2.2.2 Monotone convergence theorem

The following theorem is what is known in the literature as the "monotone convergence theorem" which is a statement about nonnegative increasing sequences \((f_n)_{n \geq 1}\) of measurable functions in a measure space \((\Omega, \Sigma, \mu)\). It has a Corollary which we call here the monotone convergence theorem for summable/integrable functions where the assumption \(f_n \geq 0\) a.e. is replaced by \(f_n \in L^1\). Contrary to the monotone convergence theorem for the Riemann integral the measurability or integrability of the pointwise (or a.e.) limit is part of the conclusion and not of an hypothesis to be put into the theorem.

**Monotone convergence theorem - general version: measurable functions.** If \((f_n)_{n \geq 1}\) is a monotone increasing sequence of functions in \(M^+(\Omega, \Sigma)\) which converges almost everywhere to a function \(f\) on \(\Omega\). Then \(f\) is measurable and

\[
\lim_{n \to \infty} \int_{\Omega} f_n(x) \, d\mu = \int_{\Omega} f(x) \, d\mu \left(= \int_{\Omega} \lim_{n \to \infty} f_n(x) \, d\mu \right) \in [0, +\infty]. \tag{2.24}
\]

In particular, if \(\lim_{n \to \infty} \int_{\Omega} f_n(x) \, \mu(dx) < +\infty\), then \(f\) is summable/integrable.

**Remark.** Note the monotonicity allows us to define

\[
f(x) := \lim_{n \to \infty} f_n(x) \in [0, +\infty]
\]

**Remark.** When the \(f_n\) are summable/integrable we can drop the assumption that the \(f_n \geq 0\) by considering in that case the non-negative sequence \(g_n = f_n - f_1\). This fact is a corollary of the monotone convergence theorem but we state it as monotone convergence for integrable functions

**Corollary: Monotone convergence - integrable functions.** If \((f_n)_{n \geq 1}\) is a monotone increasing sequence of functions in \(L^1(\Omega, d\mu)\) which converges almost everywhere to a function \(f\) on \(\Omega\). Then \(f\) is measurable and

\[
\lim_{n \to \infty} \int_{\Omega} f_n(x) \, d\mu = \int_{\Omega} f(x) \, d\mu \left(= \int_{\Omega} \lim_{n \to \infty} f_n(x) \, d\mu \right) \in [0, +\infty]. \tag{2.25}
\]

In particular, if \(\lim_{n \to \infty} \int_{\Omega} f_n(x) \, \mu(dx) < +\infty\), then \(f\) is summable/integrable.

**Proof of the monotone convergence theorem.** First of all, we prove the theorem in the case that \(f_n\) converges pointwise to \(f\). By the monotonicity of the sequence

\[
S_{f_n}(t) \subset S_{f_{n+1}}(t)
\]

for all \(t\) and \(n\). By definition

\[
S_f(t) = \bigcup_{n=1}^{\infty} S_{f_n}(t)
\]
and therefore by the continuity of measures
\[
\lim_{n \to \infty} \mu(S_{f_n}(t)) = \mu \left( \bigcup_{n=1}^{\infty} S_{f_n}(t) \right) = \mu(S_f(t)).
\]

Note that this convergence is monotone for a sequence of monotone functions. Now the theorem follows from the corresponding theorem of Riemann integration. Indeed, let \( \nu_n(t) = \mu(S_{f_n}(t)) \) and \( \nu(t) = \mu(S_f(t)) \). Then \( \nu_n(t) \leq \nu_{n+1}(t) \leq \nu(t) \) for all \( n \in \mathbb{N} \) and \( t > 0 \) and all functions are decreasing functions of \( t \), hence Riemann integrable on any compact interval. Since obviously
\[
\lim_{n \to \infty} \int_0^\infty \nu_n(t) \, dt \leq \int_0^\infty \nu(t) \, dt
\]
we have to prove the reversed inequality in the case when the sequence \( I_n = \int_0^\infty \nu_n(t) \, dt \) is bounded which implies that \( \nu_n(t) < \infty \) and therefore \( \nu(t) < \infty \) for all \( t > 0 \). Let \( b > a > 0 \). Since the \( \nu_n(t) \) are decreasing functions of \( t \) we have for any positive integer \( N \) that
\[
\int_a^b \nu_n(t) \, dt \geq \int_a^b \nu_n(t) \, dt \geq \frac{b - a}{N} \sum_{k=1}^{N} \nu_n(a_k), \quad a_k = a + \frac{k(b - a)}{N}
\]
Since \( \nu(t) \) is a decreasing function of \( t \) for any \( \epsilon > 0 \) there is a positive integer \( N \) such that
\[
\int_a^b \nu(t) \, dt \leq \frac{b - a}{N} \sum_{k=1}^{N} \nu(a_k) + \epsilon.
\]
Since \( \nu_n(t) \) converges pointwise to \( \nu(t) \) for all \( n \) sufficiently large
\[
\max_{k \in \{1, \ldots, N\}} |\nu_n(a_k) - \nu(a_k)| < \epsilon/(b - a)
\]
with \( N \) as selected before. This implies that for all \( \epsilon > 0 \) and all \( n \) sufficiently large
\[
\int_a^b \nu_n(t) \, dt \geq \int_a^b \nu(t) \, dt - 2\epsilon.
\]
which implies
\[
\lim_{n \to \infty} \int_a^b \nu_n(t) \, dt \geq \int_a^b \nu(t) \, dt
\]
and this inequality proves the claim. Finally we prove the monotone convergence theorem when \( f_n \) converges to \( f \) almost everywhere. We need the following lemma.

**Lemma 1.** Let \( f \in M^+(\Omega, \Sigma) \). Then \( f(x) = 0 \) almost everywhere if and only if \( \int f \, d\mu = 0 \).

**Proof.** If \( \int f \, d\mu = 0 \), then for all \( a > 0 \).
\[
0 = \int_0^{\infty} \mu(S_f(t)) \, dt \geq \int_0^a \mu(S_f(t)) \, dt \geq a\mu(S_f(a))
\]
where the latter inequality is called Chebychev’s inequality. Hence $\mu(S_f(a)) = 0$ for all $a > 0$ Put $a = 1/n$ and note that $S_f(0) = \bigcup_n S_f(1/n)$ which has measure zero thanks to the sigma-additivity of $\mu$.

Conversely, let $f(x) = 0$ almost everywhere. Then $\mu(S_f(t)) = 0$ for all $t \geq 0$ and therefore $\int f \, d\mu = 0$ proving the lemma.

Proof of the monotone convergence theorem in the case of almost everywhere convergence. Convergence almost everywhere means that there exists a set $N \subset \Omega$ of measure zero such that $f_n$ converges pointwise to $f$ on $E := \Omega \setminus N$. We then apply the monotone convergence theorem to $f\chi_E$. Indeed, with Lemma 1 we have

$$\int f \, d\mu = \int f\chi_E \, d\mu = \lim_{n \to \infty} \int f_n \chi_E \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

2.2.3 Fatou’s lemma

Fatou’s lemma is not only needed to prove the dominated convergence theorem below but it includes also a statement of the behaviour of the integral under pointwise (or a.e.) convergence: The integral is lower semi-continuous under a.e. convergence.

Fatou’s lemma. If $(f_n)_n$ is a sequence of functions in $M^+(\Omega, \Sigma)$. Then $f(x) := \liminf_{n \to \infty} f_n(x)$ is measurable and

$$\liminf_{n \to \infty} \int_{\Omega} f_n(x) \, \mu(dx) \geq \int_{\Omega} f(x) \, \mu(dx) \left( = \int_{\Omega} \liminf_{n \to \infty} f_n(x) \, \mu(dx) \right). \quad (2.26)$$

In particular, if $(f_n)_n$ is a sequence of summable functions, then $f$ is summable.

Remark. The hypothesis that the $f_n$’s are nonnegative is crucial (see exercises).

Remark. Fatou’s lemma means that the mapping $f \mapsto \int f$ is pointwise lower semicontinuous.

Corollary. If $f(x) = \lim_{n \to \infty} f_n(x)$ a.e., then

$$\liminf_{n \to \infty} \int_{\Omega} f_n(x) \, \mu(dx) \geq \int_{\Omega} f(x) \, \mu(dx).$$

Proof of Fatou’s lemma. We have already shown before that $f$ is measurable. For $k \geq 1$ let $F_k(x) = \inf_{n \geq k} f_n(x)$. $F_k$ is measurable and if all $f_n$ are summable it is also summable since $F_k \leq f_k$ and more generally $F_k \leq f_n$ if $n \geq k$. Therefore for all $n \geq k$

$$\int_{\Omega} f_n(x) \, \mu(dx) \geq \int_{\Omega} F_k(x) \, \mu(dx).$$
Obviously, $F_k$ is an increasing sequence (i.e. $F_{k+1} \geq F_k$) with
\[ \lim_{k \to \infty} F_k(x) = \sup \left( \inf_{k \geq n} f_n(x) \right) =: \liminf_{n \to \infty} f_n(x) = \sup_{k \geq 1} F_k(x). \]
Consequently, applying the monotonicity of the integral and the monotone convergence theorem, we have
\[ \liminf_{n \to \infty} \int_{\Omega} f_n(x) \, d\mu := \sup_{k \geq 1} \left( \inf_{n \geq k} \int_{\Omega} f_n(x) \, d\mu \right) \geq \lim_{k \to \infty} \int_{\Omega} F_k(x) \, d\mu = \int_{\Omega} f(x) \, d\mu. \]

### 2.2.4 Dominated convergence theorem

The dominated convergence theorem formulates sufficient conditions under which almost everywhere convergence yields an integrable function and limit and integral are interchangeable. Again this is an important difference between Lebesgue and Riemann integral.

**Dominated convergence theorem.** Let $(f_n)_n$ be a sequence of (summable) functions in $M(\Omega, \Sigma)$ converging a.e. to a function $f$. If there exists a nonnegative summable function $g$ on $(\Omega, \Sigma, \mu)$ such that $|f_n(x)| \leq g(x)$ for all $n$, then $f$ is summable such that $|f(x)| \leq g(x)$ and
\[ \lim_{n \to \infty} \int_{\Omega} f_n(x) \, d\mu = \int_{\Omega} f(x) \, d\mu. \quad (2.27) \]

**Proof.** Since $g - f_n \geq 0$ and $g + f_n \geq 0$ we may apply Fatou’s lemma to both sequences of functions:
\[ \int_{\Omega} g(x) - f(x) \, d\mu \leq \liminf_{n \to \infty} \int_{\Omega} g(x) - f_n(x) \, d\mu \leq \int_{\Omega} g(x) \, d\mu - \limsup_{n \to \infty} \int_{\Omega} f_n(x) \, d\mu, \]
\[ \int_{\Omega} g(x) + f(x) \, d\mu \leq \liminf_{n \to \infty} \int_{\Omega} g(x) + f_n(x) \, d\mu \leq \int_{\Omega} g(x) \, d\mu + \liminf_{n \to \infty} \int_{\Omega} f_n(x) \, d\mu. \]
Consequently,
\[ \limsup_{n \to \infty} \int_{\Omega} f_n(x) \, d\mu \leq \int_{\Omega} f(x) \, d\mu \leq \liminf_{n \to \infty} \int_{\Omega} f_n(x) \, d\mu, \]
which implies that
\[ \lim_{n \to \infty} \int_{\Omega} f_n(x) \, d\mu = \int_{\Omega} f(x) \, d\mu. \]

**Corollary - convergence in $L^1$.** Under the assumptions of the dominated convergence theorem the sequence $(f_n)_n$ converges in $L^1(\Omega, d\mu)$, that is
\[ \lim_{n \to \infty} \int_{\Omega} |f_n(x) - f(x)| \, d\mu = 0. \quad (2.28) \]
Proof. Since the sequence \((|f_n - f|)\) also satisfies the hypotheses of the dom-
inated convergence theorem (in particular, the \(|f_n - f|\) are summable since \(f\)
is summable by the dominated convergence theorem). Now \(|f_n - f| \to 0\) a.e.
by the continuity of the absolute value and applying dominated convergence we
conclude.

Application of the dominated convergence theorem - integrals de-
pending on a parameter

- If for a \(t_0 \in [a, b]\) a real-valued function \(f(x, t)\) satisfies \(f(x, t_0) = \lim_{t \to t_0} f(x, t)\)
  and is uniformly on \([a, b]\) bounded by an integrable function \(g(x)\), then we
can interchange integration and limit, i.e.
  \[
  \lim_{t \to t_0} \int \Omega f(x, t) \, \mu(dx) = \int \Omega f(x, t_0) \, \mu(dx).
  \]

- If \(f(x, t)\) is integrable and the partial derivative \(\frac{\partial f(x, t)}{\partial t}\) exists on \(\Omega \times [a, b]\)
  and is uniformly on \([a, b]\) bounded by an integrable function \(g(x)\), then
  \[
  F(t) := \int \Omega f(x, t) \, \mu(dx)
  \]
is differentiable with respect to \(t\) and
  \[
  F'(t) = \int \Omega \frac{\partial f(x, t)}{\partial t} \, \mu(dx).
  \]
that is, we can interchange integration and differentiation.