphism of smooth varieties and we equip \( X_{an} \) and \( Y_{an} \) with the complex analytic structures just defined, then \( \phi_{an} \) is holomorphic. \( \square \)

(4.6) Nomenclature. If \( P \) is any property then strongly \( P \) is the new property that we get by replacing each algebraic variety and each morphism of algebraic varieties occurring in \( P \) by their analytification. For instance, a subset \( U \subseteq X \) of an algebraic variety \( X \) is called strongly dense if \( U \) lies dense in \( X_{an} \).

5. Finite and Proper Morphisms

It is proved in [SGA 1] (in the more general context of schemes of finite type over \( \mathbb{C} \)) that the analytification of proper morphisms (cf. the definition below) is again proper (in the topological sense). For the proof of this, a very general version of Chow’s lemma is needed. We will not invoke this but instead prove a stronger version of it for morphisms with a quasi-projective domain. The trade-off is that we will only be able to prove the strong properness of proper morphisms for the case of a quasi-projective domain. However, our version of Chow’s lemma is enough to show that the analytification of a finite morphism is again finite (that is, closed with finite fibres).

(5.1) Definition. Recall that a morphism \( \phi : X \to Y \) between affine varieties gives \( \mathcal{O}_X(X) \) the structure of an \( \mathcal{O}_Y(Y) \)-algebra, via its comorphism \( \phi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) \). Now, \( \phi \) is called finite iff \( \mathcal{O}_X(X) \) is a finite \( \mathcal{O}_Y(Y) \)-algebra (i.e. it is finitely generated as an \( \mathcal{O}_Y(Y) \)-module). A general morphism \( \phi : X \to Y \) between algebraic varieties (not necessarily affine) is called finite iff \( Y \) has a cover \( Y = Y_1 \cup \ldots \cup Y_n \) by affine open subvarieties such that \( X_i := \phi^{-1}Y_i \subseteq X \) is affine and the restriction \( \phi|_{Y_i}^i : X_i \to Y_i \) is finite for all \( i \in \{1, \ldots, n\} \).

(5.2) Definition. A morphism \( \phi : X \to Y \) of algebraic varieties is called proper iff for any algebraic variety \( Z \), the product morphism

\[
\phi \times 1_Z : X \times Z \to Y \times Z
\]

is closed.
(5.3) **Example.** Finite morphisms (between affine varieties) are proper and in particular, closed immersions are proper. To wit, if \( \varphi : X \to Y \) is finite, then
\[
\mathcal{O}_X(X) = \mathcal{O}_Y(Y) f_1 + \ldots + \mathcal{O}_Y(Y) f_m
\]
for suitable \( f_1, \ldots, f_m \in \mathcal{O}_X(X) \). If now, \( Z \) is any other affine variety, then \( \mathcal{O}_{X \times Z}(X \times Z) \), as an Abelian group, is generated by all regular functions of the form
\[
f \cdot h : X \times Z \to \mathbb{C}, (x, z) \mapsto fx \cdot hz
\]
with \( f \in \mathcal{O}_X(X) \) and \( h \in \mathcal{O}_Z(Z) \). Similarly for \( \mathcal{O}_{Y \times Z}(Y \times Z) \). Hence, \( f_1 \cdot 1, \ldots, f_m \cdot 1 \) generate \( \mathcal{O}_{X \times Z}(X \times Z) \) as an \( \mathcal{O}_{Y \times Z}(Y \times Z) \)-module, meaning that \( \varphi \times 1_Z : X \times Z \to Y \times Z \) is finite and in particular closed.

(5.4) **Example.** If \( \varphi : X \to Y \) is a proper morphism and \( A \subseteq X \) a closed subvariety, then the restriction \( \varphi|_A : A \to Y \) is proper, too, which is an instance of the more general fact that the composite of two proper morphisms is proper.

(5.5) **Proposition.** A morphism \( \varphi : X \to Y \) is proper iff it is universally closed. That is, the base-change \( X \times_Y Y' \to Y' \) of \( \varphi \) along any \( \psi : Y' \to Y \) is closed.

**Proof.** The direction “\( \Leftarrow \)” is obvious, for if \( Z \) is any variety, the product map \( \varphi \times 1_Z \) is nothing but the base-change of \( \varphi \) along the standard projection \( Y \times Z \to Y \). As for “\( \Rightarrow \)”, we look at the following commutative diagram (where the pullback is taken in the category \( \text{AlgVar}_c \))
\[
\begin{array}{ccc}
X \times Y' & \xleftarrow{\varphi \times 1_Y} & X \\
\downarrow & & \downarrow \varphi \\
Y \times Y' & \xleftarrow{1_Y \times \varphi} & Y
\end{array}
\]
and observe that the left-hand horizontal maps are both closed immersions (by separatedness). But the map on the left is closed (by hypothesis) and hence, so is the middle one. 

(5.6) **Corollary.** If \( \varphi : X \to Y \) is a proper morphism and \( Y' \subseteq Y \) a subvariety, then the restriction \( \varphi^{-1}Y' \to Y' \) of \( \varphi \) is proper, too.

(5.7) **Theorem.** Let \( \varphi : X \to Y \) be a proper morphism and \( \psi : X \to Z \) any other morphism. Then \( (\varphi, \psi) : X \to Y \times Z \) is again proper.

**Proof.** For \( W \) any variety, we need to check that
\[
X \times W \xrightarrow{(\varphi, \psi) \times 1_W} Y \times Z \times W
\]
is a closed morphism. Let us write \( X \times W \) for the closure of the image in \( Y \times Z \times W \) and consider the diagonal \( X \times W \xrightarrow{\Delta} X \times W \times X \times W \), which has the standard projection to \( X \times W \) as an obvious retraction and is thus a closed immersion. But observe that because \( X \times W \hookrightarrow Y \times Z \times W \) is a closed immersion, its product with \( 1_{X \times W} \)
\[
X \times W \times X \times W \hookrightarrow X \times W \times Y \times Z \times W \cong X \times W \times Y \times W \times Z
\]
is a closed immersion, too. Now, we consider the commutative diagram
\[
\begin{array}{ccc}
X \times W & \xrightarrow{\Delta} & X \times W \times X \times W \\
\downarrow (\varphi, \psi) \times 1_W & & \downarrow (\varphi \times 1_W \times Y \times W \times Z) \\
Y \times Z \times W & \cong & Y \times W \times Z
\end{array}
\]
where \( \Delta': Y \times W \to (Y \times W)^2 \) is the diagonal and \( \varphi \times 1_{W,Y \times W \times Z} \) is closed by properness of \( \varphi \). It follows that the image of a closed \( A \subseteq X \times W \) in \( Y \times W \times Y \times W \times Z \), which we call \( A \), is closed and the injectivity of \( (\Delta' \times 1_Z) \) implies that \( ((\varphi, \psi)) 1_{W} A = (\Delta' \times 1_Z) A \). In particular, the image of \( A \) in \( Y \times Z \times W \) is closed. 

(5.8) **Corollary.** (Chow’s Lemma, Quasi-Projective Case) Let \( X \subseteq \mathbb{P}^n \) be locally closed (i.e. \( X \) quasi-projective) and \( \varphi: X \to Y \) a proper morphism of algebraic varieties. Then \( (\varphi, i): X \hookrightarrow Y \times \mathbb{P}^n \) is a closed immersion, where \( i: X \hookrightarrow \mathbb{P}^n \) denotes the inclusion.

**Proof.** By the above theorem, the image \( \overline{X} \) of \( X \) in \( Y \times \mathbb{P}^n \) is closed and the standard projection \( Y \times \mathbb{P}^n \to \mathbb{P}^n \) restricts to an inverse of the morphism \( X \to \overline{X} \), induced by \( (\varphi, i) \). 

(5.9) **Corollary.** In the same situation as in Chow’s Lemma, the analytification of \( \varphi \) is proper (in the topological sense). In particular, \( \varphi \) is strongly closed.

**Proof.** By Chow’s Lemma, \( \varphi \) factors as

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow & & \downarrow \text{pr}_Y \\
& \varphi & \\
& Y & \\
\end{array}
\]

where the horizontal morphism is a closed immersion. By the fact that analytification preserves closed immersions and products, as well as the compactness of \( \mathbb{P}^n \) in the strong topology, it follows that \( \varphi_{\text{an}} \) is proper. 

(5.10) **Corollary.** The analytification of a finite morphism is finite (in the topological sense). In particular, finite morphisms are strongly closed.

**Proof.** Let \( \varphi: X \to Y \) be finite and \( Y = Y_1 \cup \ldots \cup Y_n \) a cover of \( Y \) by affine open subvarieties, such that \( X_i := \varphi^{-1} Y_i \subseteq X \) is affine and the restriction \( \varphi_i^X: X_i \to Y_i \) is finite for all \( i \in \{1, \ldots, n\} \). Now, by the last corollary, all these restrictions are strongly closed and by (2.3) from the topological preliminaries, \( \varphi \) is itself strongly closed. 

This corollary can now be used to prove that the analytic topology on a variety, while being stronger than the Zariski topology, is not “too strong” in the sense that the situation from \( \mathbb{C}^n \) where Zariski-open subsets lie dense, generalises to the analytification of an arbitrary variety. This result can also be found in [Mumford, 1999, p. 58, Theorem 1] and in fact, the proof given here is essentially the same as the one in *loc. cit.* However, we can use the results obtained above to our advantage, whereas Mumford has to give a streamlined proof involving a minor index skirmish. For readability’s sake, we shall split off a small part of the proof as a lemma.

(5.11) **Lemma.** Let \( A \subseteq \mathbb{A}^n \) be a proper closed subset. Then \( A_{\text{an}} \subseteq \mathbb{C}^n \) is nowhere dense (i.e. it has no interior points).

**Proof.** Let \( a \in A \) arbitrary, \( b \in \mathbb{A}^n \setminus A \) and consider the straight line \( L \) through \( a \) and \( b \). Because \( L \cong \mathbb{A}^1 \) as a closed subvariety of \( \mathbb{A}^n \) and \( L \cap A \subseteq L \) is a proper closed subset, \( L \cap A \) must be finite. In particular every strongly open neighborhood of \( a \) intersects \( L \setminus A \). 

(5.12) **Proposition.** For \( X \) an irreducible algebraic variety and \( U \subseteq X \) a non-empty open subset, \( U \) is strongly dense (i.e. dense in \( X_{\text{an}} \)).
\textbf{Paragraph 5. Finite and Proper Morphisms}

\textbf{Proof.} It suffices to check this for $X$ affine, where by Noether’s normalisation lemma, we find a finite surjective morphism $\varphi : X \to \mathbb{A}^n$ for $n = \dim X$. Let us now define $A := X \setminus U$, which is a proper closed subset of $X$ and we need to check that $A_{\text{an}} \subset X_{\text{an}}$ is nowhere dense, for which we consider $a \in A_{\text{an}}$, arbitrary and $b := \varphi a \in (\varphi A)_{\text{an}}$. Because $A \subset X$ is a proper closed subset $\dim A < \dim X = n$, so that also $\dim \varphi A < n$, meaning that $\varphi A \subset \mathbb{A}^n$ is a proper closed subset again. By the last lemma, there is some sequence $(b_i)_{i \in \mathbb{N}} \subset (\mathbb{C}^n \setminus (\varphi A)_{\text{an}})^{\mathbb{N}}$ with limit $\lim_{i \to \infty} b_i = b \in (\varphi A)_{\text{an}}$.

As a next step, let us choose some $h \in \mathcal{O}_X(X)$ such that the only zero of $h$ in the fibre $\varphi^{-1} b = \varphi^{-1} \varphi a$ is $a$ itself and let $H \in \mathbb{C}[x_1, \ldots, x_n][y]$ be the minimal polynomial of $h$. That is to say, $H$ is the irreducible monic polynomial

$$H(x_1, \ldots, x_n, y) = y^d + c_1(x_1, \ldots, x_n)y^{d-1} + \ldots + c_d(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n][y]$$

such that $H(x_1, \ldots, x_n, h) = 0 \in \mathbb{C}[x_1, \ldots, x_n]$, which exists because $\varphi$ is finite. This gives us two surjective and finite morphisms (by the transitivity of integral dependence)

$$X \xrightarrow{(\varphi, h)} V_{\mathbb{A}^n + 1}(H) \xrightarrow{pr_1} \mathbb{A}^n \leftrightarrow \mathcal{O}_X(X) \supseteq \mathbb{C}[x_1, \ldots, x_n, h] \supseteq \mathbb{C}[x_1, \ldots, x_n],$$

whose composite is $\varphi$. If we abbreviate $x := (x_1, \ldots, x_n)$, $H(x, h) = 0$ implies that in particular $0 = H(\varphi a, ha) = c_d(\varphi a) = c_d(h)$ and hence $\lim_{i \to \infty} c_d(b_i) = 0$ by continuity of $c_d$. On the other hand, $c_d(b_i)$ is the product of all roots of $H(b_i, t) \in \mathbb{C}[t]$, whence we can find a sequence $(z_i)_{i \in \mathbb{N}} \subset \mathbb{C}$ such that $H(b_i, z_i) = 0$ for all $i \in \mathbb{N}$ and $\lim_{i \to \infty} z_i = 0$. Therefore, $(b_i, z_i)_{i \in \mathbb{N}}$ is a sequence in $V(H)_{\text{an}} \setminus ((\varphi, h)A)_{\text{an}}$ with limit $(\varphi a, 0)$ and hence $(\varphi a, 0)$ is no interior point of $((\varphi, h)A)_{\text{an}}$. But now, (2.9) implies that $a$ is no interior point of $A_{\text{an}}$ either because $(\varphi, h)^{-1}(\varphi a, 0) = \{a\}$. Since $a \in A_{\text{an}}$ was arbitrary, $A_{\text{an}} \subset X_{\text{an}}$ is indeed nowhere dense as claimed. \hfill \qed

(5.13) \textbf{Corollary.} Let $X$ be an irreducible algebraic variety and $U \subseteq X_{\text{an}}$ a non-empty open subset. Then $U$ is dense in the Zariski topology.

\textbf{Proof.} Assume $U \subseteq A$ for some proper closed $A \subset X$. Then $X \setminus A \subset X$ is a non-empty open subset but its closure in the strong topology is contained in $X \setminus U \neq X$. \hfill \qed

(5.14) \textbf{Corollary.} Let $X$ be a algebraic variety and $C \subseteq X$ a constructible subset. Then the closure of $C$ in Zariski topology and that in the strong topology are equal.

\textbf{Proof.} As taking the closure is distributive over taking finite unions, it suffices to check this for $C$ locally closed, meaning that $C = A \cap U$ with $A \subset X$ closed and $U \subset X$ open and by the same argument, we can even assume that $A$ is irreducible. But now, $C = A \cap U$ is an open subset of the irreducible subvariety $A \subseteq X$, so that $C \subseteq A$ is dense in the Zariski topology, as well as in the strong topology by the above corollary. This implies that $\overline{C} = A$ in the Zariski topology, as well as in the strong topology. \hfill \qed

Using this result, we immediately deduce that the converse of (5.9) also holds. That is to say, given a morphism $\varphi : X \to Y$, whose analytification is proper, $\varphi$ is itself proper. For this, one only needs to observe that analytification preserves products, that the Zariski-closure of a constructible set is the same as the closure in the strong topology (as shown in the last corollary) and finally, that images of morphisms are constructible (Chevalley’s theorem).

(5.15) \textbf{Remark.} One could now use what we just discussed to prove the converse of (5.10) but one needs the following (non-trivial) characterisation of finite morphisms: A morphism is finite iff it is proper and quasi-finite (i.e. has finite fibres).
The fact that a morphism \( \varphi: X \to Y \) with \( X \) quasi-projective is proper iff its analytification is so enables us to use the machinery of point-set topology to prove a fundamental result concerning complete varieties, which would otherwise be proved by algebraic reasoning, fiddling around with the construction of homogeneous ideals.

(5.16) **Definition.** A variety \( X \) is called **complete** iff the unique morphism \( X \to \{ \ast \} \) is proper. That is to say that for each variety \( Y \), the standard projection \( \text{pr}_Y: X \times Y \to Y \) is closed.

(5.17) **Corollary.** A quasi-projective variety \( X \) is complete iff \( X \) an is compact.  

(5.18) **Corollary.** Any projective variety (in particular, \( \mathbb{P}^n \) for any \( n \in \mathbb{N} \)) is complete.  

(5.19) **Corollary.** Projective morphisms are proper.

**Proof.** Let \( \varphi: X \to Y \) be projective, so that it factors as

\[
X \xrightarrow{\varphi} Y = X \xrightarrow{i} Y \times \mathbb{P}^n \xrightarrow{\text{pr}_Y} Y
\]

for some \( n \in \mathbb{N} \) and some closed immersion \( i: X \to Y \times \mathbb{P}^n \). But now, if \( Z \) is another variety, we get a factorisation

\[
X \times Z \xrightarrow{i \times 1_Z} Y \times \mathbb{P}^n \times Z \xrightarrow{\cong} Y \times Z \times \mathbb{P}^n \xrightarrow{\text{pr}_{Y,Z}} Y \times Z,
\]

where \( i \times 1_Z \) is obviously closed and \( \text{pr}_{Y,Z} \) is also closed, by completeness of \( \mathbb{P}^n \).

6. **Étale Morphisms**

The usual description of étale morphisms is that they are the algebraic analogue of local homeomorphisms. In most texts, the only justification for this claim is that it is obvious in the case of smooth varieties. However, in the next section, we shall justify this statement for arbitrary varieties (and it can even be justified for schemes of finite type over \( \mathbb{C} \)).

(6.1) **Definition.** For \( X \) an algebraic variety (over \( \mathbb{C} \)) and \( x \in X \), we write \( \mathcal{O}_{X,x}^{\mathbb{C}} \) for the cotangent cone at \( x \). It is defined as the graded \( \mathbb{C} \)-algebra

\[
\mathcal{O}_{X,x}^{\mathbb{C}} = \bigoplus_{n \in \mathbb{N}} \frac{m_x^n}{m_x^{n+1}}
\]

(where \( m_x^0 = \mathcal{O}_{X,x} \)) with the multiplication of two homogeneous elements \( [x] \in m_x^m/m_x^{m+1} \) and \( [y] \in m_x^n/m_x^{n+1} \) defined as \( [xy] := [xy] \in m_x^{m+n}/m_x^{m+n+1} \).

(6.2) **Definition.** A morphism \( \varphi: X \to Y \) between two varieties (over \( \mathbb{C} \)) is called **étale at** \( x \) if it induces an isomorphism of \( \mathbb{C} \)-algebras \( \varphi^*: \mathcal{O}_{Y,Y}^{\mathbb{C}} \cong \mathcal{O}_{X,x}^{\mathbb{C}} \). \( \varphi \) is called **étale** iff it is étale at every point \( x \in X \). Because of the short 5-Lemma, this is equivalent to requiring that \( \varphi^* : \mathcal{O}_{Y,Y}^{\mathbb{C}} / m_Y^{n+1} \to \mathcal{O}_{X,x}^{\mathbb{C}} / m_X^{n+1} \) is an isomorphism for all \( n \in \mathbb{N} \).

(6.3) **Remark.** Another equivalent definition (although we won’t need it) is the following: Let \( \varphi: X \to Y \) be a morphism between two varieties and \( x \in X \). Then \( \varphi \) is étale at \( x \) if it induces an isomorphism \( \hat{\varphi}^* : \hat{\mathcal{O}}_{Y,Y} \cong \hat{\mathcal{O}}_{X,x} \) between the completed stalks of \( \varphi x \) and \( x \).
Proof. The direction “⇒” can be found in [Atiyah and Macdonald, 1969, p. 112, Lemma 10.23]. As for the other direction, we have a natural isomorphism $C^x X = \text{gr}_{m_x} \mathcal{O}_{X,x} \cong \text{gr}_{m_x} \mathcal{E}_{X,x}$ by [ibid., p. 111, Proposition 10.22] and if $\tilde{\varphi}^*$ is an isomorphism, so is $\text{gr}_{m_x} \tilde{\varphi}^*$. \hfill \Box

(6.4) Observation. For $X$ a variety and $T_x X$ the tangent space at $x$ in the guise of
$$T_x X := \{ \delta : \mathcal{O}_{X,x} \to \mathbb{C} \mid \delta(fg) = f x \cdot \delta g + \delta f \cdot gx \}$$
the set of all derivations $\mathcal{O}_{X,x} \to \mathbb{C}$ at $x$, we have a natural isomorphism
$$T_x X \cong \text{Hom}_\mathbb{C}(m_x/m_x^2, \mathbb{C}), \; \delta \mapsto \left( \delta|_{m_x} : m_x/m_x^2 \to \mathbb{C}, \; [f] \mapsto \delta f \right),$$
from which we immediately conclude that if a morphism $\varphi : X \to Y$ is étale at $x \in X$, then the differential $D\varphi_x : T_x X \to T_y Y$ is an isomorphism. Conversely, if $x$ and $\varphi x$ are smooth, then $C^x X$ and $C^x Y$ are polynomial rings (in $\dim_x X$ and $\dim_{\varphi x} Y$ variables respectively) with the standard grading. In particular, an isomorphism $D\varphi_x : T_x X \cong T_{\varphi x} Y$ can be lifted to an isomorphism $C^x X \cong C^x Y$ and hence, $\varphi$ is étale at $x$.

We can use this observation to prove that the analytification of an étale morphism $\varphi : X \to Y$ between smooth varieties is a local homeomorphism. This follows readily from the fact that for $X$ and $Y$ smooth, these carry the structure of a (complex analytic) manifold and even better, for $x \in X$, the algebraic tangent spaces of $x$ and $\varphi x$ are the same as the analytic ones and we can use the inverse function theorem to prove our claim.

We now wish to prove this even for non-smooth varieties. The idea is to embed $X$ and $Y$ into affine spaces $\mathbb{A}^m$ and $\mathbb{A}^n$, so that the morphism $\varphi : X \to Y$ extends to a morphism $\mathbb{A}^m \to \mathbb{A}^n$. But these are now smooth varieties and we can use the above considerations. However, from this we usually cannot derive that the original $\varphi$ must be a local homeomorphism in the strong topology except if we can control the embeddings. As it turns out, $X$ and $Y$ can locally be embedded nicely, such that the morphism $\varphi$ between the embedded varieties is so-called “standard étale”. But we have to work a little bit to arrive at this point.

(6.5) Proposition. Étale morphisms have finite fibres.

Proof. Let $\varphi : X \to Y$ be étale and $y \in Y$. Taking the fibre above $y$, we get a commutative square as shown on the left below and taking differentials at an arbitrary point $x \in \varphi^{-1} y$, we get a commutative square as shown on the right.

Because $D\varphi_x : T_x X \cong T_y Y$ is an isomorphism, we conclude that $T_x(\varphi^{-1} y) = 0$ and hence $x \in \varphi^{-1} y$ is an isolated point. But as $\varphi^{-1} y \subseteq X$ is a closed subvariety, it is in particular quasi-compact and the claim follows. \hfill \Box

(6.6) Remark. Recall that if $Y$ is an affine variety and $X \subseteq Y$ a closed subvariety then the inclusion $X \hookrightarrow Y$ induces for each $x \in X$ an inclusion of tangent spaces
$$\text{res}^* : T_x X \hookrightarrow T_y Y, \; \delta \mapsto \delta \circ \text{res},$$
where $\text{res} : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is the restriction map. If $I_Y(X) = \langle g_1, \ldots, g_n \rangle \subseteq \mathcal{O}_Y(Y)$ for suitable $g_1, \ldots, g_n \in \mathcal{O}_Y(Y)$, then the subspace $T_x X \subseteq T_y Y$ has a more computable description: Considering $\langle g_1, \ldots, g_n \rangle : Y \to \mathbb{A}^n$ and taking the differential at $x$, it turns out that
$$T_x X = \text{Ker} \; D(g_1, \ldots, g_n)_{x}.$$
Proof. If we write \( g := (g_1, \ldots, g_n) : Y \to \mathbb{A}^n \) then \( X = g^{-1}(0) \) is the fibre of \( g \) over \( 0 \). This fibre is obviously reduced because \( g^* : \mathbb{C}[y_1, \ldots, y_n] \to \mathcal{O}_Y(Y) \), \( y_i \mapsto g_i \) and hence

\[
\mathcal{O}_X(X) \cong \mathcal{O}_Y(Y)/(g^* m_{\mathbb{C}^n,0}) \mathcal{O}_Y(Y).
\]

As localising commutes with taking quotients and the localisation of an ideal is the extended ideal in the localised ring, we conclude that \( \mathcal{O}_{X,x} \cong \mathcal{O}_{Y,x}/(g^* m_{\mathbb{C}^n,0}) \mathcal{O}_{Y,x} \) and hence \( (g^* m_{\mathbb{C}^n,0}) \mathcal{O}_{Y,x} \) is radical, i.e. the fibre \( X = g^{-1}(0) \) is reduced.

(6.7) Proposition. Let \( X, Y \) be affine varieties together with an isomorphism of \( \mathbb{C} \)-algebras \( \mathcal{O}_X(X) \cong \mathcal{O}_Y(Y)[t_1, \ldots, t_n]/(p_1, \ldots, p_n) \) for some \( n \in \mathbb{N} \), \( p_1, \ldots, p_n \in \mathcal{O}_Y(Y)[t_1, \ldots, t_n] \). Moreover, let \( \varphi : X \to Y \) be a morphism such that \( \varphi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) \) is the canonical map

\[
\mathcal{O}_Y(Y) \to \mathcal{O}_Y(Y)[t_1, \ldots, t_n] \to \mathcal{O}_Y(Y)[t_1, \ldots, t_n]/(p_1, \ldots, p_n) \cong \mathcal{O}_X(X).
\]

For \( x \in X \), the following are equivalent:

(a) the differential \( D\varphi_x : T_x X \to T_{\varphi(x)} Y \) is an isomorphism;

(b) the Jacobi matrix

\[
J_x := \text{Jac}(p_1, \ldots, p_n)|_x = \begin{bmatrix} \frac{\partial p_i}{\partial t_j}(x) \end{bmatrix}_{i,j}
\]

is invertible.

Moreover, if these two conditions are satisfied then \( \varphi_{|_{\text{an}}} \) is a local homeomorphism at \( x \).

Proof. By hypothesis, we can identify \( X \) with the subspace \( V_{X \times \mathbb{A}^n}(p_1, \ldots, p_n) \subseteq Y \times \mathbb{A}^n \) and under this identification, \( \varphi \) is nothing but the restriction of the standard projection. That is, we have a commutative triangle

\[
\begin{array}{ccc}
X & \longrightarrow & Y \times \mathbb{A}^n \\
\varphi \downarrow & & \downarrow \text{pr}_Y \\
Y & \to & 
\end{array}
\]

By the chain rule and the fact that the differential of the inclusion \( X \hookrightarrow Y \times \mathbb{A}^n \) is just the inclusion of tangent spaces, the differential \( D\varphi_x \) at \( x = (y, a) \) (with \( y \in Y \) and \( a = (a_1, \ldots, a_n) \in \mathbb{A}^n \)) is given by restricting \( (D\text{pr}_Y)_x \) to \( T_x X \). Observe that we have natural isomorphisms

\[
T_x(Y \times \mathbb{A}^n) \cong T_yY \oplus T_a \mathbb{A}^n, \quad \delta \mapsto (\delta_1, \delta_2), \quad \varepsilon \leftrightarrow (\varepsilon_1, \varepsilon_2),
\]

where \( \delta_1 g := \delta(g \cdot 1), \delta_2 h := \delta(1 \cdot h) \) and \( \varepsilon(g \cdot h) := \varepsilon_1 g \cdot h(a) + g(y) \cdot \varepsilon_2 h \); as well as

\[
T_a \mathbb{A}^n \cong \mathbb{A}^n, \quad \delta \mapsto (\delta t_1, \ldots, \delta t_n), \quad \partial_v|_a \leftrightarrow v.
\]

Putting these together, \( p = (p_1, \ldots, p_n) : Y \times \mathbb{A}^n \to \mathbb{A}^n \) gives us a commutative square

\[
\begin{array}{ccc}
T_x(Y \times \mathbb{A}^n) & \xrightarrow{D\varphi_x} & T_{\varphi(x)} \mathbb{A}^n \\
\cong & & \cong \\
T_y Y \oplus T_a \mathbb{A}^n \cong & & \cong \\
\cong & & \cong
\end{array}
\]
and chasing an element \((\varepsilon, v) \in T_y Y \oplus \mathbb{A}^n\) around the diagram, its image is
\[
(\varepsilon p_i(y, -) + \partial p_i(y, -)|_{a_i})_{i \in \{1, \ldots, n\}}.
\]
In particular for \(j \in \{1, \ldots, n\}\), the image of \((0, \varepsilon_j)\) is
\[
\left( \frac{\partial p_i(y, -)}{\partial t_j} \right)_{i,j} = \left( \frac{\partial p_i(x)}{\partial t_j} \right)_{i,j}.
\]
To sum this up, under all identifications we made above, the differential of \(x\) is
\[
Dp_x = \left[ * \begin{pmatrix} \frac{\partial p_i(x)}{\partial t_j} \end{pmatrix}_{i,j} \right] = \left[ * J_x \right] : T_y Y \oplus \mathbb{A}^n \to \mathbb{A}^n.
\]
Now, as \(I_{Y \times \mathbb{A}^n}(X) = (p_1, \ldots, p_n)\), by the above remark
\[
T_x X = \text{Ker } Dp_x = \text{Ker } \left[ * J_x \right]
\]
and again upon identifying \(T_x X(Y \times \mathbb{A}^n)\) with \(T_y Y \oplus T_x \mathbb{A}^n\), \((D \text{pr}_Y)_x\) is nothing but the standard projection \(\pi : T_y Y \oplus T_x \mathbb{A}^n \to T_y Y\). Recalling that \(D \varphi_x = (D \text{pr}_Y)_x|_{T_x X} = \pi|_{T_x X}\), we immediately see that \(D \varphi_x\) is invertible if \(J_x\) is so. To wit, if \(D \varphi_x\) is invertible then there’s for every \(\delta \in T_y Y\) exactly one \(v \in \mathbb{A}^n\) such that \((\delta, v) \in T_x X\), meaning that \(* \delta + J_x v = 0\). In particular, putting \(\delta = 0\), there is exactly one \(v \in \mathbb{A}^n\) such that \(J_x v = 0\), which must be \(v = 0\). Hence \(\text{Ker } J_x = 0\) and by the rank-nullity theorem, \(J_x\) is invertible. Conversely, if \(J_x\) is invertible then \(D \varphi_x\) is an isomorphism, for if \(\delta \in T_y Y\) and \((\delta, v) \in T_x X\) for some \(v \in \mathbb{A}^n\) then \(* \delta + J_x v = 0\) and consequently \(v = -J_x^{-1} \delta\), so that \(D \varphi_x\) is indeed bijective.

For the second claim, let’s choose a closed immersion \(Y \subseteq \mathbb{A}^m\) for some \(m \in \mathbb{N}\) and construct from this the commutative diagram
\[
\begin{array}{ccc}
Y \times \mathbb{A}^n & \xrightarrow{\text{pr}_Y} & \mathbb{A}^m \times \mathbb{A}^n \\
\downarrow & & \downarrow \text{pr}_{\mathbb{A}^m} \\
X & \xrightarrow{\varphi} & \mathbb{A}^m
\end{array}
\]
By choosing preimages \(P_1, \ldots, P_n\) of \(p_1, \ldots, p_n\) in \(\mathbb{C}[y_1, \ldots, y_m, t_1, \ldots, t_n]\) we get a regular map
\[
f : (P_1, \ldots, P_n) : \mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^n
\]
of which we know \(X = (Y \times \mathbb{A}^n) \cap f^{-1} 0\) and whose differential at \(x = (y, a) \in X \subseteq Y \times \mathbb{A}^n\) is
\[
Df_x = \left[ * | J_x \right].
\]
But \(x\) is a smooth point in \(\mathbb{A}^m \times \mathbb{A}^n\), so that by the implicit function theorem, we find \(U \subseteq \mathbb{C}^m\) and \(V \subseteq \mathbb{C}^n\) open with \(x = (y, a) \in U \times V\) as well as a unique smooth (even holomorphic) map \(g : U \to V\) such that for all \((u, v) \in U \times V\), we have \(f(u, v) = 0\) iff \(v = gu\). But this means that the restriction of \((1_U, g) : U \to U \times V\) to
\[
\psi : Y_{\text{an}} \cap U \to X_{\text{an}} \cap (U \times V)
\]
(which makes sense because \(f(y', gy') = 0\) for all \(y' \in U\) and in particular, if \(y' \in Y\) then \((y', gy') \in (Y \times \mathbb{C}^n) \cap f^{-1} 0 = X\) provides a local inverse of \(\varphi_{\text{an}}\). \(\square\)
(6.8) **Example.** Let \( \varphi: X \to Y \) be a morphism between affine algebraic varieties and suppose that there is an isomorphism of \( \mathbb{C} \)-algebras \( \mathcal{O}_X(X) \cong (\mathcal{O}_Y(Y)[t]/(p))_q \), where \( p \in \mathcal{O}_Y(Y)[t] \) is monic and \( q \in \mathcal{O}_Y(Y)[t]/(p) \) with \( [p'] \) invertible in \( (\mathcal{O}_Y(Y)[t]/(p))_q \) (e.g. \( q = [p'] \)). Moreover, this isomorphism is required to be compatible with \( \varphi \) in the sense that the following triangle commutes.

\[
\begin{array}{ccc}
\mathcal{O}_Y(Y) & \xrightarrow{\varphi^*} & \mathcal{O}_X(X) \\
\downarrow & & \cong \\
(\mathcal{O}_Y(Y)[t]/(p))_q & & ,
\end{array}
\]

where the diagonal arrow is the canonical morphism

\( \mathcal{O}_Y(Y) \hookrightarrow \mathcal{O}_Y(Y)[t] \to \mathcal{O}_Y(Y)[t]/(p) \to (\mathcal{O}_Y(Y)[t]/(p))_q; \ g \mapsto [g]/1. \)

In this case, the differential \( D\varphi_x \) is an isomorphism at every \( x \in X \) and hence, by the proposition just proved, \( \varphi_{an} \) is a local homeomorphism.

**Proof.** We identify \( \mathcal{O}_X(X) \cong (\mathcal{O}_Y(Y)[t]/(p))_q \cong \mathcal{O}_Y(Y)[t,s]/(p,qs - 1) \) and calculate for \( x \in X \)

\[
\det \begin{bmatrix}
\frac{\partial p}{\partial t}(x) & \frac{\partial p}{\partial s}(x) \\
\frac{\partial (qs - 1)}{\partial t}(x) & \frac{\partial (qs - 1)}{\partial s}(x)
\end{bmatrix} = \det \begin{bmatrix}
p'(x) & 0 \\
q(x) & p'(x)q(x) = (p'q)(x) \neq 0
\end{bmatrix}
\]

because \( p' \) and \( q \) are invertible in \( \mathcal{O}_X(X) \). As \( \varphi^* \) is the standard projection upon the above identification, the last lemma proves our claim. \( \square \)

(6.9) **Remark.** One can even show that \( \varphi \) from this example is étale and we call a morphism of this form standard étale. In fact, the conditions for \( \varphi: X \to Y \) and \( x \in X \) as in the proposition are equivalent to \( \varphi \) being étale at \( x \in X \).

(6.10) **Theorem.** Étale morphisms are locally standard étale. I.e. a morphism \( \varphi: X \to Y \) between two algebraic varieties is étale iff for every \( x \in X \) there are affine open neighborhoods \( U \ni x \) and \( V \ni \varphi x \) with \( \varphi U \subseteq V \) and \( \varphi|_U: U \to V \) standard étale.

**Proof.** [Milne, 1980, p. 26, Theorem 3.14] \( \square \)

(6.11) **Corollary.** For \( \varphi: X \to Y \) an étale morphism, the analytification \( \varphi_{an}: X_{an} \to Y_{an} \) is a local homeomorphism. \( \square \)

7. **Irreducible Algebraic Curves**

In the analytic proof that irreducible plane algebraic curves are connected in the strong topology, we investigated some restriction of the projection onto the \( x \)-axis, which turned out to be a covering map. Afterwards, we used analytic tools to study the monodromy action of this covering map and were able to show that it is transitive.

Now, our algebraic proof relies on the observation that only considering the projection onto some coordinate axis is a vast constraint and not really necessary. The intuition is that by allowing more general morphisms (namely finite or even more general, proper ones) that are still covering maps on a dense open subvariety, one should be able to collapse the covering’s sheets in one point, thus rendering all arguments for transitivity of the monodromy action in that point needless.
Lemma. Let \( \varphi : X \to Y \) be a surjective proper morphism between two irreducible affine varieties. Then there is a dense open subset \( V \subseteq Y \) such that \( Y_{\text{sing}} \cap V = X_{\text{sing}} \cap \varphi^{-1}V = \emptyset \) and the restriction \( \varphi_{\text{an}}' : (\varphi^{-1}V)_{\text{an}} \to V_{\text{an}} \) of \( \varphi_{\text{an}} \) is a covering map (in the strong topology that is). Moreover, if \( \varphi \) is of finite degree \( d = [\mathbb{C}(X) : \mathbb{C}(Y)] \) then \( V \) be chosen in such a way that \( \varphi_{\text{an}}' \) is a \( d \)-sheeted covering.

Proof. By [Kraft, 2005, p.46, Theorem 4.2] there is a dense open \( U \subseteq X \) such that \( D_\varphi \) is surjective for all \( u \in U \) and we write \( S := \varphi(X_{\text{sing}}) \cup \varphi(X \setminus U) \). This is a proper closed subset of \( Y \) because \( \dim X_{\text{sing}} \) and \( \dim(X \setminus U) \) are strictly smaller than \( \dim X = \dim Y \). Putting \( V := Y \setminus S \), one easily checks \( \varphi^{-1}V \subseteq X \setminus X_{\text{sing}} \cap U \), and thus \( V \subseteq Y \setminus Y_{\text{sing}} \) by definition of \( U \). Moreover, \( \varphi \) restricts to a surjective proper and smooth (i.e. all differentials are isomorphisms) morphism

\[
\varphi' : \varphi^{-1}V = X \setminus \varphi^{-1}S \to Y \setminus S = V.
\]

By the inverse function theorem, \( \varphi_{\text{an}}' \) is a local homeomorphism and by (5.9) it is also proper.

But then (2.7) gives us that \( \varphi_{\text{an}}' \) is a covering map. As for the last claim, if \( \varphi \) is of finite degree \( d \) then by [Kraft, 2005, p.34, Proposition 3.8] there exists a dense open \( V' \subseteq Y \) such that \( \#\varphi^{-1}y = d \) for all \( y \in V' \) and restricting \( \varphi_{\text{an}} \) even further to \( (\varphi^{-1}(V \cap V'))_{\text{an}} \) yields the desired result. \( \square \)

Theorem. Let \( C \) be an irreducible affine curve, together with a point \( a \in C \) and a proper surjective morphism \( \varphi : C \to \mathbb{A}^1 \), whose fibre above \( \varphi a \) consists of \( a \) itself only. Then \( C \) is connected in the strong topology.

Proof. By the lemma just proven, there is a dense open \( V \subseteq \mathbb{A}^1 \) such that \( C_{\text{sing}} \cap \varphi^{-1}V = \emptyset \) and the restriction \( \varphi' : \varphi^{-1}V \to V \) of \( \varphi \) is a covering map in the strong topology. Because this is a finite covering and \( V_{\text{an}} \) is path-connected, \( (\varphi^{-1}V)_{\text{an}} \) has only finitely many connected components. As \( (\varphi^{-1}V)_{\text{an}} \) lies dense in \( C_{\text{an}} \), this implies that for every \( b \in C_{\text{an}} \) there is a sequence that is contained in a single connected component of \( (\varphi^{-1}V)_{\text{an}} \) and converges to \( b \).

Hence it suffices to prove that if \( M \subseteq (\varphi^{-1}V)_{\text{an}} \) is a connected component, the closure of \( M \) in \( C_{\text{an}} \) contains \( a \). To do so, we choose a sequence \((y_n)_{n \in \mathbb{N}} \) in \( V \) converging to \( \varphi a \), which we can do because \( V_{\text{an}} \) lies dense in \( C \). But by (2.10) the restriction \( \varphi'|_M : M \to V \) is surjective, so that \((y_n)_{n \in \mathbb{N}} \) lifts to a sequence \((x_n)_{n \in \mathbb{N}} \) in \( M \). Observing that \( \varphi_{\text{an}} \) is proper, we conclude that \( \varphi_{\text{an}}^{-1}(y_n)_{n \in \mathbb{N}} \subseteq C_{\text{an}} \) is compact and therefore \((x_n)_{n \in \mathbb{N}} \) has a convergent subsequence, whose limit we call \( x \). But \( \varphi_{\text{an}} \) is continuous, \( (\varphi_{\text{an}}x_n)_{n \in \mathbb{N}} \) converges to \( \varphi_{\text{an}}a \) and the fibre above \( \varphi_{\text{an}}a \) consists only of \( a \) itself, so that \( x = a \). \( \square \)

Rather than now constructing for every irreducible affine curve \( C \) a proper surjective morphism \( \varphi : C \to \mathbb{A}^1 \) that has some one-point fibre, we start with a rational function \( f \in \mathbb{C}(C) \) as in (3.1), and then manipulate the model curve \( C \) of \( \mathbb{C}(C) \) in such a way that \( f \) becomes a finite regular function. Because \( f \) was chosen such that it behaves well under manipulation of the domain, this will then allow us to use the theorem just proved, and if our manipulation is careful enough we can still deduce the strong connectedness of \( C \) from that of the manipulated domain. But first, we shall prove the following elementary observation.

Lemma. Let \( k \) be a finitely generated field extension of \( \mathbb{C} \) and \( f_1, \ldots, f_n \) a transcendence basis. If \( A \) is the integral closure of \( \mathbb{C}[f_1, \ldots, f_n] \) in \( k \), then its field of fractions is \( \text{Frac} A = k \).

Proof. Let \( S := \mathbb{C}[f_1, \ldots, f_n] \setminus \{0\} \), which is a multiplicative system because \( k \) is an integral domain. By [Atiyah and Macdonald, 1969, p. 62, Proposition 5.12] \( S^{-1}A \) is the integral closure of \( S^{-1}\mathbb{C}[f_1, \ldots, f_n] = \mathbb{C}(f_1, \ldots, f_n) \) in \( S^{-1}k = k \). But \( k \) is integral over \( \mathbb{C}(f_1, \ldots, f_n) \) because \( f_1, \ldots, f_n \) is a transcendence basis of \( k \) over \( \mathbb{C} \) and hence \( k \subseteq S^{-1}A \subseteq \text{Frac} A \). \( \square \)
(7.4) **Theorem.** Every irreducible affine curve $C$ is connected in the strong topology.

*Proof.* Without loss of generality, we may assume that $C$ is normal, for otherwise we have a surjective finite morphism $\tilde{C} \to C$ with $\tilde{C}$ the normalisation of $C$ and by surjectivity, if $\tilde{C}$ is strongly connected, so is $C$.

Now we choose some $f \in \mathbb{C}(C) \setminus \mathbb{C}$ as in (3.1) (i.e. such that for every affine algebraic curve $C'$ together with a chosen isomorphism $\mathbb{C}(C') \cong \mathbb{C}(C)$, $f$ has at most one zero on $C'$) and consider the integral closure $A$ of $\mathbb{C}[f]$ in $\mathbb{C}(C)$, yielding a surjective finite morphism

$$A \supseteq \mathbb{C}[f] \cong \mathbb{C}[x] \quad \iff \quad \varphi: \text{Max } A \to \text{Max } \mathbb{C}[f] \cong \mathbb{A}^1,$$

where Max $A$ is the maximal spectrum of $A$, carrying the Zariski topology (which is an affine algebraic variety by [Kraft, 2005, p.51, Proposition 5.1]) and the isomorphism $\mathbb{C}[x] \cong \mathbb{C}[f]$ is given by $x \mapsto f$. By surjectivity, $\varphi^{-1}(0) \neq \emptyset$ and by definition of $f$, it is even a one-point fibre.

It therefore follows from the above theorem, that Max $A$ is strongly connected and by the last lemma (7.3) $\text{Max } A = \mathbb{C}(C)$, so that there are (non-empty) isomorphic special open subsets $U \subseteq \text{Max } A$ and $V \subseteq C$. But $\text{Max } A \setminus U$ is only a finite number of points, which are smooth because $A$ is integrally closed (i.e. $A$ is a normal curve) and therefore $U \subseteq \text{Max } A$ is still connected; hence so is $V \subseteq C$. Finally, this implies that $\overline{V} = C$ is connected, too. $\square$

8. **IRREDUCIBLE ALGEBRAIC VARIETIES**

(8.1) **Lemma.** Let $f: \mathbb{C}^n \to \mathbb{C}$ be a continuous function that is holomorphic on a non-empty Zariski-open (hence dense) subset $U \subseteq \mathbb{C}^n$. If $f$ satisfies an integral dependence relation over $\mathbb{C}[x_1, \ldots, x_n]$ (i.e. there is some monic $p(t) = t^d - \sum_{i=0}^{d-1} p_i t^i \in \mathbb{C}[x_1, \ldots, x_n][t]$ such that $p(f): \mathbb{C}^n \to \mathbb{C}$ is the zero-function) then $f$ is a polynomial function.

*Proof.* We observe that we find some $M \in \mathbb{N}$, $R \in \mathbb{R}_{\geq 1}$ such that $|p_i(x)| \leq \|x\|^M$ for all $i \in \{0, \ldots, d-1\}$ and all $x \in \mathbb{C}^n$ with $\|x\| > R$. If $x \in \mathbb{C}^n$ with $\|x\| > R$ is such that $|fx| \geq 1$ then

$$|fx|^d = \left| \sum_{i=0}^{d-1} p_i(f(x)) f(x)^i \right| \leq \sum_{i=0}^{d-1} \|x\|^M |f(x)|^i \leq \sum_{i=0}^{d-1} \|x\|^M |f(x)|^{d-1} = d \|x\|^M |f(x)|^{d-1}$$

and therefore $|fx| \leq d \|x\|^M$. On the other hand, if $x \in \mathbb{C}^n$ with $\|x\| > R$ is such that $|fx| < 1$ then trivially again $|fx| \leq d \|x\|^M$ as before.

Now we proceed by induction on $n$. If $n = 1$ then $\mathbb{C} \setminus U$ is finite and by the Riemann extension theorem $f$ is entire (i.e. holomorphic on all of $\mathbb{C}$). But by the above considerations, its Taylor series around 0 has zero-coefficients in degree $> M$, whence $f$ is polynomial. For the inductive step, consider $f: \mathbb{C}^{n+1} \to \mathbb{C}$, $U \subseteq \mathbb{C}^{n+1}$ as in the claim and let

$$V := \text{pr}_1 U = \{ x \in \mathbb{C}^n \mid U \cap (\{x\} \times \mathbb{C}) \neq \emptyset \},$$

where $\text{pr}_1: \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, $(x, y) \mapsto x$, which is open in the Zariski topology. By the inductive hypothesis $f|_{\{x\} \times \mathbb{C}}: \{x\} \times \mathbb{C} \cong \mathbb{C} \to \mathbb{C}$ is a polynomial function for every $x \in V$, so that we have a family of functions $a_0, \ldots, a_M: V \to \mathbb{C}$ (with $M \in \mathbb{N}$ as above) satisfying

$$f(x, y) = \sum_{i=0}^{M} a_i(x) y^i \quad \text{for all } x \in V, y \in \mathbb{C}$$

(the degree is $\leq M$ for all $x \in V$ by the upper bound established above). Furthermore, if we consider $\text{pr}_2: \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, $(x, y) \mapsto y$, which is again open in the Zariski topology,

$$W := \text{pr}_2 U = \{ y \in \mathbb{C} \mid U \cap (\mathbb{C}^n \times \{y\}) \neq \emptyset \}$$
is a non-empty Zariski-open (i.e. cofinite) subset of $\mathbb{C}$ and we can choose pairwise distinct $c_0, c_1, \ldots, c_M \in W$. This then gives us a system of linear equations $f(x, c_j) = \sum_{i=0}^M a_i(x)c_j^i$ for $j \in \{0, \ldots, M\}$ and $x \in V$ or in matrix notation

$$[f(x, c_j)]_{j \in \{0, \ldots, M\}} = C[a_i(x)]_{i \in \{0, \ldots, M\}}$$

where

$$C = \begin{bmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^M \\ 1 & c_1 & c_1^2 & \cdots & c_1^M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_M & c_M^2 & \cdots & c_M^M \end{bmatrix}$$

is the Vandermonde matrix defined by the $c_j$. As $C$ is invertible, we find for every $i \in \{0, \ldots, M\}$ suitable $\lambda_{i,0}, \ldots, \lambda_{i,M} \in \mathbb{C}$ (namely $\lambda_{i,j} = (C^{-1})_{i,j}$ with matrix indices starting at 0) such that $a_i(x) = \sum_{j=0}^M \lambda_{i,j} f(x, c_j)$ for all $x \in V$ and can define

$$g: \mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}, (x, y) \mapsto \sum_{i=0}^M \sum_{j=0}^M \lambda_{i,j} f(x, c_j) y^i,$$

which is a polynomial function by the inductive hypothesis. But now $f(x, y) = g(x, y)$ for all $(x, y) \in V \times \mathbb{C}$ and this is a dense open subset of $\mathbb{C}^{n+1}$, implying that $f = g$ by (2.1). \hfill $\square$

(8.2) **Remark.** In fact, our reasoning from the induction basis works in arbitrary dimension. A continuous function $f: \mathbb{C}^n \to \mathbb{C}$ that is holomorphic on a non-empty Zariski-open $U \subseteq \mathbb{C}^n$ is entire because $\mathbb{C}^n \setminus U$ is a thin subset of $\mathbb{C}^n$ in the sense of [Gunning and Rossi, 1965, p.19] and we can use the Riemann extension theorem as found in loc. cit.. Our lemma then just says that $\mathbb{C}[x_1, \ldots, x_n]$ is integrally closed in the ring $\mathcal{H}_\mathbb{C}(\mathbb{C}^n)$ of holomorphic functions on $\mathbb{C}^n$.

(8.3) **Theorem.** Let $X$ be an irreducible algebraic variety. Then $X_{\text{an}}$ is connected.

**Proof.** It suffices to consider the case where $X$ is affine. By Noether normalisation, we choose a finite surjective morphism $p: X \to \mathbb{A}^n$, yielding a finite field extension $\mathbb{C}(x_1, \ldots, x_n) \subseteq \mathbb{C}(X)$. We take the Galois closure

$$\mathbb{C}(x_1, \ldots, x_n) \subseteq \mathbb{C}(X) \subseteq K$$

with Galois group $G := \text{Aut}_{\mathbb{C}(x_1, \ldots, x_n)} K$, which is finite because $\mathbb{C}(x_1, \ldots, x_n) \subseteq \mathbb{C}(X)$ is finite, and consider the integral closure $A$ of $\mathbb{C}[x_1, \ldots, x_n]$ in $K$. By [Kraft, 2005, p.51, Proposition 5.1] this defines an irreducible normal affine variety $Z := \text{Max} A$ and because $\mathcal{O}_X(X)$ is integral over $\mathbb{C}[x_1, \ldots, x_n]$, it comes equipped with a finite surjective morphism $\varphi: Z \to X$. Now the Galois group $G$ acts faithfully on $Z$ because $\text{Frac} A = K$ by (7.3) and hence if $f \in G$ with $fa = a$ for all $a \in A$ then $A \subseteq \text{Fix}(f)$ but then also $\text{Frac} A = K = \text{Fix}\{\text{id}_K\} \subseteq \text{Fix}(f)$, so that $f = \text{id}_K$. Moreover, the composition

$$Z \xrightarrow{\varphi} \mathbb{A}^n \quad \text{such that} \quad Z \xrightarrow{\varphi} X \xrightarrow{p} \mathbb{A}^n$$

is the geometric quotient $Z/G$, i.e. $A^G = \mathbb{C}[x_1, \ldots, x_n]$. As this is surjective, it suffices to check that $Z_{\text{an}}$ is connected. For this notice that each fibre of $q$ contains exactly one closed $G$-orbit (cf. [Kraft, 1984, p.96, Bemerkung 1]) and because $q$ is finite, each fibre of $q$ is a single $G$-orbit (or put differently, $G$ acts transitively on the fibres of $q$) and the surjectivity of $q$ implies that it is in fact a quotient of $Z$ as a $G$-set. By (7.1) there is some dense open $U \subseteq \mathbb{A}^n$ such that $Z_{\text{sing}} \cap q^{-1}U = \emptyset$ and the restriction $q': q^{-1}U \to U$ of $q$ is a covering map in the strong topology.

Now let $Z_{\text{an}} = Z_1 \cup \ldots \cup Z_d$ be the decomposition of $Z_{\text{an}}$ into its connected components. These are only finitely many because $q_{\text{an}}$ is a finite covering and $(q^{-1}U)_{\text{an}} \subseteq Z_{\text{an}}$ is dense. If we write $q_i := q_{\text{an}}|Z_i: Z_i \to \mathbb{C}^n$ for $i \in \{1, \ldots, d\}$ then the $q_i$ are surjective by (2.10)
because $Z_i \cap (q^{-1}U)_{\text{an}}$ contains a connected component of $(q^{-1}U)_{\text{an}}$ and $U_{\text{an}} \subseteq \mathbb{C}^n$ is dense and path-connected. Furthermore, the $q_i$ are proper because the $Z_i$ are are closed in $Z$.

As seen above, $G$ acts transitively on the fibres of $q'$, implying that for every two $i, j \in \{1, \ldots, d\}$ there is some $g \in G$ such that $gZ_i = Z_j$ (in particular $Z_i \cong Z_j$). For $i \in \{1, \ldots, d\}$ let us define

$$N_i := \text{Norm}_{O_i}(Z_i) = \{g \in G \mid gZ_i = Z_i\} \triangleleft G,$$

which has $\#N_i = n/d$ and acts transitively on the fibres of $q_i$. In particular, $q_i$ induces a continuous bijection $Z_i/N_i \to \mathbb{C}^n$ that is also closed (because $q_i$ is so), hence a homeomorphism.

Let us write $C(Z_i)$ for the $\mathbb{C}$-algebra of continuous functions $Z_i \to \mathbb{C}$ (with all operations defined pointwise). If $i \in \{1, \ldots, d\}$ then $Z_i \subseteq Z_{\text{an}}$ is open and therefore $Z_i \subseteq Z$ lies dense (in the Zariski topology) by (5.13). Hence the restriction map

$$\varphi_i : \mathcal{O}_Z(Z) \to C(Z_i), f \mapsto f|_{Z_i}$$

is an injective morphism of $\mathbb{C}$-algebras, so that this equips $C(Z_i)$ with the structure of a faithful $\mathcal{O}_Z(Z)$-algebra. By restriction of scalars, $C(Z_i)$ becomes a $\mathbb{C}[x_1, \ldots, x_n]$-algebra and each $\varphi_i$ is a morphism of $\mathbb{C}[x_1, \ldots, x_n]$-algebras. Moreover, each $\varphi_i$ is $N_i$-equivariant and in particular $\varphi_i(\mathcal{O}_Z(Z))_{N_i} \subseteq C(Z_i)_{N_i}$, implying that each $f \in \mathcal{O}_Z(Z)_{N_i}$ gives rise to a continuous function

$$\bar{f} : \mathbb{C}^n \cong Z_i/N_i \to \mathbb{C}$$

induced by $\varphi_i f$. Furthermore, $(q^{-1}U)_{\text{an}}$ is naturally a complex analytic manifold, $(f|_{(q^{-1}U)})_{\text{an}}$ is holomorphic, $q'_{\text{an}}$ a local homeomorphism and we have a commutative diagram

$$
\begin{array}{ccc}
(q^{-1}U)_{\text{an}} & q'_{\text{an}} & U_{\text{an}} \\
\downarrow & & \downarrow \\
Z_{\text{an}} & q_{\text{an}} & \mathbb{C}^n \\
\downarrow f_{\text{an}} & & \downarrow f \\
\mathbb{C} & & \mathbb{C}
\end{array}
$$

so that $\bar{f}$ is holomorphic on $U_{\text{an}}$. Now $f$ is integral over $\mathbb{C}[x_1, \ldots, x_n]$ (because $q$ is finite) but as $\varphi_i$ is a morphism of $\mathbb{C}[x_1, \ldots, x_n]$-algebras, so is $\bar{f}$. By the last lemma then, $f$ is a polynomial function, so that we conclude $(\varphi_i \mathcal{O}_Z(Z))_{N_i} = \mathbb{C}[x_1, \ldots, x_n]$.

But now we are done because each $\varphi_i$ induces an isomorphism $\mathcal{O}_Z(Z) \cong \varphi_i \mathcal{O}_Z(Z)$ that is $N_i$-equivariant, hence restricting to $\mathcal{O}_Z(Z)_{N_i} \cong (\varphi_i \mathcal{O}_Z(Z))_{N_i} = \mathbb{C}[x_1, \ldots, x_n]$ for every $i \in \{1, \ldots, d\}$. Therefore also $\mathcal{O}_Z(Z)_{N_i} = \mathbb{C}[x_1, \ldots, x_n]$, implying $N_i = G$ for every $i \in \{1, \ldots, d\}$ and thus $d = 1$ as claimed.

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