A characterization of Kerr-Newman space-times and some applications

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Introduction

- Question: how can we tell whether an electro-vacuum space-time is Kerr-Newman, at least locally?
- Partial answer: in the stationary case, the simultaneous vanishing of two tensor quantities
- Construction based on that of Marc Mars [Mar99]
- Large body of existing literature (list is by no means complete): [HS73, BS81, Sim84a, Sim84b, DKM84, FS99, BJM01, BCJM04]... + recent work on initial data sets (e.g. Valiente-Kroon and Garcia-Parrado [GLK08]).
• No hair theorem? Requires real analyticity
  • In the smooth class, (stationary, non-extremal) black-hole uniqueness reduces to a problem of rigidity of the bifurcate sphere [IK09, Won09a]
  • Or, allowing deformations at the bifurcate sphere, perturbative no-hair theorem (W & Yu, in prep, see also [AIK09]), strengthening [Car73]

• Suitable notion of almost-Killing vector can lead to application in stability of Kerr-Newman
Basic assumptions

We consider a space-time \((\mathcal{M}, g_{ab})\) and a two-form \(H_{ab}\) on \(\mathcal{M}\):

- \(\mathcal{M}\) is 4d, orientable, paracompact, simply-connected manifold; \(g_{ab}\) is a Lorentzian metric tensor \((- + + +)\)
- Einstein-Maxwell field equations are satisfied:
  \[
  Ric(g)_{ab} = 2H_{ac}H^c_b - \frac{1}{2}g_{ab}H_{cd}H^{cd}
  \]
  \[
  = (H + i^*H)_{ac}(H - i^*H)_b^c
  \]
  \[
  \nabla_{[c}(H + i^*H)_{ab]} = 0
  \]
- There exists a non-trivial vector-field \(t^a\) such that \(\mathcal{L}_t g_{ab} = 0, \mathcal{L}_t H_{ab} = 0\).

Note we don’t (yet) need to assume \(\mathcal{M}\) is asymptotically flat nor \(t^a\) is time-like.
Notations

- For an arbitrary tensor $Z$, $Z^2 := g(Z, Z)$ the induced Lorentzian norm (which can be arbitrarily signed); for complex tensors, “norm” is extended linearly.
- $\ast$ acts on forms as the Hodge dual. On a two-form $\omega_{ab}$ we write $\ast \omega_{ab} = \frac{1}{2} \epsilon_{abcd} \omega^{cd}$ where $\epsilon_{abcd}$ is the volume form.
- On $\text{Sym}_2(\Lambda^2 T^* M)$ (e.g. a curvature tensor), $\ast$ naturally extends to a left- and a right- Hodge dual, which are in general not equal. We write $\ast R_{abcd} = \frac{1}{2} \epsilon_{abef} R_{efcd}$, and $R^*_{abcd} = \frac{1}{2} R_{abef} \epsilon^{ef}_{\quad cd}$.
- Define the complex tensor
  \[
  I_{abcd} = \frac{1}{4} (g_{ac} g_{bd} - g_{ad} g_{bc} + i \epsilon_{abcd})
  \]
Review of ASD forms

- On a 4d, Lorentzian manifold, $** = -1$ for two-forms
- Factor $\Lambda^2 T^* \mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}$, the space of complex two-forms, into eigenspaces $\Lambda_\pm$ of eigenvalues $\pm i$, the self-dual $(\pm)$ and anti-self-dual $(\mp)$ spaces.
- An arbitrary (real-valued) two-form $H_{ab}$ can be decomposed
  $$H_{ab} = \mathcal{H}_{ab} + \bar{\mathcal{H}}_{ab}$$
  where $\mathcal{H}_{ab} \in \Lambda_-$ is given by $2\mathcal{H} = H + i^* H$.
- In this notation, Einstein-Maxwell equations can be written
  $$Ric(g)_{ab} = 4 \mathcal{H}_{ac} \bar{\mathcal{H}}_b^c \quad \nabla^a \mathcal{H}_{ab} = 0$$
- The tensor $\mathcal{I}_{abcd}$ is the projection to $\Lambda_-$ of the induced Lorentzian product on $\Lambda^2 T^* \mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}$
Review of ASD forms II

- The tensors $Sym_2(\Lambda^2 T^* M)$ naturally also factors into diagonal components taking $\Lambda_\pm \to \Lambda_\pm$, and the anti-diagonal component taking $\Lambda_\pm \to \Lambda_\mp$.

- By a calculation first due to Singer and Thorpe [ST69], for a Riemann curvature tensor, the diagonal components are given by the Weyl curvature and the scalar curvature, the anti-diagonal component is given by the trace-free Ricci curvature.

- Einstein-Maxwell: scalar curvature is zero

- The left- and right- Hodge duals of the Weyl tensor are equal, the left- and right- Hodge duals of the Ricci part of the curvature differs by a minus sign.

- So for the Weyl tensor $W_{abcd}$ we can also define an ASD version $C_{abcd} = \frac{1}{2}(W_{abcd} + i^* W_{abcd})$. 
Symmetric spinor product

For two ASD two-forms $\mathcal{X}_{ab}, \mathcal{Y}_{ab}$, define

$$(\mathcal{X} \tilde{\otimes} \mathcal{Y})_{abcd} := \frac{1}{2} (\mathcal{X}_{ab} \mathcal{Y}_{cd} + \mathcal{Y}_{ab} \mathcal{X}_{cd}) - \frac{1}{3} I_{abcd} \mathcal{X}_{ef} \mathcal{Y}^{ef}$$

- a symmetric bilinear form acting on two-forms
- trace free
- maps $\Lambda_-$ into itself, and annihilates $\Lambda_+$

$\Rightarrow$ is algebraically ASD Weyl

Call it a symmetric spinor product because, if we use $x_{AB}$ and $y_{AB}$ (symmetric in the spinor indices) to represent $\mathcal{X}, \mathcal{Y}$, then $\mathcal{X} \tilde{\otimes} \mathcal{Y}$ is represented by $x_{(AB} y_{CD)}$
Ernst two-form and friends

• $t^a$ is Killing $\iff \hat{F}_{ab} := 2\nabla_a t_b$ is antisymmetric

• $\hat{F} = \frac{1}{2}(F + i^*F)$

• $\mathcal{H}$ is closed (Maxwell) $\implies \iota_t \mathcal{H}$ is closed (Cartan) $\implies \exists \Xi$ a complex scalar s.t. $d\Xi = \iota_t \mathcal{H}$ up to a constant $c_\Xi$.

• Define the ASD Ernst two-form $\mathcal{F} := \hat{F} - 4\Xi \mathcal{H}$. It is Maxwell (Einstein eq. + Jacobi eq. for $t$) $\implies \ldots \implies \exists \sigma$, the (complex) Ernst potential, s.t. $d\sigma = \iota_t \mathcal{F}$, defined up to a constant $c_\sigma$.

• Note: I write $c_\Xi$ and $c_\sigma$ just to emphasize the fact that we have a freedom in normalization.
The characterization tensors

- The tensors are defined up to four normalizing constants: the complex parameters $c_\Xi$ and $c_\sigma$ implicitly from before, and a real number $\mu \neq 0$ and a complex number $\kappa$

- $B := \kappa F + 2\mu \mathcal{H}$

- $Q := C + \frac{6\kappa \bar{\Xi} - 3\mu}{2\mu\sigma} (F \bar{\otimes} F)$

- The vanishing of $B$ and $Q$ will be the characterizing conditions for a space-time to be Kerr-Newman
Characterization theorems

Local characterization

Under the “basic assumptions”, let $U \subset \mathcal{M}$ be a connected open subset, and suppose $c_{\Xi}, c_{\sigma}, \mu, \kappa$ can be taken so that on $U$: $\sigma \neq 0$, $B = 0$, and $Q = 0$. Then we have that on $U$,

$$t^2 + \sigma + \bar{\sigma} + \frac{|\kappa \sigma|^2}{\mu^2} = \text{const.} \quad \mathcal{F}^2 + 4\mu^2 \sigma^4 = \text{const.}$$

If, furthermore, the first expression evaluates to -1, the second to 0, then $U$ is locally isometric to a Kerr-Newman domain with charge $\kappa$, mass $\mu$, and angular momentum $\mu \sqrt{\mathcal{A}}$, where

$$\mathcal{A} = \left| \frac{\mu}{\sigma} \right|^2 \left( \mathcal{S} \nabla \frac{1}{\sigma} \right)^2 + \left( \mathcal{S} \frac{1}{\sigma} \right)^2$$

is constant on $U$. 
Characterization theorems

Global characterization

Under the “basic assumptions”, assume in addition that $\mathcal{M}$ is asymptotically flat and $t$ approaches a unit time translation at infinity. Normalize the potentials $\Xi$ and $\sigma$ to vanish at infinity, and define $(\mu, \kappa) = (M, q)$ to be the (positive) mass and charge respectively at the asymptotic end. Then if $B = 0$ everywhere, and $Q = 0$ whenever $\sigma \neq 0$, we can conclude that $\mathcal{M}$ is locally isometric to Kerr-Newman with angular momentum $M \sqrt{\mathcal{A}}$, where $\mathcal{A}$ is as defined on the previous slide.

Remark: by asymptotic decay and the positive mass, $\sigma \neq 0$ in a neighborhood of infinity, so the hypothesis is not empty. We can “push in” using the fact that $1/\sigma$ cannot blow-up in finite Riemannian distance.
About the proof

- The vanishing of $B$ allows us to compute $\nabla F^2$ by re-expressing $F$ in terms of $\hat{F}$, the derivative for which we have an expression by virtue of the Jacobi equation for $t$. From the Jacobi equation we pick up the contribution from the curvature tensor, and using the vanishing of $Q$ connect it back to an algebraic statement about $F$.
- From the above we obtain $\nabla F^2 = -4\mu^2\nabla \sigma^4$.
- This immediately implies that $(\nabla \frac{1}{\sigma})^2 = -t^2$, and that $\nabla t^2 = \nabla (\sigma + \bar{\sigma}) + \nabla \frac{|\kappa \sigma|^2}{\mu^2}$.
- The derivations are “algebraic”, in the sense that if $B$ and $Q$ were non-vanishing, the three equalities pick up error terms due to only $B$, $\nabla B$, and $Q$. (This fact is useful in applications.)
About the proof II

- The main step in the proof of local isometry, after making the assumption about the constants of integration, is to show

**Main Lemma**

Write $\frac{1}{\sigma} = y + iz$ for real numbers $y, z$, then $(\nabla y) \cdot (\nabla z) = 0$ and

\[
(\nabla z)^2 = \frac{\mathcal{A} - z}{y^2 + z^2} \quad \quad (\nabla y)^2 = \frac{\mathcal{A} + y^2 + |\kappa|^2 - 2\mu y}{y^2 + z^2}
\]

where $\mathcal{A}$ is the non-negative constant as defined before.

- Remark: In the AHP paper the lemma was proven using a tetrad calculus similar to GHP, but the statement can also be obtained tensorially and algebraically (in the sense above).
About the proof III

- Notice that the vanishing of $\mathcal{B}$ and $\mathcal{Q}$, along with the condition that $\sigma \neq 0$, gives that the space-time is type D. Let $l, l'$ denote the two PNDs normalized so $g(l, l') = -1$ and $g(t, l) = 1$ (that this can always be done requires proof, but is true outside bifurcate sphere).

- Define the vector fields

$$n = (A + y^2)t + (y^2 + z^2)(t \cdot l l' + t \cdot l' l) \quad b = \nabla z / (\nabla z)^2$$

- By computation: the four vector fields $t, n, b, l$ form a holonomic basis (are linearly independent and commute), so they can be attached to coordinates.

- The metric in the coordinates can be computed from the inner-products of those four vector fields, and verified to be the Kerr-Newman metric in Kerr coordinates.
Wave-like property of the characterization tensors

- For analysis of PDEs, the most useful property of this characterization is the fact that the tensors solve good hyperbolic equations.
- Evident for the tensor $B = \kappa F + 2\mu H$ since it is a constant coefficient linear combination of Maxwell fields.
- That $Q = C + \frac{6\kappa\bar{\Xi} - 3\mu}{2\mu\sigma} (F \tilde{\otimes} F)$ solves some hyperbolic equation is also evident: The Weyl tensor $C$ solves a divergence-curl system with source, $\sigma$ solves a wave equation, and $F$ is Maxwell.
- The key is in the word “good”.
Wave-like property II: Good equations

For our purpose (obtaining analytical estimates for perturbations of the 0 solution), a good equation should look like

**Good wave equation**

\[ \square_g S = J(x, S, \nabla S) \cdot (S, \nabla S) \]

Where \( S \) is some vector-valued function, and \( J \) some matrix-valued “potential” representing interaction with some background. In particular the source is “at least” linear in \( S \) and its first derivative (no exterior forcing).
Wave-like property III

- Since $\mathcal{B}$ is Maxwell, it has a wave equation
  \[ \Box g\mathcal{B}_{ab} = -C_{abcd}\mathcal{B}^{cd} \]

- Commuting the equation with the connection, we see that
  $\nabla\mathcal{B}$ also solves a good wave equation.

- The one for $\mathcal{Q}$ is more complicated; unlike $\mathcal{B}$’s equation, it
  does not decouple:
  \[ \Box g\mathcal{Q} = J \cdot (\mathcal{Q}, \nabla\mathcal{Q}, \mathcal{B}, \nabla\mathcal{B}, \nabla^2\mathcal{B}) \]
  (see [Won09a] for the actual demonstration).

- So need to take $S = (\mathcal{B}, \nabla\mathcal{B}, \mathcal{Q})$ to have a closed system of
  good wave equation.
Evolutionary statements?

- Given that the tensors solve wave equations, one might ask about the evolutionary aspect of the tensors.
- Trivially we have that

Finite speed of propagation

Given a space-like hypersurface $\Sigma$, assume $B$, $\nabla B$, $Q$ and their first normal derivatives vanish at $\Sigma$, then they vanish in the domain of dependence of $\Sigma$.

So the characterization theorem can have vanishing not on an open set, but on a Cauchy slice.

- In practice, this is not very useful, since whenever $t$ is transverse to the Cauchy slice, just the vanishing of $B$ and $Q$ themselves are enough get vanishing along the orbits.
Rigidity of Kerr-Newman solutions

- Classical proofs of No-Hair theorem require analyticity of the space-time, in order to invoke Hawking’s Rigidity Theorem

Hawking Rigidity

A real-analytic, stationary solution to the Einstein-Maxwell equations with a Killing horizon must be also axially symmetric.

- Assuming that the space-time is stationary and axisymmetric, with a non-degenerate event horizon, the proofs of [Car73, Rob75, Bun83, Maz82] (see also [Cos10]) show that the space-time must be sub-extremal Kerr-Newman. (See also the recent work of Chrusciel and Nguyen for the extremal case.)
Rigidity of Kerr-Newman solutions II

- Can we get out of the analytic category? Is just smoothness enough?
- Yes, if we pose some additional conditions:
  - If a stationary black hole is non-degenerate, it admits a bifurcate sphere. If, roughly speaking, the induced metric on the bifurcate sphere and its first jet in the directions of the horizons are identical to that of Kerr-Newman, then the domain of outer communications is everywhere locally isometric to Kerr-Newman.
  - If a stationary black hole is $C^6$-close (in the metric) to Kerr-Newman, then it must be axially symmetric.
  - There exist no stationary two-black hole solutions "close" to Kerr-Newman space-time.
- Each of the above uses the characterization presented in the previous section.
Large-data, conditional rigidity

- Three kinds of constraints on the bifurcate sphere:
  - Structural constraints. We need the bifurcate sphere to “look like the one in Kerr-Newman”. So we impose that $B^2 = 0$ and $F^2 + 4M^2\sigma^4 = 0$ (the latter $\Rightarrow Q^2 = 0$).
  - Characterization constraints. The bifurcate sphere is disconnected from infinity, so need to apply local characterization. $\mu, \kappa$ are fixed by asymptotic mass and charge, and $c_\Xi, c_\sigma$ are fixed at infinity. One derived constant fixed above, the remaining one $t^2 + \sigma + \bar{\sigma} + |\kappa\sigma|^2/\mu^2 = -1$ needs to be fixed at a point.
  - Technical constraints. We need a lower bound condition $y > M$ on the bifurcate sphere: this is to ensure that we are on the event, and not Cauchy horizon, so that if we go in the direction of increasing $y$, we head into the domain of outer communications.

- Structural constraints $\Rightarrow$ (via null structure equations) $B$, $\nabla B$, and $Q$ vanish on the event horizon.
Large-data, conditional rigidity II

- Ionescu-Klainerman’s first uniqueness theorem for wave equations implies that in a neighborhood of the bifurcate sphere $Q, B$ vanishes identically.
- The first uniqueness theorem is a statement about wave equations near a bifurcate null boundary, doesn’t use stationarity.
- Bootstrap outwards using Ionescu-Klainerman’s second uniqueness theorem for wave equations and the fact that we already have a local isometry in the region where $Q, B$ vanishes.
- The second uniqueness theorem makes use of stationarity, via considering a foliation by level surfaces of the function $y$ and using its nice properties: hence a bootstrapping is needed.
Perturbative rigidity

• Assume the stationary black-hole space-time is “close” to Kerr-Newman in the sense that $B, Q$ are close to zero. This allows the global construction of the function $y$.

• “Algebraic” construction implies $y$ still has all the nice properties, up to an error term defined by the smallness of $B, Q$.

• Since $y$ has no critical points except “at the bifurcate sphere”, a mountain-pass-type lemma or a topological argument gives that a non-extremal event horizon can only have one connected component.

• Once $y$ is constructed, and that the event horizon is sub-extremal and has only one component, then we can apply the argument of [AIK09] to show that the space-time must admit an axial Killing vector field.
Further questions

- Mars simplified his characterization to a global one using purely alignment of principal null directions [Mar00]: can the same be done here? (Presumed yes)
- Can the construction be made using a suitable notion of almost-Killing vectors? Ideally the notion of “suitable” should demand the tensors $B$ and $Q$ still solve good PDEs.
- Do the PDEs satisfied by $B$ and $Q$ have good decay properties? $B$, probably not (stationary mode); $Q$ maybe (first step: see whether that equation admits a stationary solution). Possible application to linear stability.
- Can we remove bifurcate sphere condition to obtain a full proof of black hole uniqueness? (Perhaps using local rigidity?)
Thank you for your attention.


References II


