**2.1.4 Measurable functions**

**Definition - measurable function.** Given a measurable space \((\Omega, \Sigma)\) a function \(f : \Omega \to \mathbb{R}\) is called measurable (with respect to \(\Sigma\)) if for every real number \(t\) the level set

\[
S_f(t) := S_{f>}(t) = \{x \in \Omega : f(x) > t\}
\]

(2.13)

is measurable, i.e. \(S_f(t) \in \Sigma\). Note that \(S_f(t) = f^{-1}([t, \infty])\).

More generally we may define measurable functions between two measurable spaces:

**Definition - measurable function functions between measurable spaces.** Let \((\Omega_j, \Sigma_j), j = 1, 2\) be two measurable spaces. A function \(f : \Omega_1 \to \Omega_2\) is called \((\Sigma_1 - \Sigma_2)\) measurable if for all \(E \in \Sigma_2\) the set \(f^{-1}(E) = \{x \in \Omega_1 : f(x) \in E\}\) is measurable (with respect to \(\Sigma_1\)).

**Example.** A measurable function \(f : \Omega \to \mathbb{R}\) in the sense of the first definition is \((\Sigma_1 - \mathcal{B})\) measurable. Indeed, the definition of a measurable function does not depend on the use of the \(>\) sign (2.13) as shows the following lemma:

**Lemma.** The following statements are equivalent for a function \(f : \Omega \to \mathbb{R}\):

1. For every \(t \in \mathbb{R}\), \(S_f(t) = S_{f>}(t) = \{x \in \Omega : f(x) > t\} \in \Sigma\).
2. For every \(t \in \mathbb{R}\), \(S_{f\leq}(t) = \{x \in \Omega : f(x) \leq t\} \in \Sigma\).
3. For every \(t \in \mathbb{R}\), \(S_{f\geq}(t) = \{x \in \Omega : f(x) \geq t\} \in \Sigma\).
4. For every \(t \in \mathbb{R}\), \(S_{f<}(t) = \{x \in \Omega : f(x) < t\} \in \Sigma\).

**Proof:** Obviously 1 \(\iff\) 2, 3 \(\iff\) 4 since the corresponding sets are complements of each other. To see 1 \(\implies\) 3 note that

\[
\{x \in \Omega : f(x) \geq t\} = \bigcap_{n=1}^{\infty} \{x \in \Omega : f(x) > t - \frac{1}{n}\}
\]

and the implication 3 \(\implies\) 1 follows from

\[
\{x \in \Omega : f(x) > t\} = \bigcup_{n=1}^{\infty} \{x \in \Omega : f(x) \geq t + \frac{1}{n}\}
\]

which proves the lemma.

**Lemma.** \(S_f(t)\) is measurable for all \(t \in \mathbb{R}\) if and only if \(S_f(t)\) is measurable for all \(t \in \mathbb{Q}\).

**Lemma.** The first implication is obvious. Suppose that \(S_f(t)\) is measurable for all \(t \in \mathbb{Q}\). Since \(\mathbb{Q} \subset \mathbb{R}\) is dense for any \(t \in \mathbb{R}\) there is a decreasing sequence of \(t_n \in \mathbb{Q}, t_n > t\) converging towards \(t\). Then \(S_f(t) = \bigcup_n S_f(t_n) \in \Sigma\).
Examples.
1. Any constant function \( f(\omega) = c \) on \( \Omega \) is measurable since \( S_f(t) = \Omega \) (if \( t < c \)) or \( S_f(t) = \emptyset \) (if \( t \geq c \)).
2. For any \( E \in \Sigma \) its characteristic function is measurable.
3. If \( \Sigma = \mathcal{B} \) is the Borel-algebra any continuous function on \( \Omega \) is measurable since \( S_f(t) \) is open.
4. If \( \Sigma = \mathcal{B} \) is the Borel-algebra any lower semicontinuous function on \( \Omega \) is measurable since \( S_f(t) \) is open. Indeed, \( f : \Omega \to \mathbb{R} \) is lower semicontinuous at \( x_0 \in \Omega \) if \( \lim_{x \to x_0} \inf f(x) \geq f(x_0) \) which implies that \( S_f(t) \) is open.

Lemma. Let \( f, g : \Omega \to \mathbb{R} \) be measurable and \( c \in \mathbb{R} \). Then
1. \( cf \),
2. \( f^2 \),
3. \( f + g \),
4. \( fg \)
5. \(|f|\)
6. \( \min(f, g) \) and \( \max(f, g) \)

are measurable.

Proof:
1. \( cf \): trivial, if \( c = 0 \). For \( c > 0 \) note that \( S_{cf}(t) = S_f(t/c) \) and for \( c < 0 \) \( S_{cf}(t) = (\bigcap_{n=1}^{\infty} \{ x \in \Omega : f(x) > t/c - \frac{1}{n} \})^c \)
2. \( f^2 \): \( S_{f^2}(t) = S_f(\sqrt{t}) \cup \{ x \in \Omega : f(x) < -\sqrt{t} \} \)
3. \( f + g \): For all rational \( t \)
   \[ S_{f+g}(t) = \bigcup_{q \in \mathbb{Q}} (S_f(q) \cap S_g(t-q)). \]
4. \( fg \): \( 4fg = (f + g)^2 - (f - g)^2 \)
5. \(|f|\): if \( t \geq 0 \) then \( S_{|f|}(t) = S_f(t) \cup \{ x \in \Omega : f(x) < -t \} \) otherwise \( S_{|f|}(t) = \Omega \)
6. \( S_{\min(f,g)}(t) = S_f(t) \cap S_g(t) \) and \( S_{\max(f,g)}(t) = S_f(t) \cup S_g(t) \)

Complex valued functions. A function \( f := f_1 + i f_2 : \Omega \to \mathbb{C} \) with \( f := f_1, f_2 : \Omega \to \mathbb{R} \) is measurable if and only if \( f_1 \) and \( f_2 \) are measurable.
Extended real-valued functions. All conclusions also hold for extended real-valued functions $F : \Omega \to [-\infty, \infty]$ since
\[
\{x \in \Omega : f(x) = +\infty\} = \bigcap_{n=1}^{\infty} \{x \in \Omega : f(x) > n\}
\]
\[
\{x \in \Omega : f(x) = -\infty\} = \left(\bigcup_{n=1}^{\infty} \{x \in \Omega : f(x) > -n\}\right)^c
\]
are measurable for a measurable function. We denote the collection of measurable extended real-valued functions by $M(\Omega, \Sigma)$.

Lemma. Let $f_+(x) = \sup(f(x), 0)$ et $f_-(x) = -\inf(f(x), 0)$. Then $f$ is measurable if and only if $f_+$ and $f_-$ are measurable.

Proof. Use $f = f_+ - f_-$ and $|f| = f_+ + f_-$. when $f_+$ and $f_-$ are measurable and $2f_+ = |f| + f$ and $2f_- = |f| - f$ are measurable when $f$ (and hence $|f|$) is measurable.

Lemma. Let $f_n : \Omega \to [-\infty, \infty], n \in \mathbb{N}^*$, be measurable (extended real-valued) functions. Then the following functions are measurable
1. $f(x) = \inf f_n(x)$,
2. $F(x) = \sup f_n(x)$,
3. $f^*(x) = \lim \inf f_n(x)$,
4. $F^*(x) = \lim \sup f_n(x)$.

Proof:
1. $\{x \in \Omega : f(x) \geq t\} = \bigcap_{n=1}^{\infty} \{x \in \Omega : f_n(x) \geq t\}$
2. $\{x \in \Omega : f(x) > t\} = \bigcup_{n=1}^{\infty} \{x \in \Omega : f_n(x) > t\}$
3. $f^*(x) = \sup \left(\inf f_m(x)\right)$,
4. $F^*(x) = \inf \left(\sup f_m(x)\right)$.

Corollary. If $(f_n)$ is a sequence of functions that converges (pointwise) to $f$ on $\Omega$, then $f$ is measurable.

Functions and measure space. We shall say that a property holds ($\mu$-almost everywhere if there exists a subset $N \subset \Omega$ with measure zero, such that the property holds on the complement of $N$. In particular, two functions $f, g$ are equal for ($\mu$-)almost everywhere when $f(x) = g(x)$ if $x \notin N$. We write
\[
f = g \quad \mu \text{- a.e.}
\]
A sequence \((f_n)\) of measurable functions on \(\Omega\) converges \((\mu-)\)almost everywhere (or for \((\mu-)\)almost every \(x \in \Omega\) to a function \(f = \lim_{n \to \infty} f_n(x)\) for \(x \notin N\). We write
\[
f = \lim_{n \to \infty} f_n, \quad \mu -a.e.
\]
We say that a nonnegative function \(f\) is a strictly positive measurable function on a measurable set \(E\), if \(\mu(\{x \in E : f(x) = 0\}) = 0\).

Essential support of a function on \(\mathbb{R}^d\). If \(\mu\) is a Borel measure (that is, a measure defined on all open sets) and \(f\) is Borel measurable let \(E\) be the collection of open sets \(E\) such that \(f(x) = 0\) a.e. in \(E\). Let \(E^*\) the union of all \(E\)'s in \(E\) which is open. We define the essential support of \(f\), ess supp\{\(f\}\) to be the complement of \(E^*\). Hence ess supp\{\(f\}\} is a closed, and therefore measurable set. Note that ess supp\{\(f\}\} depends on the measure. If \(\mu\) is the Lebesgue measure and \(f\) is continuous, then ess supp\{\(f\}\} = supp\{\(f\}\}, where supp\{\(f\}\} is the closure of points where \(f\) is nonzero.

### 2.2 Integration on measure spaces

Let \((\Omega, \Sigma, \mu)\) be a measure space. Let \(M^+(\Omega, \Sigma)\) denote the collection of nonnegative real-valued \((\Sigma)\)-measurable functions on \(\Omega\).

#### 2.2.1 The Integral

**Definition of the integral.** For \(f \in M^+(\Omega, \Sigma)\) we define
\[
F_f(t) = \mu(S_f(t)).
\]
Then \(F_f(t)\) is a nonincreasing function of \(t\) on \([0, \infty]\). Hence its Riemann integral exists (its value might be \(+\infty\)) and therefore we define the integral of \(f\) over \(\Omega\) with respect to the measure \(\mu\) by
\[
\int_{\Omega} f(x) \mu(dx) := \int_0^\infty F_f(t) \, dt = \int_0^\infty \mu(S_f(t)) \, dt. \tag{2.14}
\]

Frequently used abbreviation: \(\int f \, d\mu\).

If \(f \in M^+\) \(\text{and} \int f \, d\mu < \infty\), we say that \(f\) is a summable (or integrable) function.

If \(f \in M\), we define its integral by \(\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu \) (when \(\int f_+ \, d\mu, \int f_- \, d\mu < \infty\) or at least one of the two integrals finite allowing \(+\infty\) or \(-\infty\) for the resulting integral) and similarly for complex-valued functions.

**Discussion of the definition.** When \(\Omega = \mathbb{R}^d\) or \(\Omega\) any Lebesgue-measurable subset of \(\mathbb{R}^d\) and \(\Sigma, \mu\) are the sigma-algebra of Lebesgue measurable function and the Lebesgue measure, respectively then the the integral defined above is called the Lebesgue integral, a notion we also keep in the abstract framework. The definition of the (Lebesgue) integral above yields absolutely convergent integrals if finite since using \(|f| = f_+ + f_-\) either \(\int |f| \, d\mu < \infty\) or \(\int |f| \, d\mu = \infty\).

For example, the continuous function \(f\) given by \(f(x) = \sin x / x\) for \(x \neq 0\) and
f(0) = 1 is Borel-measurable on \( \mathbb{R} \) but it is not covered by the above definition of the integral since the integrals of \( f_+ \), \( f_- \) are both infinite and the integral of \(|f|\) is equal to \( +\infty \). However, its Riemann improper integral is finite (see exercises series 2) but not absolutely convergent.

Example - integral of a characteristic function. For any \( E \in \Sigma \) we have \( 1_E \in M^+(\Omega, \Sigma) \) and

\[
\int_\Omega 1_E \, \mu(dx) = \int_0^1 \mu(S_{1_E}(t)) \, dt = \mu(E). \tag{2.15}
\]

Linearity of the integral for characteristic function. Let \( c > 0, E \in \Sigma \). Then, by the definition of the integral and a change of variables in the Riemann integral

\[
\int_\Omega c1_E \, \mu(dx) = \int_0^\infty \mu(S_{c1_E}(t)) \, dt = \int_0^\infty \mu(S_{1_E}(t/c)) \, dt = c \int_0^\mu(S_{1_E}(t)) \, dt = c \int_\Omega 1_E \, \mu(dx)
\]

and similarly for \( c = 0 \) using \( \mu(S_0(t)) = 0 \) for all \( t \geq 0 \) and for \( c < 0 \) since \( c1_E = -(c1_E)_\) Therefore for all \( c \in \mathbb{R} \)

\[
\int_\Omega c1_E \, \mu(dx) = c \int_\Omega 1_E \, \mu(dx). \tag{2.16}
\]

When \( E, F \in \Sigma \) disjoint then \( 1_E + 1_F = 1_{E \cup F} \) and therefore

\[
\int_\Omega 1_E + 1_F \, \mu(dx) = \mu(E \cup F) = \mu(E) + \mu(F) = \int_\Omega 1_E \, \mu(dx) + \int_\Omega 1_F \, \mu(dx).
\]

When \( E \cap F \neq \emptyset \) then write \( E \cup F \) as union of three disjoint sets and apply the inclusion-exclusion principle: \( 1_E + 1_F = 1_{E \cup F} + 1_{E \cap F} \). Therefore the integral is linear for linear combinations of characteristic functions:

\[
\int_\Omega 1_E + 1_F \, \mu(dx) = \int_\Omega 1_E \, \mu(dx) + \int_\Omega 1_F \, \mu(dx). \tag{2.17}
\]

Definition of the integral by simple functions. Alternatively, one can define the integral by an approximation procedure using "simple" functions. This approach is motivated by the following (formal) identity for \( f \in M^+(\Omega, \Sigma) \):

\[
f(x) = \int_0^{f(x)} dt = \int_0^\infty 1_{\{f(x) > t\}} dt = \int_0^\infty 1_{S_f(t)} dt \tag{2.18}
\]

where for each \( t \) the integrand can be viewed also as the indicator function of the measurable set \( S_f(t) \). A simple measurable function is of the form

\[
\phi = \sum_{j=1}^n c_j 1_{E_j}
\]
with \( c_j \in \mathbb{R} \) and \( E_j \in \Sigma \). In its unique standard representation the \( c_j \in \mathbb{R} \) are distinct and the \( E_j \in \Sigma \) are disjoint. For a simple function \( \phi \in M^+(\Omega, \Sigma) \) we define the integral by

\[
\int_{\Omega} \phi(x) \, \mu(dx) := \sum_{j=1}^{n} c_j \mu(E_j)
\]  

(2.19)

with the convention that the integral of \( \phi \equiv 0 \) is always zero even when the space has infinite measure. If we start with our definition of the integral this property has been shown above. The integral of an arbitrary function \( f \in M^+(\Omega, \Sigma) \) is then defined by

\[
\int_{\Omega} f(x) \, \mu(dx) := \sup \int_{\Omega} \phi(x) \, \mu(dx)
\]  

(2.20)

where the supremum is extended over all simple functions \( \phi \in M^+(\Omega, \Sigma) \) satisfying \( 0 \leq \phi(x) \leq f(x) \) for all \( x \in \Omega \). Indeed, there is an increasing sequence of simple functions \( \phi_k \in M^+(\Omega, \Sigma) \) which converges pointwise towards \( f \) (see exercises, series 5). Applying the monotone convergence theorem below it is then easy to show that both definitions coincide. However, the definition of the integral by means of simple functions is more appropriate to prove, for example, the linearity of the integral. Also simple functions are useful to show that all Riemann integrable functions (i.e. absolutely convergent integrals) on \( \mathbb{R}^d \) or cubes/intervalles are Lebesgue integrable with the same value of the integral.

**Linearity of the integral.** If \( f, g \in M^+(\Omega, \Sigma) \), then for all \( \alpha, \beta \geq 0 \):

\[
\int_{\Omega} \alpha f(x) + \beta g(x) \, \mu(dx) = \alpha \int_{\Omega} f(x) \, \mu(dx) + \beta \int_{\Omega} g(x) \, \mu(dx).
\]

(2.21)

This can be easily extended to \( f, g \in M(\Omega, \Sigma) \) and \( \alpha, \beta \in \mathbb{R} \) (on \( \mathbb{C} \)).

**Monotonicity of the integral.** If \( f, g \in M^+(\Omega, \Sigma) \) and \( f \leq g \), then

\[
\int_{\Omega} f(x) \, \mu(dx) \leq \int_{\Omega} g(x) \, \mu(dx).
\]

(2.22)

Indeed, this follows immediately from \( S_f(t) \subset S_g(t) \) for all \( t \geq 0 \) and the monotonicity of the measure \( \mu \) and the monotonicity of the Riemann integral.

**Definition - Space of integrable functions.**

\[
L^1 := L^1(\Omega, d\mu) = \{ f \in M(\Omega, \Sigma) : \int_{\Omega} |f(x)| \, \mu(dx) < \infty \}.
\]

(2.23)

It is easy to show that \( f \in L^1 \) if and only if \( f_+ \), \( f_- \in L^1 \). In addition, if \( f \in M(\Omega, \Sigma) \), \( g \in L^1 \) such that \( |f| \leq |g| \), then \( f \in L^1 \) and inequality (2.22) holds. By means of the triangle inequality for the absolute value \( |\alpha f(x) + \beta g(x)| \leq |\alpha| \cdot |f(x)| + |\beta| \cdot |g(x)| \) it follows that \( L^1(\Omega, d\mu) \) is a vector space. Any \( f \in L^1 \) is called integrable or summable.