Parafermionic observables in discrete models and Schramm–Loewner evolution

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1 Introduction

In recent years, there has been great interest in the study of scaling limits of discrete models in two-dimensional statistical physics at criticality. The meaning of the word "criticality" varies from one model to another, but it generally signifies a parameter value that corresponds to a sudden change in the large-scale behaviour of the system. In particular, many such models have scaling limits at criticality that are either conjectured or proven to be *conformally invariant*. This means that observable properties of the models, such as crossing probabilities, are invariant under conformal transformations of the domain.

There are a number of techniques for proving such conformal invariance. A major technique exploited recently by Stanislav Smirnov and his collaborators is the use of *parafermionic observables*. Roughly speaking, these are functions defined on a domain as expectations weighted by $\exp(i\sigma w)$, where σ is the *spin* parameter and w is the total winding angle of a curve occurring in the model. These parafermionic observables, for appropriate parameter choices, satisfy local relations such as discrete harmonicity or holomorphicity. The spin $\frac{1}{2}$ case is called *fermionic*.

We will discuss two discrete models. For self-avoiding walk on the honeycomb lattice, we define a parafermionic observable which yields the value of the connective constant (but not, at this stage, the conjectured conformally invariant scaling limit). For the FK Ising model on the square lattice in a simply connected planar domain, we define a fermionic observable satisfying a certain strong preholomorphicity property and a discrete Riemann boundary value problem. These properties allow us to show that as the lattice mesh goes to zero, the observable converges to a holomorphic function invariant under conformal transformations of the domain. Moreover, the observable is a martingale with respect to the exploration process of an interface observed in the model, which in principle should allow one to prove that the interface converges in law to Schramm–Loewner evolution. We will not do so, but we will give a basic introduction to Schramm–Loewner evolution.

2 The connective constant of the honeycomb lattice

Definition 2.1. By a *lattice* $\mathcal{L}$ we mean an infinite graph embedded in the plane by points and line segments, such that the automorphisms of $\mathcal{L}$ act transitively on vertices.

For a self-avoiding path γ in a lattice $\mathcal{L}$ denote by $\ell(\gamma)$ its length, that is, the number of edges it traverses. If we denote by $N_{\mathcal{L}}(n)$ the number of self-avoiding paths in $\mathcal{L}$ of length n starting at a given point, then by concatenation $N_{\mathcal{L}}(m+n) \le N_{\mathcal{L}}(m)N_{\mathcal{L}}(n)$ for all m, n.

Lemma 2.2. *If a function $f : \mathbb{N} \to [0,\infty)$ satisfies $f(m+n) \le f(m)f(n)$ for all m, n, then*

$$\lim_{n\to\infty} f(n)^{1/n} = \inf_n f(n)^{1/n}.$$

Definition 2.3. The *connective constant* of a lattice $\mathcal{L}$ is $\lim_{n\to\infty} N_{\mathcal{L}}(n)^{1/n}$. The *partition function* of self-avoiding walk on $\mathcal{L}$ is $Z(x) = \sum x^{-\ell(\gamma)}$ for $x > 0$, where the sum runs over all finite self-avoiding paths γ starting from a given vertex.

It is easy to see that $Z(x) < \infty$ if and only if $x > x_c$, where x_c is the connective constant.

Denote by $\mathcal{H}$ the honeycomb lattice in the plane consisting of tiled regular hexagons.

Theorem 2.4 (Smirnov, Duminil-Copin)**.** *The connective constant of $\mathcal{H}$ is $\sqrt{2+\sqrt{2}}$.*

Let D be a bounded subdomain of $\mathcal{H}$ (that is, a finite subset of the vertices of $\mathcal{H}$). Denote by ∂D (resp. $\widetilde{D}$) the set of midpoints of edges which adjoin exactly one (resp. at least one) vertex of D. Fix $a \in \partial D$; for topological reasons, we require that a lie on the outer boundary. Consider a *parafermionic observable* F defined on $\widetilde{D}$ by

$$F(z) = \sum x^{-\ell(\gamma)} \exp\bigl(-i\sigma \operatorname{winding}(\gamma, a \to z)\bigr),$$

where the sum runs over all self-avoiding paths γ in D from a to z, and the spin σ and edge-weight $x > 0$ are constants to be determined. Note that $F(a) = 1$.

Lemma 2.5. *For $\sigma = \frac{5}{8}$ and $x = \sqrt{2+\sqrt{2}}$, F satisfies the relation*

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0 \tag{1}$$

for any interior vertex v with adjacent edge midpoints p, q, r, where we regard points as complex numbers.

Proof (sketch). Let v, p, q, r be as in the statement. The set of self-avoiding paths in D from a to p, q or r can be partitioned into pairs and triples, in the following manner: a path γ which ends at p having never passed through q or r forms a triple with the paths γ_q and γ_r obtained by appending to γ the segments $[p, v, q]$ and $[p, v, r]$ respectively; and a path η which ends at r having passed via subpaths α, $[p, v, q]$ and β in that order is paired with the path η' which ends at q having passed via subpaths α, $[p, v, r]$ and β's reversal, in that order. For (1) to be satisfied, it suffices that each pair and each triple contribute 0 to the left-hand side. For each pair of paths to contribute 0, it suffices that $\cos\bigl(\frac{2\pi}{3} + \sigma\frac{4\pi}{3}\bigr) = 0$ (that is, $\sigma \in \frac{5}{8} + \frac{3}{4}\mathbb{Z}$), and for each triple of paths to contribute 0, it suffices that $1 + 2x^{-1}\cos\bigl(\frac{2\pi}{3} + \sigma\frac{\pi}{3}\bigr) = 0$. Possible solutions include $x = \sqrt{2-\sqrt{2}}, \sigma = -\frac{1}{8}$ and $x = \sqrt{2+\sqrt{2}}, \sigma = \frac{5}{8}$, but only the latter gives the estimates needed for Theorem 2.4. □

Henceforth we fix σ and x as in Lemma 2.5. By summing (1) over all vertices $v \in D$ we see that

$$\sum_{m\in\partial D} (m - v(m))F(v) = 0, \tag{2}$$

where $v(m)$ is the unique vertex of D adjacent to $m \in \partial D$.

Proof of Theorem 2.4 (sketch). For $h, w \in \mathbb{N}$, consider the trapezoidal domain $D_{h,w}$ whose left side is vertical and consists of $2h + 1$ hexagons, the middle of which adjoins a, and whose symmetrical top and bottom sides each consist of w hexagons inclined from the left side at an interior angle of $\frac{2\pi}{3}$. The symmetry of $D_{h,w}$ allows us to rewrite equation (2) as

$$A_{h,w} \cos \frac{3\pi}{8} + B_{h,w} + E_{h,w} \cos \frac{\pi}{4} = 1, \tag{3}$$

where $A_{h,w}$, $B_{h,w}$ and $E_{h,w}$ are the three sums $\sum x^{-\ell(\gamma)}$ for γ ranging over self-avoiding paths in D from a to, respectively, the left boundary (minus a), the right boundary, and the top or bottom.

First we prove that the connective constant $x_c \geq x$. As $h \uparrow \infty$, $A_{h,w}$ and $B_{h,w}$ increase to finite limits A_w and B_w, since the coefficients in (3) are positive; thus also $E_{h,w}$ decreases to a limit E_w. If $E_w > 0$ for some w, then $Z(x) \geq \sum_{h\in\mathbb{N}} E_{h,w} \geq \sum_{h\in\mathbb{N}} E_w = \infty$. Otherwise, $A_w \cos \frac{3\pi}{8} + B_w = 1$ for all w; but then the estimate $A_{w+1} - A_w \leq x B_{w+1}^2$ (obtained by breaking up paths in a vertical half-strip that go exactly w columns to the right before returning to the left boundary into two left-to-right paths across $w + 1$ columns, minus two redundant half-edges) yields the bound $B_w > c/w$ by an elementary induction, for a suitable constant $c > 0$. Hence $Z(x) \geq \sum_{w\in\mathbb{N}} B_w = \infty$.

Now, we prove that $x_c \leq x$. Let $x' > x$, and set $B'_w = \sum (x')^{-\ell(\gamma)}$ summed over the same set of paths γ as B_w. We call a self-avoiding path an *excursion* of width w if it passes from one side to the other of a strip of width w; thus, it gives a summand in B'_w. Any self-avoiding path can be decomposed into two sequences of excursions of decreasing width by taking the leftmost vertex in the path and taking successive maximal excursions alternating to the right, then the left, and so on. These sequences of excursions, moreover, characterise the initial path up to a choice of direction. Thus

$$Z(x') \leq 2\Big(\sum_{S\subset\mathbb{N}_0} \prod_{s\in S} B'_s\Big)^2 = 2\Big(\prod_{n\in\mathbb{N}_0} (1 + B'_n)\Big)^2 < \infty,$$

where the last inequality holds because $B'_n \leq \left(\frac{x}{x'}\right)^n B_n \leq \left(\frac{x}{x'}\right)^n$ and therefore $\sum_{n\in\mathbb{N}_0} B'_n < \infty$. $\square$

3 A fermionic observable in the FK Ising model

3.1 The random-cluster model

We consider the Fortuin-Kasteleyn random cluster model on a finite simply connected domain Ω in the mesh δ square lattice $\delta\mathbb{Z}^2 \subset \mathbb{C}$. The lattice vertices in Ω consist of the *black vertices* of a square lattice Γ together with the *white vertices* of its dual $\widehat{\Gamma}$. A *configuration* is a partition of the non-boundary edges of Γ into *open* and *closed edges*; the *dual open edges* are the edges of $\widehat{\Gamma}$ dual to the closed edges of Γ. The *random-cluster model* on Γ with parameters $p \in (0, 1)$ and $q \geq 0$ is the probability measure P on configurations ω given by

$$P(\omega) \propto p^{\#\text{ open edges}} (1 - p)^{\#\text{ closed edges}} q^{\#\text{ open clusters}}, \tag{4}$$

where an *open cluster* is a maximal set of black vertices connected by open edges. The *FK Ising model* arises as the special case $q = 2$, and can be related to the more usual *spin Ising model* by a coupling argument. We consider *Dobrushin boundary conditions*, under which the boundary consists of a *wired* arc ba in Γ (with all edges open) and a *dual wired* arc ab in $\widehat{\Gamma}$ (with all edges dual open), with points a and b halfway between the endpoints of the boundary arcs.

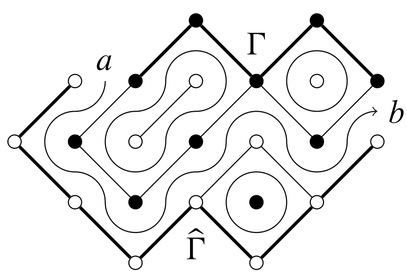
Since # open clusters + # open edges = # dual open clusters + const, P agrees with the dual measure $\widehat{P}$ defined by

$$\widehat{P}(\omega) \propto \left(\frac{p}{(1-p)q}\right)^{\#\text{ open edges}} q^{\#\text{ dual open clusters}}$$
$$\propto \hat{p}^{\#\text{ dual open edges}} (1-\hat{p})^{\#\text{ dual closed edges}} q^{\#\text{ dual open clusters}} \tag{5}$$

where $\hat{p}/(1-\hat{p}) = (1-p)q/p$, that is, the random cluster measure on $\widehat{\Gamma}$ with parameters $\hat{p}$ and q. We consider only the self-dual parameter $p = p_{\mathrm{sd}} = \sqrt{q}/(1+\sqrt{q})$, for which $p = \hat{p}$; this is known to be the critical parameter value in the FK Ising case ([2], Theorem 9.53), although we do not need that fact. Multiplying (4) by (5), we obtain

$$P(\omega) \propto \sqrt{q}^{\#\text{ open clusters} + \#\text{ dual open clusters}} \propto \sqrt{q}^{\#\text{ loops}},$$

where the loops considered are those under the *loop representation* of ω. This representation fills each atomic square of $\Gamma \cup \widehat{\Gamma}$ in with if the corresponding edge is open for ω and otherwise. This represents ω as a collection of disjoint loops and one interface $\gamma : a \to b$ covering the *directed medial graph*, whose *medial vertices* are the midpoints of edges in Γ and $\widehat{\Gamma}$, and whose *medial edges* join adjacent medial vertices in the direction which keeps black vertices on the left. We round the corners of paths on the medial lattice for clarity.



3.2 The strongly preholomorphic observable and its primitive

The *parafermionic observable* for the random-cluster model is the function F defined on midpoints of medial edges by

$$F(z) = \mathbb{E}\big[1\{z \in \gamma\} \exp\big(i\sigma \operatorname{winding}(\gamma, z \to b)\big)\big], \tag{6}$$

where the spin parameter σ is to be determined.

Proposition 3.1. *Suppose $q \le 4$. For the spin parameter $\sigma = \frac{2}{\pi} \arccos \frac{\sqrt{q}}{2}$, F satisfies the local relation $F(n) + F(s) = F(e) + F(w)$ for midpoints n, e, s, w of medial edges that adjoin the interior medial vertex z in the cardinal directions.*

Remark 3.2. Thus, we may define F on medial vertices z by $F(z) = F(n) + F(s) = F(e) + F(w)$.

Proof. Fixing z, n, e, s, w as in the statement, we group those configurations which contribute to the sum $F(n) - F(e) + F(s) - F(w)$ into pairs (ω, ω') where ω' is obtained from ω by opening or closing the edge that passes through z. By a simple calculation, in order that each pair (ω, ω') contribute 0 to the sum, it suffices that $\pm\big((1+\sqrt{q}) \sin \frac{\sigma\pi}{4} - \sin \frac{3\sigma\pi}{4}\big) = 0$, that is, $\cos \frac{\sigma\pi}{2} = \sqrt{q}/2$. $\square$

Henceforth we specialize to the FK Ising case $q = 2$, $\sigma = \frac{1}{2}$. This value of σ is special because of the following lemma. By a rotation we assume that γ arrives at b facing right.

Lemma 3.3. *For any medial edge midpoint m with neighbouring black vertex v and white vertex w, $F(m) \parallel [i(w-v)]^{-1/2}$.*

Proof. The interface γ passes through m, if at all, only in the direction $i(w-v)$. It follows that $i(w-v)\exp\big(i \text{ winding}(\gamma, m \to b)\big) > 0$ and so $\exp\big(\frac{i}{2} \text{ winding}(\gamma, m \to b)\big) \parallel [i(w-v)]^{-1/2}$. □

Corollary 3.4. *F is* strongly preholomorphic *in the following sense: for each pair (x, y) of adjacent medial vertices, oriented so that the vertex to the left when traversing $\overrightarrow{xy}$ is black, the projections of $F(x)$ and $F(y)$ onto $(y-x)^{-1/2}$ are equal.*

Remark 3.5. Strong preholomorphicity implies a more common notion of discrete holomorphicity, which states that for any lattice square with vertices a, b, c, d labelled anticlockwise, $F(d) - F(b) = i(F(c) - F(a))$.

Remark 3.6. We may define F uniquely at medial vertices $v \in \partial\Omega$ so that the above property also holds there. Moreover, $F(v) \parallel \tau(v)^{-1/2}$ for the tangent vector $\tau(v)$ to $\partial\Omega$ at v in the orientation from a to b, by reasoning similar to that of Lemma 3.3.

To show convergence of F as $\delta \to 0$, we consider a discrete primitive H defined on vertices so that, for any adjacent black vertex v and white vertex w separated by medial edge e, $H(v) - H(w) = |F(e)|^2$. That H is well-defined follows from Corollary 3.4. We may choose the additive constant so that $H = 0$ on ab and $H = 1$ on ba. A calculation yields:

Proposition 3.7. *H is subharmonic when restricted to black vertices and superharmonic when restricted to white vertices.*

3.3 Convergence as $\delta \to 0$

Consider a simply connected subdomain $\Omega \subset \mathbb{C}$ with marked boundary points $a, b \in \partial\Omega$. We consider discrete approximations (Ω_j, a_j, b_j) to (Ω, a, b) on the square lattice $\delta_j\mathbb{Z}^2$, where the mesh $\delta_j \to 0$ as $j \to \infty$. Smirnov's theorem is:

Theorem 3.8 (Smirnov). *Suppose that as $j \to \infty$, the discrete domains Ω_j converge to Ω in the Carathéodory sense of compact convergence of normalized uniformization maps, and $a_j \to a$, $b_j \to b$. Then the fermionic observables F_j defined on (Ω_j, a_j, b_j) by* (6)*, appropriately normalized, converge uniformly away from $\partial\Omega$: $F_j/\sqrt{2\delta_j} \to \sqrt{\Phi'}$, where Φ is a conformal map from Ω to* $(-\infty, \infty) \times (0, 1)$ *sending a to $-\infty$ and b to ∞.*

Here we regard F_j and H_j as defined on all of Ω_j by, for instance, piecewise constant extension.

Remark 3.9. This can be stated as a theorem about convergence of strongly preholomorphic solutions of the discrete Riemann boundary value problem $F(v) \parallel \tau(v)^{-1/2}$ to their continuous counterpart. Smirnov conjectures an extension to the case $\sigma \neq \frac{1}{2}$; the key missing ingredient in that case is the strong preholomorphicity of F.

Before proving the theorem, it is advantageous to prove the convergence of H_j. We follow the approach outlined by Werner in [9], chapter 11.

Proposition 3.10. *Under the hypotheses of Theorem 3.8, the discrete primitives H_j converge uniformly away from $\partial\Omega$ to their continuous counterpart $h := \operatorname{Im}\Phi$, which is the harmonic function on Ω taking the values* 0 *and* 1 *on ab and ba, and continuous on $\Omega \setminus \{a, b\}$.*

Proof (sketch). Let $K \subset \Omega$ be compact. Consider the discrete random walk X_t on the black vertices starting at some $y \in \Gamma_j$ nearest to a given $x \in K$, and put $T := \inf\{t > 0 : X_t \notin \operatorname{int}\Gamma_j\}$. For $r > 0$ denote by $\alpha_{r,j}$ the black vertices adjacent to the arc $a_j b_j$ which lie at least r away from a and b, and denote by β_j the black vertices of the arc $b_j a_j$. As $H_j|_{\Gamma_j}$ is subharmonic and $0 \le H_j \le 1$,

$$\begin{aligned} H_j(x) &\le H_j(y) \le \mathbb{E}[H_j(X_T)] \\ &\le P\{X_T \in \alpha_{r,j}\} \max_{z\in\alpha_{r,j}} H_j(z) + P\{X_T \in \beta_j\} + P\{|X_T - a| < r\} + P\{|X_T - b| < r\}. \end{aligned}$$

The fact that with probability one the critical FK Ising model on $\mathbb{Z}^2$ has no infinite connected component ([9], Proposition 10.4) implies that for fixed r, the first term converges to zero. The Donsker Invariance Principle ([6], Theorem 5.22) or estimates for random walk [4] imply that we may choose $r > 0$ such that the last two terms are uniformly small for large j. Moreover, $P\{X_T \in \beta_j\} \to h(x)$ since $h(x)$ is the probability that a standard Brownian motion started at x first leaves Ω through ba. We conclude that $\liminf_{j\to\infty} H_j(x) \le h(x)$. The same argument applied to white vertices shows that $\limsup_{j\to\infty} H_j(x) \ge h(x)$. These limits are all easily seen to be uniform over $x \in K$. □

We can prove the convergence of $F_j/\sqrt{2\delta_j}$ to $\sqrt{\Phi'}$ by first proving that the sequence is precompact in the topology of compact convergence, then showing that the only possible subsequential limit is $\sqrt{\Phi'}$. To establish precompactness we use the following lemma from [5].

Lemma 3.11. *If a pointwise bounded family $\mathcal{G}$ of discrete harmonic functions g, each on a lattice of mesh δ_g, satisfies*

$$\sup_{g\in\mathcal{G}} \sum_{x\sim y} |g(x) - g(y)|^2 < \infty$$

where the sum is over the lattice edges, then $\mathcal{G}$ is precompact in the uniform topology.

Proof (sketch). Interpolate $g \in \mathcal{G}$ bilinearly on each lattice square to obtain $\tilde{g}$. The hypothesis implies that $\sup_{g\in\mathcal{G}} \|D\tilde{g}\|_2 < \infty$, which gives a uniform bound on the oscillation of every $\tilde{g}$ over small circles. Combined with the maximum principle, this yields an equicontinuity estimate. □

Proof of Theorem 3.8 (sketch). Let $K \subset \Omega$ be compact, with discretization K_j. By Lemma 3.11 applied to the discrete primitives of $F_j/\sqrt{2\delta_j}$, to establish precompactness of F_j in the uniform norm on K it suffices to show that $\sup_j \delta_j \sum_{K_j} |F_j|^2 < \infty$. (The discrete primitives are only harmonic on half-lattices, but the uniform bound $|F_j| \le 1$ together with $\delta_j \to 0$ recovers convergence on the whole lattice.) Indeed, using discrete Green's function estimates [4] and sub/superharmonicity of H_j one can show that $\sup_j \delta_j \sum_{K_j} |\nabla H_j| < \infty$, which is enough.

Now let f be a subsequential limit of $\{F_j/\sqrt{2\delta_j}\}$. Since $F_j/\sqrt{2\delta_j}$ is discrete harmonic, its contour integrals along grid rectangles vanish. Therefore f's contour integrals along axis-aligned rectangles vanish, so f is holomorphic by Morera's theorem. But one can check that $2\delta_j H_j(z) = \operatorname{Im}\int F_j(z)^2\,dz$ for some choice of integration constant, so $\operatorname{Im}\int f(z)^2\,dz = h(z)$. Thus $\int f(z)^2\,dz = \Phi(z)$ up to a real constant, so $f = \sqrt{\Phi'}$. The theorem follows. □

Remark 3.12. Since the FK Ising law conditioned upon an initial segment of the interface γ equals the FK Ising law in the domain with that segment removed, F is a martingale with respect to the exploration process which traces out the interface γ one square at a time. Thus one would expect f to be a martingale with respect to the scaling limit of the interface, if such a limit curve exists. Indeed, a discrete martingale such as F that is conformally invariant in the limit has the potential to show convergence of the interface to SLE, as defined in section 4.

4 Schramm–Loewner evolution

Schramm's motivation in introducing SLE was to find a possible scaling limit for the loop-erased random walk. He conjectured that the limit should be a probability measure $\mu(a, b; D)$ on curves γ in a simply connected domain D from $a \in \partial D$ to $b \in \partial D$ satisfying two properties:

- *conformal invariance*, that is, $\mu\big(f(a), f(b); f(D)\big) = f \circ \mu(a, b; D)$ for any conformal map f; and
- *the domain Markov property*, that is, the measure from a to b in D conditioned on the initial segment $\gamma' : a \to a'$ equals the measure from a' to b in the slit domain $D \setminus \gamma'$.

By mapping any simply connected domain (D, a, b) to the half-plane $(\mathbb{H}, 0, \infty)$, we obtain from the second condition the *conformal Markov property*, which states that, if $g_t : \mathbb{H} \setminus \gamma[0, t] \to \mathbb{H}$ is the conformal map normalized so that $g_t(z) = z + o(1)$ as $z \to \infty$, then the random curve $s \mapsto g_t(\gamma(t + s)) - g_t(\gamma(t))$ is independent of $\gamma[0, t]$ and has the same law as γ.

If the curve γ is regular enough and is parametrised by ($\frac{1}{2}$ of) *half-plane capacity* (the coefficient of $1/z$ in the Laurent expansion of g_t at ∞, which is real by Schwarz reflection), the evolution of the maps g_t is described by the *chordal Loewner equation*

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}$$

where $U_t = g_t(\gamma(t))$ is the *driving function*. The conformal Markov property then implies that $U_{t+s} - U_t$ is independent of $\{U_r : r \le t\}$ and its distribution depends only upon s. It follows that U_t is a Brownian motion; moreover, scaling invariance shows that it has zero drift. This motivates the following definition, whose claims follow from [3], Theorem 4.6:

Definition 4.1. *Chordal Schramm–Loewner evolution* with parameter $\kappa \ge 0$ in $\mathbb{H}$ from 0 to ∞ parametrized by half-plane capacity is the random family of conformal maps $g_t : H_t \to \mathbb{H}$ given by

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z$$

for $z \in H_t$, the subset of $\mathbb{H}$ on which the solution exists up to time t, where B_t is a standard Brownian motion.

We need the following nontrivial theorem ([3], Theorem 7.4) to see SLE$_\kappa$ as a random curve:

Theorem 4.2. *With probability one, chordal SLE$_\kappa$ is generated by a curve γ, that is, for all t the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$ is H_t.*

The random curve is rather irregular, despite the assumptions used to derive the Loewner equation.

Proposition 4.3. *If* $\kappa \leq 4$*, then* γ *is a simple curve with* $\gamma(0,\infty) \subset \mathbb{H}$*. If* $4 < \kappa < 8$*, then* $\mathrm{dist}(0, H_t) \to \infty$*. If* $\kappa \geq 8$*, then* γ *is space-filling, that is,* $\gamma[0,\infty) = \overline{\mathbb{H}}$*.*

Proof (sketch). We call a point $z \in \overline{\mathbb{H}}$ *killed* at the time when the solution to the Loewner equation starting at z ceases to exist. For $z \in \overline{\mathbb{H}}$, writing $\hat{g}_t(z) = (g_t(z) - \sqrt{\kappa} B_t)/\sqrt{\kappa}$, we obtain the stochastic differential equation

$$d[\hat{g}_t(z)] = \frac{2/\kappa}{\hat{g}_t(z)} dt - dB_t.$$

If $z > 0$, this is the Bessel SDE, which is known to have the property that with probability one, z is killed in finite time if and only if $\kappa > 4$. This implies that γ is simple if and only if $\kappa \leq 4$. If $\kappa > 4$, by waiting for 1 and -1 both to be killed and taking a half-ball about the origin inside $\mathbb{H} \setminus H_t$, and then scaling space and time, we see that $\mathrm{dist}(0, H_t) \to \infty$. Moreover, the killing times of points $z > 0$ are strictly increasing in z if and only if $\kappa \geq 8$, which shows that in that case every boundary point lies on γ. With some extra estimates, in [3], Theorem 7.9 it is proved that every interior point also lies on γ. □

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