4.3 The dual space of $L^p$ and weak convergence

In order to establish a weaker form of the Bolzano-Weierstrass theorem we consider the weak topology on $L^p$ which is generated by the seminorms defined by the continuous linear functionals on $L^p$. We recall the following result about the continuity of linear mappings between normed spaces.

**Lemma.** Let $(B_1, ||·||_1), (B_2, ||·||_2)$ be two Banach spaces and $A : B_1 \to B_2$ a linear mapping. $A$ is continuous if and only if $A$ is bounded, that is there exists a constant $C > 0$ such that $||Ax||_2 \leq C||x||_1$ for all $x \in B_1$.

**Proof.** Suppose that $A$ is bounded. Then for any $x, h \in B_1$:

$$||A(x + h) - Ax||_2 = ||Ah||_2 \leq C||h||_1$$

which implies that $A$ is continuous. Suppose now that $A$ is continuous. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that $||h||_1 < \delta$ implies $||Ah||_2 < \epsilon$. We fix such a pair ($\epsilon, \delta$). If $A$ is not bounded then for any $C > 0$ there is a sequence $x_n$ such that $||Ax_n||_2 > C||x_n||_1$. Take $C = \frac{2\epsilon}{\delta}$ and define $\tilde{x}_n = \frac{\delta x_n}{2||x_n||_1}$. Then $||\tilde{x}_n||_1 < \delta$ but $||A\tilde{x}_n||_2 > \frac{2\epsilon}{\delta}||\tilde{x}_n||_1 = \epsilon$ which is a contradiction.

**The dual space.** The set of continuous linear functionals on $L^p$ is a normed vector space, denoted by $(L^p)'$ and called the dual of $L^p$, equipped with the norm

$$||L||_p = \sup\{||L(f)|| : ||f||_p \leq 1\} = \sup\{||L(f)|| : ||f||_p = 1\}$$

Obviously, if $g \in L^p'$ then

$$L_g(f) = \int_{\Omega} f(x)g(x) \mu(dx)$$

defines a linear continuous functional on $L^p$ (by Hölder’s inequality) so that $L^p \subset (L^p)'$. In fact we have the following theorem:

**Theorem (F. Riesz).** If $1 < p < \infty$ then the dual of $L^p$ is $L^p'$.

**Remark.** The theorem holds for any $L^p$ over a measure space if $p > 1$. If $p = 1$ we have to assume in addition that $\Omega$ is sigma-finite which we present here without proof (see Lieb-Loss, pp. 61-63 or exercises):

**Theorem.** If $\Omega$ is sigma-finite then the dual of $L^1$ is $L^\infty$.

**Proof - the case $1 < p < \infty$.** The proof relies on a property called uniform convexity. To illustrate this property we consider first of all the case $p = 2$.
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The parallelogram identity for \( L^2 \). For all \( f, g \in L^2(\Omega, d\mu) \) the following identity holds

\[
||f - g||_2^2 + ||f + g||_2^2 = 2||f||_2^2 + 2||g||_2^2.
\]

This follows obviously from the a.e. identity \(|f - g|^2 + |f + g|^2 = 2|f|^2 + 2|g|^2\).

As a consequence \( L^2 \) has the uniform convexity property if \( \lim_{n \to \infty} ||f_n + g_n||_2 \) has the identity holds

Uniform convexity property. Let \( (f_n), (g_n) \) be two sequences in a Banach space \( (B, || \cdot ||) \) such that \( ||f_n|| = ||g_n|| = 1 \). Then \( (B, || \cdot ||) \) has the uniform convex property if \( \lim_{n \to \infty} ||\frac{f_n + g_n}{2}|| = 1 \) implies \( \lim_{n \to \infty} ||f_n - g_n|| = 0 \).

Norm inequalities for \( L^p \) spaces and uniform convexity. For \( 1 < p < \infty \) the \( L^p \) spaces are uniform convex in view of the following norm inequalities: If \( 1 < p \leq 2 \) then

\[
||f - g||_p^p + ||f + g||_p^p \leq 2(||f||_p^p + ||g||_p^p)^{\frac{p}{p-1}}.
\]

and if \( 2 \leq p < \infty \), then

\[
||f - g||_p^p + ||f + g||_p^p \leq 2^{p-1}||f||_p^p + 2^{p-1}||g||_p^p.
\]

for all \( f, g \in L^p \).

Proof of the norm inequalities. For \( p \geq 2 \) apply the convex function \( B(u) = |u|^{p/2} \) to the identity \(|f - g|^2 + |f + g|^2 = 2|f|^2 + 2|g|^2\). It follows

\[
B(|f - g|^2) + B(|f + g|^2) \leq B(|f - g|^2 + |f + g|^2)
\]

\[
= B(2|f|^2 + 2|g|^2)
\]

\[
\leq \frac{1}{2} B(4|f|^2) + \frac{1}{2} B(4|g|^2)
\]

\[
= 2^{p-1} B(|f|^2) + 2^{p-1} B(|g|^2).
\]

Integrating with respect to \( d\mu \) yields the desired inequality. If \( 1 < p \leq 2 \) then we need the following lemma:

Lemma. For all \( z \in [0,1] \),

\[
(1 + z)^{p/2} \leq \frac{1}{2} (1 - z)^p + \frac{1}{2} (1 + z)^p
\]

Proof of the lemma. The statement is trivial for \( z = 1 \) with equality sign. Therefore suppose \( 0 \leq z < 1 \). Consider the binomial series

\[
(1 + z)^p = 1 + pz + \sum_{n=2}^{\infty} (-1)^n \frac{a_p(n)}{n!} z^n, \quad (1 - z)^p = 1 - pz + \sum_{n=2}^{\infty} \frac{a_p(n)}{n!} z^n
\]

where \( a_p(n) = (-1)^n p \cdot (p-1) \cdots (p-n+1) \). Note that \( a_p(n) > 0 \) if \( 1 < p < 2 \). Finally

\[
(1 + z)^{p-1} = 1 + \sum_{n=1}^{\infty} \frac{b_{p-1}(n)}{n!} z^n
\]
We have the integral inequality

\[
\|f - g\|^{p'} + \|f + g\|^{\frac{p}{p'}} = \left( \int |f + g|^p \right)^{\frac{p'}{p}} + \left( \int |f - g|^p \right)^{\frac{p'}{p}} = \left( \sup |b_1| \int |f + g|^p + |b_2| \int |f - g|^p \right)^{\frac{p'}{p}}
\]

with \( p' = \frac{p}{p-1} \) and \( b_{p-1}(n) = (p-1) \cdot (p-2) \cdot \ldots \cdot (p-n) = \frac{1}{p} (-1)^{n+1} a_p(n+1). \) Hence

\[
(1 + z^{p'})^{p-1} - \frac{1}{2}(1 + z)^p - \frac{1}{2}(1 - z)^p
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{p} \frac{a_p(n+1)}{n!} z^{p'n} - \frac{1}{2} \sum_{n=2}^{\infty} \frac{a_p(n)(1 + (-1)^n)}{n!} z^n
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{p} \frac{a_p(n+1)}{n!} z^{p'n} - \sum_{k=1}^{\infty} \frac{a_p(2k)}{(2k)!} z^{2k}
\]

Splitting the the first sum which is absolutely convergent into a two series summing over odd and even indices, respectively, we get

\[
(1 + z^{p'})^{p-1} - \frac{1}{2}(1 + z)^p - \frac{1}{2}(1 - z)^p
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{p} \frac{a_p(2k)}{(2k - 1)!} z^{(2k-1)p'} - \sum_{k=1}^{\infty} \frac{1}{p} \frac{a_p(2k+1)}{(2k)!} z^{(2k)p'} - \sum_{k=1}^{\infty} \frac{a_p(2k)}{(2k)!} z^{2k}
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{p} \frac{a_p(2k)}{(2k)!} z^{(2k-1)p'} - (2k - p)z^{2k'p'} - \sum_{k=1}^{\infty} \frac{a_p(2k)}{(2k)!} z^{2k}
\]

where we have used that \( a_p(2k+1) = a_p(2k)(2k-p). \) The term in the brackets is always negative by Young’s inequality since

\[
z^{(2k-1)p'} = z^p z^{(2k-p)p'} \leq \frac{p}{2k} z^{2k} + \frac{2k-p}{2k} z^{2k'p'}.
\]

Put then \( z = \frac{a-b}{a+b} \) the lemma implies

\[
(a + b)^{p'} + (a - b)^p \leq 2(a^p + b^p)^{p'-1}
\]

and the same inequality for the choice \( z = \frac{a+b}{a-b} \) and eventually commuting \( a \) and \( b. \) Therefore we have the a.e inequality

\[
(\|f + g\|^{p'} + |f - g|^{p'})^{\frac{p}{p'}} \leq 2^{\frac{p}{p'}} (\|f\|^p + |g|^p).
\] (4.5)

We have the integral inequality

\[
\left( \int |f + g|^p \right)^{\frac{p'}{p}} + \left( \int |f - g|^p \right)^{\frac{p'}{p}} \leq \left( \int (|f + g|^{p'} + |f - g|^{p'})^{\frac{p}{p'}} \right)^{\frac{p}{p'}}
\]

Indeed, by concavity of the function \( u \mapsto u^{\frac{p}{p'}} \) we first note that

\[
(\|f + g\|^{p'} + |f - g|^{p'})^{\frac{p}{p'}} \leq |f + g|^{p'} + |f - g|^{p'} \in L^1
\]

Then

\[
\|f - g\|^{\frac{p}{p'}} + \|f + g\|^{\frac{p'}{p}} = \left( \int |f + g|^p \right)^{\frac{p'}{p}} + \left( \int |f - g|^p \right)^{\frac{p'}{p}}
\]

\[
= \left( \sup |b_1| \int |f + g|^p + |b_2| \int |f - g|^p \right)^{\frac{p}{p'}}
\]
where the supremum is taken over all \((b_1, b_2)\) such that \(b_1^l + b_2^l = 1\) with \(l\) the dual exponent of \(b_2^l\). Indeed, for vectors in \(\vec{v} \in \mathbb{R}^2\) the \(r\)-norm can be written as \(|\vec{v}|_r = \sup \vec{b} \cdot \vec{v}\) where "." denotes the scalar product in \(\mathbb{R}^2\). This follows from Hölder’s inequality. We conclude by applying Hölder’s inequality with exponents \(l, \frac{l}{l'}\) in \(\mathbb{R}^2\):

\[
\left( \int |f + g|^l \right)^{\frac{1}{l'}} + \left( \int |f - g|^l \right)^{\frac{1}{l'}} = \sup \left( \int |b_1| \int |f + g|^l + |b_2| \int |f - g|^l \right)^{\frac{1}{l'}}
\]

\[
= \sup \left( \int |b_1||f + g|^l + |b_2||f - g|^l \right)^{\frac{1}{l'}}
\]

\[
\leq \sup \left( \int (|b_1|^l + |b_2|^l) \left( |f + g|^l + |f - g|^l \right)^{\frac{l}{l'}} \right)^{\frac{1}{l'}}
\]

\[
= \left( \int (|f + g|^l + |f - g|^l)^{\frac{l}{l'}} \right)^{\frac{1}{l'}}
\]

and the last term is bounded by \(\left( 2^{\frac{l'}{l}} \int (|f|^l + |g|^l) \right)^{\frac{1}{l'}}\) which follows from the a.e. inequality (4.5). This proves the norm inequality.

**L^1(\mathbb{R}^d) is not uniformly convex.** Let \(f\) be the characteristic functions of the unit cube centered at the origin and define \(f_n(x) = f(x - n), g_n(x) = f(x + n)\). Then \(f_n, g_n\) have disjoint supports for \(n \geq 1\) and \(||f_n||_1 = ||g_n||_1 = \|\frac{1}{2}(f_n + g_n)\|_1\) but \(||f_n - g_n||_1 = 2\).

**L^{\infty}(\mathbb{R}^d) is not uniformly convex.** Let \(A \subset B\) and consider their characteristic functions. Then \(||\chi_A||_\infty = ||\chi_B||_\infty = ||\frac{1}{2}(\chi_A + \chi_B)||_\infty\) but \(||(\chi_A - \chi_B)||_\infty = 1\).

**Proof of Riesz theorem.** We suppose there is a bounded linear functional \(L \neq 0\) on \(L^p\) having no integral representation. Consider the kernel of \(L\), i.e.

\[
K := \{ g \in L^p : L(g) = 0 \}
\]

Then there is \(f \in L^p\) such that \(L(f) \neq 0\). Define \(D = \inf_{g \in K} ||f - g||_p\). Let \((g_n)\) be a sequence in \(K\) such that \(D = \lim_{n \to \infty} ||f - g_n||_p\). We want to show that \(g_n\) is a Cauchy sequence. Let \(\epsilon > 0\) be arbitrary. Then there is \(N \in \mathbb{N}\) such that \(||f - g_n||_p < D + \epsilon\) for all \(n \geq N\). First note that \(K\) is a linear subspace and therefore \(\frac{1}{2}(g_n + g_m) \in K\) for all \(n, m \in \mathbb{N}\). By the triangle inequality and the definition of \(D\) we have

\[
D \leq ||f - \frac{1}{2}(g_n + g_m)||_p \leq \frac{1}{2}||f - g_n||_p + \frac{1}{2}||f - g_m||_p \leq D + \epsilon
\]

for all \(n, m \geq N\).
1. The case $p \geq 2$. Then for all $n,m \geq N$ we apply Clarkson’s inequality with $g \mapsto f - g_n$, $f \mapsto f - g_m$:

$$
\|g_n - g_m\|_p^p \leq 2^{p-1} \|f - g_n\|_p^p + 2^{p-1} \|f - g_m\|_p^p - 2^p \|f - \frac{1}{2} (g_n + g_m)\|_p^p
$$

\[ \leq 2^p (D + \epsilon)^p - 2^p D^p \leq 2^p \epsilon(D + \epsilon)^{p-1}. \]

Since the last term can be made arbitrarily small $(g_n)$ is a Cauchy sequence.

2. The case $p \leq 2$ follows in the same way applying Clarkson’s inequality. Since $K$ is a closed subspace (why?) there is $h \in K$ such that $D = \|f - h\|_p$. Since $h + tk \in K$ for all $k \in K$ we have

$$
\frac{d}{dt}|_{t=0} \|f - h - tk\|_p^p = 0,
$$

i.e.

$$
\int |f - h|^{p-2}(f - h)k \mu(dx) = 0.
$$

For any $g \in L^p$ we define

$$
g_1 = \frac{L(g)}{L(f - h)}(f - h), \quad g_2 = g - g_1
$$

which are well defined since $L(f - h) = L(f) \neq 0$. Then

$$
L(g_1) = L(g), \quad L(g_2) = 0,
$$

that is $g_2 \in K$, and therefore

$$
\int |f - h|^{p-2}(f - h)g \mu(dx) = \int |f - h|^{p-2}(f - h)g_1 \mu(dx) = \frac{L(g)}{L(f - h)} \|f - h\|_p^p.
$$

Hence

$$
L(g) = \frac{L(f - h)}{D^p} \int |f - h|^{p-2}(f - h)g \mu(dx) = \int v g \mu(dx).
$$

with

$$
v = \frac{L(f - h)}{D^p}|f - h|^{p-2}(f - h) \in L^{p'}.
$$

Uniqueness of $v$ follows from the fact that if $f(v - w)g = 0$ for all $g \in L^p$ then choosing $g = ((v) - (W))v - w|^{(2-p)/(p-1)}X_{v \neq w} \in L^p$ to obtain $|v - w|_{p'} = 0$.

**Weak convergence.** Let $(f_n)$ be a sequence in $L^p$ and $f \in L^p$. We say that $(f_n)$ converges weakly to $f$ if

$$
\lim_{n \to \infty} L(f_n) = L(f),
$$

or equivalently

$$
\lim_{n \to \infty} L(f_n - f) = 0
$$

for every $L \in (L^p)'$. By the above theorem this means that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^d} (f_n - f)g = 0
$$

for every $g \in L^{p'}$. 

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**Unicity of weak limits.** Let $1 < p < \infty$. If $L(f) = 0$ for all $L \in (L^p)'$, then $f = 0$. Consequently, weak limits are unique.

**Proof.** Let $1 < p < \infty$. Define $g = |f(x)|^{p-2} f(x) \cdot \chi_{f \neq 0}(x)$ Then $g \in L^p$ and

$$L_g(f) = \int_{\Omega} f(x) g(x) \mu(dx) = \|f\|_p^p = 0$$

since $L_g \in (L^p)'$. Hence $f = 0$. If $p = 1$ take $g = |f(x)|^{-1} f(x) \cdot \chi_{f \neq 0}(x)$. Then $g \in L^\infty$ and the above argument applies.

**Remark.** If $p = \infty$ then the conclusion holds under the technical assumption that $(\Omega, \Sigma, \mu)$ is a sigma-finite measure space. Since then the set $\{|f| > 0\}$ has a subset $B$ of finite measure. Then define $g = |f(x)|^{-1} f(x) \cdot \chi_B(x) \in L^1$ and argue as before.

**The norm is weakly lower semicontinuous.** Let $1 < p < \infty$. If $(f_n)$ converges weakly to $f \in L^p$, then

$$\liminf_{n \to \infty} \|f_n\|_p \geq \|f\|_p$$

**Remark.** Again, if $p = \infty$ then the conclusion holds under the technical assumption that $(\Omega, \Sigma, \mu)$ is a sigma-finite measure space.

**Proof.** Take $g, L_g$ as before. By Hölder’s inequality

$$\|f\|_p = L_g(f) = \lim_{n \to \infty} L_g(f_n) \leq \|g\|_p \liminf_{n \to \infty} \|f_n\|_p$$

Since $\|g\|_p = \|f\|_p^{p-1}$ we conclude.

**Relation between weak and strong convergence.**

1. strong convergence implies weak convergence: If $(f_n)$ converges strongly to $f \in L^p$, then by continuity $(L(f_n))$ converges to $L(f)$ for every $L \in (L^p)'$.

2. If $(f_n)$ converges weakly to $f \in L^p$ and if, in addition,

$$\lim_{n \to \infty} \|f_n\|_p = \|f\|_p,$$

then $(f_n)$ converges strongly to $f \in L^p$. In particular, strong and weak limits agree, if they exist.

**Proof of 2.** Since $f_n + f$ converges weakly to $2f$ we have by the lower semicontinuity of the norm,

$$\liminf_{n \to \infty} \|f + f_n\|_p \geq 2\|f\|_p$$

and by Minkowski’s inequality that $\|f + f_n\|_p \leq \|f\|_p + \|f_n\|_p$. Since, by hypothesis, $\lim_{n \to \infty} \|f_n\|_p = \|f\|_p$ we get

$$\lim_{n \to \infty} \|f_n + f\|_p = 2\|f\|_p.$$
We now apply Clarkson’s inequalities with $g = f_n$. If $p \geq 2$, then
\[ \|f - f_n\|_p \leq 2^{p-1}\|f\|_p^p + 2^{p-1}\|f_n\|_p^p - \|f + f_n\|_p^p. \]
and the r.h.s tends to zero as $n \to \infty$ proving the strong convergence. If $1 < p \leq 2$ the proof is similar.