Representations of $GL(2, F)$ for finite and local fields F

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Chapter 1

Introduction

The representations of the general linear group are intrinsically interesting, and they are also relevant to many other areas of mathematics. In particular, they are connected to the theory of automorphic forms via an object known as the *adèle ring* (see [1], Chapter 3). We will not explore these connections, but will instead strive to understand the representations of $GL(2, F)$ for certain fields F.

The definition and basic properties of local fields are stated in Section 4.1. It turns out that local fields come in two classes, *Archimedean* and *non-Archimedean.* Of these two classes, the non-Archimedean local fields are in some senses simpler, and they have a surprising amount in common with finite fields; indeed, for each non-Archimedean local field there exists a naturally defined *residue field*, which is finite. In this essay we construct representations of $GL(2, F)$ where F is a finite field, and where F is a non-Archimedean local field. It turns out that these two cases may be discussed in a way that is similar in broad outline. Indeed, in this essay we reuse many of the propositions from the finite field case in the non-Archimedean local field case.

For finite fields F, we construct all the irreducible representations of $GL(2, F)$. They fall into four categories: characters, special representations, principal series representations and cuspidal representations. All of these representations are given in more or less explicit form, except for the special representations (for the sake of brevity). By counting arguments we verify that all the irreducibles have been found.

For non-Archimedean local fields F, we construct a large family of representations of $GL(2, F)$, known as *dihedral supercuspidal* representations, which correspond to the cuspidal representations of the finite field case. We do not construct any other families of representations, although a slight variant of our construction yields the principal series representations in a way we shall not discuss (see [1], p. 543). The representations we construct are irreducible, but the verification of this fact requires theory that is beyond the scope of this essay (see [1], Sections 4.7 and 4.8). Because our constructions depend heavily on the diagonalisation of symmetric bilinear forms, we shall not consider those non-Archimedean local fields whose residue fields have characteristic 2. Indeed, in such cases, the standard categorisation of irreducible representations breaks down in a way described by the *local Langlands conjecture* (see [1], Section 4.9).

There are certainly many ways to construct representations of $GL(2, F)$, and the method adopted in this essay is not the most efficient method possible. An alternative, perhaps more elementary, method to construct the irreducible representations of $GL(2, F)$ for finite fields F is described by Piatetski-Shapiro [7]. The irreducible representations of $GL(2, F)$ are in some sense parametrised by the characters of its *tori*, a viewpoint expounded (in greater generality) by Deligne and Lusztig [3].

The route we take to construct representations of $GL(2, F)$ is the one outlined by Bump [1]. Most of this essay is concerned with constructing representations of $GL(2, F)$ by means of the *Weil representation*. This is a construction introduced in 1964 in a famous paper of Weil [10], which is applicable in considerably greater generality than we shall consider in this essay. We give a brief treatment of the principal series representations as induced representations in Section 3.4, but these representations are subsequently rediscovered by means of the Weil representation in Section 3.6.

In Chapter 2, we give some basic definitions, introduce the induced representation, develop Mackey's theory of intertwiners, introduce the *Heisenberg group*, and provide a presentation for $SL(2, F)$. In Chapter 3, we prove the Finite Stone–von Neumann Theorem, and we use that as a starting point to construct representations of $SL(2, F)$ for finite fields F by means of the Weil representation. We arrive at a complete description of the irreducible representations of $GL(2, F)$. In Chapter 4, we introduce local fields, introduce an appropriate notion of *admissible representations*, and proceed to construct a large family of representations of $GL(2, F)$ for non-Archimedean local fields F by means of the Weil representation.

1.1 Index of notation

The table below provides a guide to the notation and nomenclature that will be used. Terms introduced in the text appear in SMALL CAPS when they are first defined. Blank entries in matrices always denote zero. We frequently use 0 and 1 to represent the trivial subspace $\{0\}$ and trivial subgroup $\{1\}$ respectively.

Notation	Meaning
$\mathbb{N}$	The natural numbers $\{1, 2, 3, \ldots\}$
I	The identity matrix
$A \sqcup B$	Disjoint union of A and B
$\mathbb{T}$	The group $\{t \in \mathbb{C} \mid \lvert t\rvert = 1\}$
$R^\times$	The group of invertible elements of the ring R
$Z(G)$	The centre of the group G
$H_1 \backslash G / H_2$	The set of double cosets $\{H_1 g H_2 \mid g \in G\}$
$\mathbf{1}_S$	The characteristic function of the set S
$\operatorname{char} F$	The characteristic of the field F
$\lvert a\rvert_p$	The p-adic absolute value of a
$\operatorname{supp} \Phi$	The support $\overline{\{v \in V \mid \Phi(v) \neq 0\}}$ of the function $\Phi : V \to \mathbb{C}$

Chapter 2

Background theory

In this chapter we establish the basic theoretical tools we will need to generate representations of $GL(2, F)$ for finite and local fields F. We will often construct representations by the construction known as the *induced representation*, and *Mackey theory* provides the tools we need to analyse intertwiners between induced representations. We introduce the *Heisenberg group*, which we will use to construct certain projective representations of $SL(2, F)$. Finally, we give a presentation of $SL(2, F)$, which is needed to check that the projective representations of $SL(2, F)$ we construct can be lifted to true representations.

2.1 Representations and intertwiners

Definition 2.1. Let G be a group. A REPRESENTATION of G is a group homomorphism $\pi : G \to GL(V)$, for some complex vector space V. We often denote the representation by (π, V), to emphasise the space on which G acts. A PROJECTIVE REPRESENTATION of G is a homomorphism $\rho : G \to PGL(V)$, for some complex vector space V.

Definition 2.2. A CHARACTER of a topological group G is a continuous group homomorphism $\chi : G \to \mathbb{T}$. If G is finite, then we assume no topology on G, so a character of G is in that case any homomorphism $\chi : G \to \mathbb{T}$.

An ADDITIVE CHARACTER of a ring R is a character of R considered as a group under addition; a MULTIPLICATIVE CHARACTER of R is a character of the group of units $R^\times$. Any topology on R is taken to apply also to these groups.

A QUASICHARACTER of a topological group G is a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$.

Remarks 2.3. If χ is a quasicharacter of a finite group G, then $\chi(g)$ has finite order for all $g \in G$. Thus $\operatorname{im} \chi \subseteq \mathbb{T}$, so χ is in fact a character.

Often we will regard a character of a group as a one-dimensional representation of that group on the space $\mathbb{C}$ (by multiplication).

We will not use the word *character* to mean a map giving the traces of the representing matrices of a representation.

Definition 2.4. If (π_1, V_1) and (π_2, V_2) are representations of a group G, we say a linear map $T : V_1 \to V_2$ is an INTERTWINER from π_1 to π_2 if $T \circ \pi_1(g) = \pi_2(g) \circ T$ for all $g \in G$. The space of intertwiners from π_1 to π_2 will be denoted $\mathrm{Hom}_G(\pi_1, \pi_2)$. This is not to be confused with the (larger) space $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ of vector space homomorphisms from V_1 to V_2. The representations π_1 and π_2 are EQUIVALENT if there exists an intertwiner $T : V_1 \to V_2$ from π_1 to π_2 which is also a vector space isomorphism.

We will often want to understand the space of all intertwiners between two given representations, because this space gives us useful information about the structure of those representations.

The complete reducibility of each representation of a finite group G on a finite-dimensional complex vector space, known as Maschke's theorem, is the key ingredient in the following results. For the remainder of this section, we assume all representation spaces are finite-dimensional.

Proposition 2.5. *Let G be a finite group, and let (π, V) and (ρ, W) be representations of G. Let $V = \bigoplus_{i=1}^r V_i$ and $W = \bigoplus_{j=1}^s W_j$ be decompositions of V and W into irreducible invariant subspaces of π and ρ respectively, and let $\pi_i = \pi|_{V_i}$ and $\rho_j = \rho|_{W_j}$ be the corresponding subrepresentations for all i and j. For any irreducible representation σ of G, let $n(\sigma, \pi)$ be the number of subrepresentations in the above decomposition of V that are equivalent to σ, and similarly for $n(\sigma, \rho)$. Then*

$$\dim \mathrm{Hom}_G(\pi, \rho) = \sum_{\sigma} n(\sigma, \pi) n(\sigma, \rho),$$

where the sum is over all irreducible representations σ of G (up to equivalence).

Proof. By Schur's lemma,

$$\dim \mathrm{Hom}_G(\pi_i, \rho_j) = \begin{cases} 1 & \text{if } \pi_i \cong \rho_j; \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

It is a general property of modules over a ring R that $\mathrm{Hom}_R(A \oplus B, C) = \mathrm{Hom}_R(A, C) \oplus \mathrm{Hom}_R(B, C)$ and $\mathrm{Hom}_R(A, B \oplus C) = \mathrm{Hom}_R(A, C) \oplus \mathrm{Hom}_R(B, C)$ for R-modules A, B and C. Since representations of a group G are just modules over the group algebra $\mathbb{C}G$ and intertwiners are just $\mathbb{C}G$-module homomorphisms, we have

$$\mathrm{Hom}_G(\pi, \rho) = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s} \mathrm{Hom}_G(\pi_i, \rho_j).$$

It now follows from (2.1) that

$$\dim \mathrm{Hom}_G(\pi, \rho) = \dim \bigoplus_{\substack{1 \le i \le r \\ 1 \le j \le s \\ \pi_i \cong \rho_j}} \mathrm{Hom}_G(\pi_i, \rho_j) = \sum_{\sigma} \sum_{\substack{1 \le i \le r \\ 1 \le j \le s \\ \pi_i \cong \rho_j \cong \sigma}} 1 = \sum_{\sigma} n(\sigma, \pi) n(\sigma, \rho),$$

where σ runs over all irreducible representations of G (up to equivalence). □

Some simple consequences of this proposition are especially useful.

Corollary 2.6. *Let (π, V) be a representation of a finite group G. If $\dim \operatorname{Hom}_G(\pi, \pi) = 1$ then π is irreducible. If $\dim \operatorname{Hom}_G(\pi, \pi) = 2$ then π is a direct sum of two inequivalent irreducible representations.*

Proof. Apply Proposition 2.5 with $\rho = \pi$. □

Corollary 2.7. *Let G be a finite group and let (π, V) and (ρ, W) be representations of G. If $\dim \operatorname{Hom}_G(\pi, \rho) = 0$ then π and ρ have no equivalent nontrivial subrepresentations.*

Proof. If they had a nontrivial equivalent subrepresentation, they must have an equivalent irreducible subrepresentation, which is impossible if $\dim \operatorname{Hom}_G(\pi, \rho) = 0$ by Proposition 2.5. □

2.2 Induced representations and Mackey theory

Definition 2.8. Let G be a group, H a subgroup of G and (π, V) a representation of H. Let

$$V^G = \big\{ f : G \to V \;\big|\; f(hg) = \pi(h) f(g) \text{ for all } h \in H \text{ and } g \in G \big\}$$

and define the representation $\pi^G : G \to GL(V^G)$ by

$$\big(\pi^G(k) f\big)(g) = f(gk)$$

for all $g, k \in G$. Since $h(gk) = (hg)k$, we have $\pi^G(k) f \in V^G$ for all $f \in V^G$ and $k \in G$. Also,

$$\big(\pi^G(k)\pi^G(l) f\big)(g) = \big(\pi^G(l) f\big)(gk) = \pi^G(gkl) = \big(\pi^G(kl) f\big)(g)$$

for all $g, k, l \in G$. So π^G is indeed a representation of G on V^G. We call (π^G, V^G) the INDUCED REPRESENTATION of G from (π, V).

Induced representations are our main tool for creating representations of groups from representations of their subgroups. We would often like to know about the structure of induced representations, and a good way to do this is to analyse intertwiners between induced representations.

Proposition 2.9. *Let H_1 and H_2 be subgroups of a finite group G. Let (π_1, V_1) and (π_2, V_2) be representations of H_1 and H_2 respectively. Let*

$$\mathcal{D} = \left\{ \Delta : G \to \operatorname{Hom}_{\mathbb{C}}(V_1, V_2) \;\middle|\; \begin{array}{c} \Delta(h_2 g h_1) = \pi_2(h_2) \circ \Delta(g) \circ \pi_1(h_1) \\ \text{for all } h_1 \in H_1,\ h_2 \in H_2 \text{ and } g \in G \end{array} \right\}.$$

Then $\operatorname{Hom}_G(\pi_1^G, \pi_2^G)$ and $\mathcal{D}$ are isomorphic as complex vector spaces.

Proof. For any function $f : G \to V_1$ and any $\Delta \in \mathcal{D}$, we can consider the CONVOLUTION $\Delta * f : G \to V_2$ defined by

$$(\Delta * f)(g) = \sum_{x \in G} \Delta(gx^{-1}) f(x)$$

for all $g \in G$. It is easily seen from the definition of $\mathcal{D}$ that $\Delta * f \in V_2^G$. For any $\Delta \in \mathcal{D}$ we define the map $L(\Delta) : V_1^G \to V_2^G$ by $L(\Delta)(f) = \Delta * f$. Clearly $L(\Delta)$ is $\mathbb{C}$-linear, and for all $g, k \in G$ we have

$$\begin{aligned}\big(\pi_2^G(k)(\Delta * f)\big)(g) = (\Delta * f)(gk) &= \sum_{x \in G} \Delta(gkx^{-1}) f(x) \\ &= \sum_{y \in G} \Delta(gy^{-1}) f(yk) = \big(\Delta * \pi_1^G(k) f\big)(g),\end{aligned}$$

so $L(\Delta) \in \mathrm{Hom}_G(\pi_1^G, \pi_2^G)$.

Now we consider the map $L : \mathcal{D} \to \mathrm{Hom}_G(\pi_1^G, \pi_2^G)$ given by $\Delta \mapsto L(\Delta)$ as defined above. We claim this is a vector space isomorphism. It follows from the definition of convolution that L is $\mathbb{C}$-linear. Suppose first that $L(\Delta) = 0$ for some $\Delta \in \mathcal{D}$. For any $v \in V_1$ define $e_v \in V_1^G$ by $e_v(h_1) = \pi_1(h_1)v$ for $h_1 \in H_1$ and $e_v(g) = 0$ for $g \in G \setminus H_1$. Now, since $L(\Delta) = 0$,

$$0 = \big(\Delta * e_v\big)(g) = \sum_{h_1 \in H_1} \Delta(gh_1^{-1}) \pi_1(h_1) v = |H_1| \Delta(g) v$$

for all $v \in V_1$ and $g \in G$. Thus $\Delta = 0$, and so L is injective.

Finally, let $T \in \mathrm{Hom}_G(\pi_1^G, \pi_2^G)$, and define $\Delta : G \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ by

$$\Delta(g) v = |H_1|^{-1} T(e_v)(g)$$

for all $g \in G$ and $v \in V_1$, where e_v is as defined above. $\Delta(g)$ is a $\mathbb{C}$-linear map because $v \mapsto e_v$ is itself $\mathbb{C}$-linear. Also,

$$\begin{aligned}\Delta(h_2 g h_1) v &= |H_1|^{-1} \pi_2(h_2) T(e_v)(gh_1) && \text{as } T(e_v) \in V_2^G \\ &= |H_1|^{-1} \pi_2(h_2) \pi_2^G(h_1)\big(T(e_v)\big)(g) \\ &= |H_1|^{-1} \pi_2(h_2) T\big(\pi_1^G(h_1) e_v\big)(g) && \text{as } T \text{ is an intertwiner} \\ &= |H_1|^{-1} \pi_2(h_2) T\big(e_{\pi_1(h_1)v}\big)(g) \\ &= \pi_2(h_2) \Delta(g) \pi_1(h_1) v\end{aligned}$$

for all $h_1 \in H_1$, $h_2 \in H_2$ and $g \in G$, so $\Delta \in \mathcal{D}$. Now we can check that

$$\begin{aligned}(\Delta * f)(g) &= \sum_{x \in G} |H_1|^{-1} L(e_{f(x)})(gx^{-1}) = |H_1|^{-1} \sum_{x \in G} \pi_2^G(x^{-1})\big(T(e_{f(x)})\big)(g) \\ &= |H_1|^{-1} \sum_{x \in G} T\big(\pi_1^G(x^{-1}) e_{f(x)}\big)(g) = T\left(|H_1|^{-1} \sum_{x \in G} \pi_1^G(x^{-1}) e_{f(x)}\right)(g)\end{aligned}$$

for all $f \in V_1^G$ and $g \in G$. But it is easy to see that

$$\left(\sum_{x\in G} \pi_1^G(x^{-1})e_{f(x)}\right)(g) = \sum_{x\in G} e_{f(x)}(gx^{-1}) = \sum_{x\in H_1 g} \pi_1(gx^{-1})f(x) = |H_1|f(g)$$

for all $f \in V_1^G$ and $g \in G$, so $\Delta * f = T(f)$ for all $f \in V_1^G$. Thus $L(\Delta) = T$. Since $T \in \mathrm{Hom}_G(\pi_1^G, \pi_2^G)$ was arbitrary, L is surjective, and the result follows. □

For $\Delta \in \mathcal{D}$ as in Proposition 2.9, the map $\Delta(g)$ determines the maps $\Delta(g')$ for all g' in the double coset $H_1 g H_2$. It is therefore natural to identify Δ with its values on a set of double coset representatives, and that is what Mackey's theorem does.

Theorem 2.10 (Mackey's theorem)**.** *Let H_1 and H_2 be subgroups of a finite group G, and let $x_1, \dots, x_r$ be a complete set of double coset representatives for $H_2\backslash G/H_1$. Let (π_1, V_1) and (π_2, V_2) be representations of H_1 and H_2, respectively. For $i = 1, \dots, r$, define representations $(\pi_{1,i}, V_1)$ and $(\pi_{2,i}, V_2)$ of $S_i := x_i H_1 x_i^{-1} \cap H_2$ by $\pi_{1,i}(s) = \pi_1(x_i^{-1} s x_i)$ and $\pi_{2,i}(s) = \pi_2(s)$. Then*

$$\mathrm{Hom}_G(\pi_1^G, \pi_2^G) \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i})$$

as complex vector spaces.

Proof. Let $\mathcal{W} = \bigoplus_{i=1}^r \mathrm{Hom}_{S_i}(\pi_{1,i}, \pi_{2,i})$. By Proposition 2.9, it will suffice to prove that $\mathcal{D} \cong \mathcal{W}$. Define the linear map $M : \mathcal{D} \to \mathcal{W}$ by $M\Delta = \big(\Delta(x_i)\big)_{i=1}^r$. We need to check that each $\Delta(x_i)$ is an intertwiner from $\pi_{1,i}$ to $\pi_{2,i}$, which is true because

$$\Delta(x_i) \circ \pi_{1,i}(s) = \Delta(x_i) \circ \pi_1(x_i^{-1} s x_i) = \Delta(s x_i) = \pi_{2,i}(s) \circ \Delta(x_i)$$

for all i and all $s \in S_i$. M is clearly $\mathbb{C}$-linear, and is injective since $\Delta(x_i)$ determines the values of Δ on $H_1 x_i H_2$ by Proposition 2.9.

We would like to show M is surjective. Let $(T_i)_{i=1}^r \in \mathcal{W}$, and tentatively define $\Delta : G \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ by

$$\Delta(h_2 x_i h_1) = \pi_2(h_2) \circ T_i \circ \pi_1(h_1)$$

for all i and all $h_1 \in H_1$, $h_2 \in H_2$. This gives a candidate definition of $\Delta(g)$ for all $g \in G$, since we have a complete set of double coset representatives x_i. We need to check this definition is valid. Since the double cosets in $H_2\backslash G/H_1$ are disjoint and have union G, the only possible conflicting definitions occur when $h_2 x_i h_1 = h_2' x_i h_1'$ for some i, some $h_1, h_1' \in H_1$ and some $h_2, h_2' \in H_2$. If that is the case, then $h_2^{-1} h_2' = x_i h_1 (h_1')^{-1} x_i^{-1} \in S_i$. Using the fact that $T_i \circ \pi_{1,i}\big(x_i h_1 (h_1')^{-1} x_i^{-1}\big) = \pi_{2,i}\big(h_2^{-1} h_2'\big) \circ T_i$, we have

$$\begin{aligned}\pi_2(h_2) \circ T_i \circ \pi_1(h_1) &= \pi_2(h_2) \circ \pi_{2,i}(h_2^{-1} h_2') \circ T_i \circ \pi_{1,i}(x_i h_1' h_1^{-1} x_i^{-1}) \circ \pi_1(h_1) \\ &= \pi_2(h_2 h_2^{-1} h_2') \circ T_i \circ \pi_1(h_1' h_1^{-1} h_1) \\ &= \pi_2(h_2') \circ T_i \circ \pi_1(h_1'),\end{aligned}$$

which shows that Δ is well-defined. It is clear from the definition of Δ that $\Delta \in \mathcal{D}$ and $M\Delta = (T_i)_{i=1}^r$. So M is surjective, and the theorem is proved. □

2.3 The Heisenberg group

In order to construct the Weil representation for $SL(2,F)$, we need to consider representations of a simpler group called the Heisenberg group, which is in some respects similar to the group of matrices of the form

$$\begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & I & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}.$$

Definition 2.11. Suppose we are given a field F not of characteristic 2, a finite-dimensional vector space V over F, and a nondegenerate symmetric bilinear form $B : V \times V \to F$. The HEISENBERG GROUP is the set $H = V \times V \times F$ equipped with the multiplication

$$(u, v, x)(u', v', x') = (u + u', v + v', x + x' + B(u, v') - B(v, u')).$$

We need to check this is a group: multiplication is associative since

$$\begin{aligned}(u + u', v + v', x + x' + B(u, v') - B(v, u'))(u'', v'', x'')\\ = \big(u + u' + u'', v + v' + v'', x + x' + x'' + B(u, v')\\ + B(u, v'') + B(u', v'') - B(v, u') + B(v, u'') + B(v', u'')\big)\\ = (u, v, x)(u' + u'', v' + v'', x' + x'' + B(u', v'') - B(v', u'')).\end{aligned}$$

Furthermore, $(0, 0, 0)$ is an identity element, and (u, v, x) has an inverse $(-u, -v, -x)$.

Remark 2.12. To find the centre of H, note that (u, v, x) is in the centre if and only if

$$\big(u+u', v+v', x+x'+B(u, v')-B(v, u')\big) = \big(u'+u, v'+v, x'+x+B(u', v)-B(v', u)\big)$$

for all $(u', v', x') \in H$. This is equivalent to requiring that $B(u, v') = B(v, u')$ for all $u', v' \in V$. By taking $v' = 0$ this implies that $B(v, u') = 0$ for all $u', v' \in V$, which implies that $v = 0$, since B is nondegenerate. Similarly we require that $u = 0$. Conversely, $(0, 0, x)$ is in the centre of H for all $x \in F$. So the centre of H is $\{(0, 0, x) \mid x \in F\}$.

Proposition 2.13. *$SL(2,F)$ acts on H by group automorphisms, letting*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (u, v, x) = (au + bv, cu + dv, x)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,F)$ *and* $(u, v, x) \in H$.

Proof. It is very straightforward to check that this is a group action. To check that $SL(2,F)$ acts by automorphisms, we must check that

$$\begin{aligned}(au + bv, cu + dv, x)(au' + bv', cu' + dv', x')\\ = (a(u + u') + b(v + v'), c(u + u') + d(v + v'), x + x' + B(u, v') - B(v, u')).\end{aligned}$$

The first two components clearly agree, so we need only check that

$$x + x' + B(au + bv, cu' + dv') - B(cu + dv, au' + bv') = x + x' + B(u, v') - B(v, u').$$

After expansion and cancellation, the left hand side reduces to

$$x + x' + (ad - bc)B(u, v') + (bc - ad)B(v, u'),$$

which equals the right hand side since $ad - bc = 1$. □

2.4 The structure of $SL(2,F)$

We will need a presentation for $SL(2,F)$ in order to lift a projective representation to a true representation. Our presentation is inspired by the Bruhat decomposition: we only need a single representative, $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$, of the nontrivial double coset in $B\backslash SL(2,F)/B$, where B is the subgroup of $SL(2,F)$ whose elements are upper triangular.

Lemma 2.14. *$SL(2,F)$ is generated by $\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)$ for $b \in F$, $\left(\begin{smallmatrix} a & \\ & 1/a \end{smallmatrix}\right)$ for $a \in F^\times$, and $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$.*

Proof. If $\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \in SL(2,F)$ then $d = 1/a$ and so

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} = \begin{pmatrix} 1 & ab \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & 1/a \end{pmatrix}.$$

If $c \neq 0$ then $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -a/c \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $SL(2,F)$ and has zero as its bottom-left entry, so it can be expressed in terms of the generators by the previous case. This gives the decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & cd \\ & 1 \end{pmatrix} \begin{pmatrix} -c & \\ & -1/c \end{pmatrix} \quad \text{if } c \neq 0.$$ □

Before we state our presentation of $SL(2,F)$, we recall the formal definition of a group given by generators and relations. Let X be a set, $\mathcal{F}$ the free group on X and R a subset of $\mathcal{F}$. R can be thought of as a set of words in the elements of X and their inverses; a relation stated in the form $x_1 \ldots x_m = y_1 \ldots y_n$ translates to the word $x_1 \ldots x_m y_n^{-1} \ldots y_1^{-1}$ in R. Let $\mathcal{N}$ be the intersection of all normal subgroups of $\mathcal{F}$ containing R; this is itself a normal subgroup of $\mathcal{F}$. The GROUP GENERATED BY X SUBJECT TO THE RELATIONS R is defined to be $G = \mathcal{F}/\mathcal{N}$. There is a natural map $i : X \to G : x \mapsto x\mathcal{N}$. The universal property of this construction states that, if H is a group and $f : X \to H$ any map such that $f(x_1)^{\varepsilon_1} \ldots f(x_n)^{\varepsilon_n} = 1$ for every word $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} \in R$ ($x_i \in X$, $\varepsilon_i \in \{\pm 1\}$), then there exists a unique homomorphism $\alpha : G \to H$ such that $f = \alpha \circ i$.

Proposition 2.15. *Let F be a field. Consider the group G generated by the symbols $t(a)$ for $a \in F^\times$, $n(b)$ for $b \in F$ and w, subject to the relations*

$$\begin{aligned} t(a_1)t(a_2) &= t(a_1 a_2), & n(b_1)n(b_2) &= n(b_1 + b_2), \\ t(a)n(b) &= n(a^2 b)t(a), & wt(a)w &= t(-1/a), \\ \textit{and} \quad wn(b)w &= n(-1/b)wn(-b)t(-b) & \textit{if} \quad b &\neq 0. \end{aligned} \tag{2.2}$$

Then there exists a unique isomorphism $\alpha : G \to SL(2,F)$ *such that*

$$t(a) \mapsto \begin{pmatrix} a & \\ & 1/a \end{pmatrix}, \quad n(b) \mapsto \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad \text{and} \quad w \mapsto \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{2.3}$$

Proof. By the universal property of a group defined by generators and relations, there exists a unique homomorphism α satisfying (2.3) if and only if the images of the generators in (2.3) satisfy the relations (2.2). It is easy to see that

$$\begin{aligned} \begin{pmatrix} a_1 & \\ & 1/a_1 \end{pmatrix}\begin{pmatrix} a_2 & \\ & 1/a_2 \end{pmatrix} &= \begin{pmatrix} a_1a_2 & \\ & 1/a_1a_2 \end{pmatrix}, \\ \begin{pmatrix} 1 & b_1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & b_2 \\ & 1 \end{pmatrix} &= \begin{pmatrix} 1 & b_1+b_2 \\ & 1 \end{pmatrix}, \\ \begin{pmatrix} a & \\ & 1/a \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} &= \begin{pmatrix} a & ab \\ & 1/a \end{pmatrix} = \begin{pmatrix} 1 & a^2b \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1/a \end{pmatrix}, \\ \text{and} \quad \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & \\ & 1/a \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} &= \begin{pmatrix} -1/a & \\ & -a \end{pmatrix}. \end{aligned}$$

The last relation is less obvious, but we can calculate that

$$\begin{aligned} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} &= \begin{pmatrix} -1 & \\ b & -1 \end{pmatrix} = \begin{pmatrix} 1/b & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} -b & 1 \\ & -1/b \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1/b \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & -b \\ & 1 \end{pmatrix}\begin{pmatrix} -b & \\ & -1/b \end{pmatrix}. \end{aligned}$$

Thus there is a unique homomorphism α satisfying (2.2). By Lemma 2.14, α is surjective.

Before showing α is injective, we will make some observations about G. Since $n(0)n(0) = n(0)$ and $t(1)t(1) = t(1)$, we see that $n(0) = t(1) = 1$, and hence $t(-1)^2 = t(1) = 1$. Note that the fourth relation with $a = 1$ gives $w^2 = t(-1)$, which shows that

$$wt(a) = wt(a)t(-1)^2 = wt(a)w^4 = t(-1/a)w^3 = t(-1/a)t(-1)w = t(1/a)w$$

for all $a \in F^\times$. This will allow us to "bump" $t(a)$ past w in our calculations.

To show α is injective, we will construct an inverse map $\beta : SL(2,F) \to G$. As we saw in Lemma 2.14, there are two cases to consider, so we define

$$\begin{aligned} \beta\begin{pmatrix} a & b \\ & d \end{pmatrix} &= n(ab)t(a) \\ \text{and} \quad \beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= n(a/c)wn(cd)t(-c) \quad \text{if } c \neq 0. \end{aligned}$$

To check this is a homomorphism, we must check that

$$\beta\begin{pmatrix} A & B \\ C & D \end{pmatrix}\beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta\begin{pmatrix} Aa+Bc & Ab+Bd \\ Ca+Dc & Cb+Dd \end{pmatrix} \tag{2.4}$$

whenever $AD-BC=ad-bc=1$.

If $C=c=0$, we can use the third relation to "bump" t past n as follows:

$$n(AB)t(A)n(ab)t(a)=n(AB)n(A^2ab)t(A)t(a)=n\big(Aa(Ab+Bd)\big)t(Aa)$$

since $ad=1$, which proves (2.4). If $c=0$ but $C\neq 0$, checking (2.4) is just as easy:

$$\begin{aligned}n(A/C)wn(CD)t(-C)n(ab)t(a)&=n(Aa/Ca)wn(CD)n(C^2ab)t(-C)t(a)\\&=n(Aa/Ca)wn\big(Ca(Cb+Dd)\big)t(-Ca)\end{aligned}$$

since $ad=1$. If $C=0$ but $c\neq 0$, we have to do a little more work. Thus we can "bump" t past n and w to get

$$\begin{aligned}n(AB)t(A)n(a/c)wn(cd)t(-c)&=n(AB)n(A^2a/c)t(A)wn(cd)t(-c)\\&=n(A^2a/c+AB)wt(1/A)n(cd)t(-c)\\&=n(A^2a/c+AB)wn(cd/A^2)t(1/A)t(-c)\\&=n\big((Aa+Bc)/Dc\big)wn(D^2cd)t(-Dc)\end{aligned}$$

since $D=1/A$, which proves (2.4).

If C and c are both nonzero, we first calculate that

$$\begin{aligned}\beta\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\beta\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)&=n(A/C)wn(CD)t(-C)n(a/c)wn(cd)t(-c)\\&=n(A/C)wn(CD)n(C^2a/c)wn(cd/C^2)t(c/C)\\&=n(A/C)wn\big(C(Ca+Dc)/c\big)wn(cd/C^2)t(c/C).\end{aligned}$$

There are now two subcases, depending on whether $Ca+Dc$ is zero or not. If $Ca+Dc\neq 0$, then we can use the fifth relation to get rid of one w:

$$\begin{aligned}\beta\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\beta\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)&=n\big(\tfrac{A}{C}\big)n\big(\tfrac{-c}{C(Ca+Dc)}\big)wn\big(-\tfrac{C(Ca+Dc)}{c}\big)t\big(-\tfrac{C(Ca+Dc)}{c}\big)n\big(\tfrac{cd}{C^2}\big)t\big(\tfrac{c}{C}\big)\\&=n\big(\tfrac{A(Ca+Dc)-c}{C(Ca+Dc)}\big)wn\big(-\tfrac{C(Ca+Dc)}{c}\big)n\big(\tfrac{d(Ca+Dc)^2}{c}\big)t(-Ca-Dc)\\&=n\big(\tfrac{ACa+(1+BC)c-c}{C(Ca+Dc)}\big)wn\big((Ca+Dc)\tfrac{d(Ca+Dc)-C}{c}\big)t(-Ca-Dc)\\&=n\big(\tfrac{Aa+Bc}{Ca+Dc}\big)wn\big((Ca+Dc)\tfrac{(1+bc)C+cdD-C}{c}\big)t(-Ca-Dc)\\&=n\big(\tfrac{Aa+Bc}{Ca+Dc}\big)wn\big((Ca+Dc)(Cb+Dd)\big)t(-Ca-Dc)\\&=\beta\left(\begin{smallmatrix}Aa+Bc&Ab+Bd\\Ca+Dc&Cb+Dd\end{smallmatrix}\right).\end{aligned}$$

If $Ca+Dc=0$, we observe that $-c(Cb+Dd)=C(1-ad)-Dcd=C-d(Ca+Dc)=C$, and pick up from the same point we left off:

$$\begin{aligned}\beta\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\beta\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)&=n(A/C)w^2n(cd/C^2)t(c/C)\\&=n(A/C)n(cd/C^2)t(-c/C)\qquad\text{as } w^2=t(-1)\\&=n\big(\tfrac{AC+cd}{C^2}\big)t\big(-\tfrac{c}{C}\big)\\&=n\big(\tfrac{-Ac(Cb+Dd)+cd}{-Cc(Cb+Dd)}\big)t\big(\tfrac{1}{Cb+Dd}\big)\\&=n\big(\tfrac{-ACb-(1+BC)d+d}{-C(Cb+Dd)}\big)t\big(\tfrac{1}{Cb+Dd}\big)\\&=n\big((Aa+Bc)(Ab+Bd)\big)t\big(Aa+Bc\big)\big)\\&=\beta\left(\begin{smallmatrix}Aa+Bc&Ab+Bd\\&Cb+Dd\end{smallmatrix}\right),\end{aligned}$$

since $(Aa + Bc)(Cb + Dd) = 1$. We have shown β is a homomorphism.

It is now easy to see that

$$\begin{aligned} \beta\big(\alpha(t(a))\big) &= \beta\big(\begin{smallmatrix} a & \\ & 1/a \end{smallmatrix}\big) = n(0)t(a) = t(a), \\ \beta\big(\alpha(n(b))\big) &= \beta\big(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\big) = n(b)t(1) = n(b), \\ \text{and} \qquad \beta\big(\alpha(w)\big) &= \beta\big(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\big) = n(0)wn(0)t(1) = w, \end{aligned}$$

and hence that $\beta \circ \alpha = \mathrm{id}$ (since it is true on the generators of G). Therefore α is injective and thus an isomorphism. □

Chapter 3

Representations over finite fields

In this chapter, we construct all the irreducible representations of $GL(2, F)$ for a finite field F. We first prove the Finite Stone–von Neumann Theorem. Applied to the Heisenberg group H, this establishes the existence of a unique irreducible representation of H with certain properties. The action described in Proposition 2.13 allows us to construct a family of equivalent irreducible representations of H, and we then use Schur's lemma to show that the intertwiners among the members of this family themselves form a projective representation of $SL(2, F)$. From this point on, a fair amount of calculation is needed, first to find the projective representation of $SL(2, F)$ in explicit form, and then to check that the projective representation can be lifted to a true representation of $SL(2, F)$, called the Weil representation.

As an interlude, we construct the principal series representations as induced representations from the standard Borel subgroup of $GL(2, F)$. This provides motivation to specialise the parameters of the Weil representation in a particular way, so that the representation extends to a representation of $GL(2, F)$. We introduce *cuspidality* as a defining property of the new representations, and use it to prove they are irreducible. Once we verify all our irreducible representations are inequivalent, a simple counting argument shows that we have found all the irreducible representations of $GL(2, F)$.

3.1 The Finite Stone–von Neumann Theorem

Definition 3.1. A group G is TWO-STEP NILPOTENT if $G/Z(G)$ is abelian.

Throughout this section, let H be a finite two-step nilpotent group with centre Z, and let χ_0 be a character of Z. We denote the quotient space H/Z by $\overline{H}$ and the corresponding quotient map by $x \mapsto \overline{x}$.

Proposition 3.2. *There exists a bilinear, skew-symmetric pairing* $\langle\, ,\, \rangle : \overline{H} \times \overline{H} \to \mathbb{C}^\times$ *satisfying* $\langle \overline{x}, \overline{y} \rangle = \chi_0(xyx^{-1}y^{-1})$ *for all* $x, y \in H$*. (*Bilinear *and* skew-symmetric *have the meanings appropriate when considering* $\overline{H}$ *and* $\mathbb{C}^\times$ *as* $\mathbb{Z}$*-modules).*

Proof. First observe that $\overline{xyx^{-1}y^{-1}} = \overline{x}\,\overline{y}\,\overline{x}^{-1}\overline{y}^{-1} = 1$ as $\overline{H}$ is abelian, so $xyx^{-1}y^{-1} \in Z$ for all $x, y \in H$. For any $z \in Z$ we have $\chi_0(xzy(xz)^{-1}y^{-1}) = \chi_0(xyx^{-1}y^{-1}zz^{-1}) =$

$\chi_0(xyx^{-1}y^{-1})$, and similarly we see $\chi_0(xyzx^{-1}(yz)^{-1}) = \chi_0(xyx^{-1}y^{-1})$, so the map $(\overline{x}, \overline{y}) \mapsto \chi_0(xyx^{-1}y^{-1})$ is well-defined. Now

$$\begin{aligned}\langle \overline{x_1 x_2}, \overline{y} \rangle &= \chi_0\big(x_1 x_2 y x_2^{-1} x_1^{-1} y^{-1}\big) \\ &= \chi_0\big(x_1 (x_2 y x_2^{-1}) x_1^{-1} (x_2 y x_2^{-1})^{-1} \cdot x_2 y x_2^{-1} y^{-1}\big) \\ &= \big\langle \overline{x_1}, \overline{x_2 y x_2^{-1}} \big\rangle \langle \overline{x_2}, \overline{y} \rangle \\ &= \langle \overline{x_1}, \overline{y} \rangle \langle \overline{x_2}, \overline{y} \rangle \qquad \text{as } x_2 y x_2^{-1} y^{-1} \in Z.\end{aligned}$$

and similarly $\langle \overline{x}, \overline{y_1 y_2} \rangle = \langle \overline{x}, \overline{y_1} \rangle \langle \overline{x}, \overline{y_2} \rangle$, so $\langle\ ,\ \rangle$ is bilinear. Obviously $\langle \overline{x}, \overline{x} \rangle = 1$, which implies that

$$1 = \langle \overline{xy}, \overline{xy} \rangle = \langle \overline{x}, \overline{x} \rangle \langle \overline{x}, \overline{y} \rangle \langle \overline{y}, \overline{x} \rangle \langle \overline{y}, \overline{y} \rangle = \langle \overline{x}, \overline{y} \rangle \langle \overline{y}, \overline{x} \rangle$$

and so it is skew-symmetric. □

Henceforth we assume that χ_0 is GENERIC (with respect to H), which means that the map $\overline{H} \to \mathbb{C}^\times : \overline{x} \mapsto \langle \overline{x}, \overline{y} \rangle$ is nontrivial for every $\overline{y} \neq \overline{1}$ in $\overline{H}$.

Remark 3.3. A standard result in the character theory of finite groups states that a finite abelian group of order n has n distinct characters. Since χ_0 is generic, the characters $\overline{x} \mapsto \langle \overline{x}, \overline{y} \rangle$ are distinct, and hence these are all the characters of $\overline{H}$.

Definition 3.4. With H and χ_0 as above, a subgroup $\overline{A}$ of $\overline{H}$ is said to be ISOTROPIC if $\langle \overline{x}, \overline{y} \rangle = 1$ for all $\overline{x}, \overline{y} \in \overline{A}$, and is said to be POLARISING if, for each $\overline{x} \in \overline{H}$, $\langle \overline{x}, \overline{y} \rangle = 1$ for all $\overline{y} \in \overline{A}$ if and only if $\overline{x} \in \overline{A}$.

If we define $\overline{A}^\perp = \big\{ \overline{x} \in \overline{H} \mid \langle \overline{x}, \overline{y} \rangle = 1 \text{ for all } \overline{y} \in \overline{A} \big\}$, then $\overline{A}$ is polarising if and only if $\overline{A} = \overline{A}^\perp$. Polarising subgroups are clearly also isotropic.

Proposition 3.5. *Maximal isotropic subgroups are polarising.*

Proof. Let $\overline{A}$ be a maximal isotropic subgroup of $\overline{H}$, and take any $\overline{x} \in \overline{H}$ such that $\langle \overline{x}, \overline{y} \rangle = 1$ for all $\overline{y} \in \overline{A}$. Define $\overline{B} = \langle \overline{x} \rangle$ and consider $\overline{B}\,\overline{A}$, which is a subgroup of $\overline{H}$ since $\overline{H}$ is abelian. Any element of $\overline{B}\,\overline{A}$ can be written as $\overline{x}^k \overline{y}$ for some k and some $\overline{y} \in \overline{A}$, so the pairing applied to any two elements of $\overline{B}\,\overline{A}$ evaluates to

$$\big\langle \overline{x}^k \overline{y}, \overline{x}^l \overline{z} \big\rangle = \langle \overline{x}, \overline{x} \rangle^{kl} \langle \overline{x}, \overline{z} \rangle^k \langle \overline{y}, \overline{x} \rangle^l \langle \overline{y}, \overline{z} \rangle = 1^{kl} 1^k 1^l 1 = 1$$

as $\overline{x} \in \overline{A}^\perp$. Thus $\overline{B}\,\overline{A}$ is isotropic and contains $\overline{A}$; by assumption we obtain $\overline{B}\,\overline{A} = \overline{B}$. It follows that $\overline{x} \in \overline{A}$, so $\overline{A}$ is polarising. □

Lemma 3.6. *Let $F \leq G$ be finite abelian groups, and let χ be a character of F. Then there exist exactly $[G : F]$ characters of G which extend χ.*

Proof. We proceed by induction on $[G : F]$. If $[G : F] = 1$, then $G = F$, so the only character of G extending χ is χ itself.

Now suppose the result is true for all pairs $F \leq G$ with $[G : F] < n$, and take a pair $F \leq G$ with $[G : F] = n$. Take any $x \in G \backslash F$ and let $X = \langle x \rangle$. Let $k = \min\{i \mid x^i \in F\}$, let r be any kth root of $\chi(x^k)$ in $\mathbb{C}$, and tentatively define a homomorphism

$$\psi : FX \to \mathbb{C}^\times : fx^i \mapsto \chi(f) r^i.$$

To check that this is well-defined, note that if $f_1 x^{i_1} = f_2 x^{i_2}$ for $f_1, f_2 \in F$ and $i_1, i_2 \in \mathbb{Z}$ then $x^{i_1 - i_2} = f_1^{-1} f_2 \in F$, so $i_1 - i_2 = mk$ for some $m \in \mathbb{Z}$. It follows that $\chi(f_1^{-1} f_2) = \chi(x^{i_1 - i_2}) = \chi(x^{mk}) = \chi(x^k)^m = r^{km} = r^{i_1 - i_2}$ and hence that $\chi(f_1) r^{i_1} = \chi(x_2) r^{x_2}$, so ψ is well-defined, and it is clearly a character of FX. On the other hand, every character of FX extending χ is of the form $fx^i \mapsto \chi(f)\psi(x)^i$ where $\psi(x)^i$ is a kth root of $\chi(x^k)$. It follows that there are exactly k characters of FX extending χ, one for each choice of r as a kth root of $\chi(x^k)$.

Since $[G : FX] < [G : F]$, each character ψ of FX extending χ can be extended to a character of G in exactly $[G : FX]$ ways, by the inductive assumption. We also observe that every character of G extending χ restricts to a character of FX extending χ, so there are exactly $k[G : FX]$ characters of G extending χ. Finally, note that $[FX : F] = k$, since every coset of F in FX is of the form Fx^i, and $Fx^{i_1} = Fx^{i_2}$ if and only if $k \mid i_1 - i_2$. We conclude that the number of extensions of χ to G is $[G : FX][FX : F] = [G : F]$. □

Proposition 3.7. *Let $\overline{B} \leq \overline{A}$ be isotropic subgroups of $\overline{H}$, let A and B be their preimages in H, and let χ_B be a character of B extending χ_0. Then χ_B can be extended to a character of A in exactly $[A : B]$ ways. In particular, since Z is itself isotropic, χ_0 can be extended to a character of A.*

Proof. Let $Z_0 = \ker \chi_0$. Since $\overline{A}$ is isotropic, $\chi_0(xyx^{-1}y^{-1}) = 1$ for all $x, y \in A$. In particular, if $a \in A$ and $z \in Z_0$ then $aza^{-1} = (aza^{-1}z^{-1})z \in Z$ and $\chi_0(aza^{-1}) = \chi_0(z) = 1$, so $aza^{-1} \in Z_0$. So Z_0 is normal in A and A/Z_0 is abelian.

Consider the character χ'_B of B/Z_0 defined by $\chi'_B(bZ_0) = \chi_B(b)$ for all $b \in B$; this is a well-defined character since $Z_0 \leq \ker \chi_B$. We can extend this to a character ψ of A/Z_0 in $[A/Z_0 : B/Z_0] = [A : B]$ ways by Lemma 3.6 and thence derive a character $\varphi = \psi \circ q$ of A, where $q : A \to A/Z_0$ is the quotient map. Since every character of A extending χ_0 must factor through q, we are done. □

Lemma 3.8. *Let $\overline{A}$ be a subgroup of $\overline{H}$. Then $\overline{A}^{\perp\perp} = \overline{A}$.*

Proof. Clearly $\langle \overline{a}, \overline{x} \rangle = 1$ for all $\overline{a} \in \overline{A}$ and $\overline{x} \in \overline{A}^\perp$ and so $\overline{A} \subseteq \overline{A}^{\perp\perp}$. To prove the reverse inclusion, let $\overline{b} \notin \overline{A}$ and let χ be the trivial character on $\overline{A}$. Let $k = \min\{i \mid \overline{b}^i \in \overline{A}\}$ and let r be any kth root of 1 except 1 itself. By the same argument as in Lemma 3.6 there exists a character χ' of $\overline{H}$ which is trivial on $\overline{A}$ and satisfies $\chi'(\overline{b}) = r$. Since χ_0 is generic, χ' is given by $\overline{x} \mapsto \langle \overline{x}, \overline{y} \rangle$ for some $\overline{y} \in \overline{H}$. We then have $\overline{y} \in \overline{A}^\perp$ but $\langle \overline{b}, \overline{y} \rangle = r \neq 1$, so $\overline{b} \notin \overline{A}^{\perp\perp}$ as we wanted. □

Proposition 3.9. *Let $\overline{A}$ and $\overline{B}$ be polarising subgroups of $\overline{H}$ and let A and B be their preimages in H. Let χ_A and χ_B be characters of A and B extending χ_0, and let π_A and π_B be the representations of H induced from χ_A and χ_B respectively. Then* $\dim \operatorname{Hom}_H(\pi_A, \pi_B) = 1$.

Proof. For all $a \in A$ and $h \in H$, $hah^{-1} = hah^{-1}a^{-1}a \in A$, since $hah^{-1}a^{-1} \in Z \subseteq A$. In other words, A is normal in H. Similarly, B is normal in H. Hence AB is also a normal subgroup of H.

From each double coset $Bx_iA \in B\backslash H/A$ choose a representative x_i, and define the character $\chi_{A,i} : A \cap B \to \mathbb{C}$ by $\chi_{A,i}(s) = \chi_A\big(x_i^{-1}sx_i\big)$ for all $s \in A \cap B$. We wish to apply Theorem 2.10 to find the dimension of $\operatorname{Hom}_H(\pi_A, \pi_B)$. To do so we will need to consider the intertwiners from $\chi_{A,i}$ to $\chi_B|_{A\cap B}$ for each i.

Since $\chi_{A,i}$ and $\chi_B|_{A\cap B}$ are characters of $A \cap B$, a nonzero intertwiner T from $\chi_{A,i}$ to $\chi_B|_{A\cap B}$ is simply a nonzero linear map $\mathbb{C} \to \mathbb{C}$ such that $T \circ \chi_{A,i}(s) = \chi_B(s) \circ T$ for all $s \in A \cap B$. Since scalar transformations commute, a nonzero intertwiner exists if and only if $\chi_{A,i}(s) = \chi_B(s)$ for all $s \in A \cap B$. Since $\chi_{A,i}(s)\chi_A(s)^{-1} = \chi_A\big(x_i^{-1}sx_is^{-1}\big) = \big\langle \overline{x_i}^{-1}, \overline{s}\big\rangle = \big\langle \overline{s}, \overline{x_i}\big\rangle$, a nonzero intertwiner exists if and only if

$$\chi_B(s)\chi_A^{-1}(s) = \big\langle \overline{s}, \overline{x_i}\big\rangle \tag{3.1}$$

for all $s \in A \cap B$.

Since χ_A and χ_B agree on Z, $\chi_B\chi_A^{-1}$ factorises to a character of $\overline{A \cap B}$. This can be extended to a character of $\overline{H}$ by Lemma 3.6, which can in turn be written as $\overline{y} \mapsto \langle \overline{y}, \overline{x_k}\rangle$ for some $x_k \in H$ (which we take as the representative of the kth double coset) by Remark 3.3. For that choice of x_k, (3.1) is satisfied, and hence $\dim \operatorname{Hom}_{A\cap B}(\chi_{A,k}, \chi_B) = 1$.

If there is some other coset Ax_jB with a nonzero intertwiner $\chi_{A,j}^H \to \chi_B^H$, then by the same argument $(\chi_B\chi_A^{-1})(s) = \langle \overline{s}, \overline{x_j}\rangle$ and hence $\big\langle \overline{s}, \overline{x_kx_j^{-1}}\big\rangle = 1$ for all $s \in A \cap B$. Thus $\overline{x_kx_j^{-1}} \in (\overline{A \cap B})^\perp = (\overline{A} \cap \overline{B})^\perp$. Note that if $\overline{x} \in \overline{A} \cap \overline{B}$ then $\langle \overline{x}, \overline{a}\overline{b}\rangle = \langle \overline{x}, \overline{a}\rangle\langle \overline{x}, \overline{b}\rangle = 1$ for all $\overline{a} \in \overline{A}$ and $\overline{b} \in \overline{B}$, so $\overline{x} \in (\overline{AB})^\perp$. So $\overline{x_kx_j^{-1}} \in (\overline{AB})^{\perp\perp} = \overline{AB}$ by Lemma 3.8, which means that $Ax_kB = x_kAB = x_jAB = Ax_jB$. In other words, for all double cosets Ax_jB other than Ax_kB we have $\dim \operatorname{Hom}_{A\cap B}(\chi_{A,j}, \chi_B) = 0$.

It is now clear from Theorem 2.10 that

$$\dim \operatorname{Hom}_H(\pi_A, \pi_B) = \sum_i \dim \operatorname{Hom}_{A\cap B}(\chi_{A,i}, \chi_B) = 1. \qquad \square$$

Before we state the main theorem, we need the following definition.

Definition 3.10. If π is an irreducible representation of a finite group G and $z \in Z(G)$, then $\pi(z)\pi(g) = \pi(g)\pi(z)$ for all $g \in G$, so $\pi(z)$ is an intertwiner from π to π. By Schur's Lemma, $\pi(z)$ is a scalar multiple, $\chi(z)$ say, of the identity map, and $\chi : Z(G) \to \mathbb{C}^\times$ is clearly a character. It is called the CENTRAL CHARACTER of G.

We restate the standing assumptions for the sake of completeness.

Theorem 3.11 (The Finite Stone–von Neumann Theorem)**.** *Let H be a finite two-step nilpotent group with centre Z, and let χ_0 be a generic character of Z. There exists an irreducible representation π of H with central character χ_0, and π is unique up to equivalence. It may be constructed as the representation π_A induced from the character χ_A, where $\overline{A}$ is any polarising subgroup of $\overline{H}$, A is its preimage in H and χ_A is any extension of χ_0 to A.*

Proof. That π_A is irreducible follows from Proposition 3.9 and Corollary 2.6, taking $A = B$ and $\chi_A = \chi_B$. In addition, any different choice of polarising subgroup $\overline{A'} \leq \overline{H}$ (with preimage A' in H) and character $\chi_{A'}$ of A' extending χ_0 yields an irreducible representation $\pi_{A'}$ of H induced from $\chi_{A'}$, and $\pi_{A'}$ is equivalent to π_A by Proposition 3.9 and Schur's Lemma.

Let π be any irreducible representation of H with central character χ_0 and let

$$B = \big\{ b \in H \;\big|\; \pi(b) \text{ is a scalar transformation} \big\},$$

observing that $Z \leq B$. For $b_1, b_2 \in B$ we have

$$\langle \overline{b_1}, \overline{b_2} \rangle = \chi_0(b_1 b_2 b_1^{-1} b_2^{-1}) = \pi(b_1)\pi(b_2)\pi(b_1)^{-1}\pi(b_2)^{-1} = 1$$

as these transformations are scalar, so B is isotropic. There is an obvious character χ_B of B given by $\chi_B(b)v = \pi(b)v$ for all v in the space of π. By Proposition 3.5, we can let $\overline{A'}$ be a polarising subgroup of $\overline{H}$ containing $\overline{B}$, with preimage A' in H. By Proposition 3.7, we can let $\chi_1, \ldots, \chi_l$ be distinct characters of A' extending χ_B, where $l = [A' : B]$. Then the induced representations $\pi_i = \chi_i^H$ are irreducible for all i, by Proposition 3.9. It will suffice to prove that π is equivalent to π_i, for some i.

Denote by V the space of π, and denote by V_i the space of π_i for all i.

We then fix any nonzero $v_0 \in V$ and, for each i, define the map $T_i : V_i \to V$ by

$$T_i(f) = \sum_{h \in H} f(h)\pi(h^{-1})(v_0).$$

For all i and all $g \in H$,

$$\begin{aligned}\big(T_i \circ \pi_i(g)\big)(f) &= \sum_{h \in H} \big(\pi_i(g)f\big)(h)\pi(h^{-1})(v_0) \\ &= \sum_{h \in H} f(hg)\pi(h^{-1})(v_0),\end{aligned}$$

whereas

$$\begin{aligned}\big(\pi(g) \circ T_i\big)(f) &= \pi(g)\left(\sum_{h \in H} f(h)\pi(h^{-1})(v_0)\right) \\ &= \sum_{h \in H} f(h)\pi(gh^{-1})(v_0) \\ &= \sum_{h \in H} f(hg)\pi(h^{-1})(v_0),\end{aligned}$$

which shows that T_i is an intertwiner from π_i to π. Since π_i (for all i) and π are irreducible, it will suffice to show that T_{i_0} is nonzero for some i_0, as this will show π is equivalent to π_{i_0} and hence (by Proposition 3.9) to π_i for all i.

Suppose by way of contradiction that T_i is zero for all i. Fix a set of coset representatives $a_1, \dots, a_l$ for B in A'. For each i we take the function $f_i : H \to \mathbb{C}$ given by

$$f_i(h) = \begin{cases} \chi_i(h) & \text{if } h \in A' \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f_i \in V_i$. Now

$$T_i(f_i) = \sum_{a \in A'} \chi_i(a)\pi(a^{-1})(v_0),$$

and since

$$\chi_i(ba)\pi\big((ba)^{-1}\big)(v_0) = \chi_B(b)\chi_i(a)\chi_B(b^{-1})\pi(a^{-1})(v_0) = \chi_i(a)\pi(a^{-1})(v_0)$$

for all $a \in A'$ and $b \in B$, it follows that

$$0 = T_i(f_i) = |B| \sum_{j=1}^{l} \chi_i(a_j)\pi(a_j^{-1})(v_0) \tag{3.2}$$

for $i = 1, \dots, l$.

By Schur's orthogonality relations for characters,

$$\delta_{i_1,i_2}|A'| = \sum_{a \in A'} \chi_{i_1}(a)\overline{\chi_{i_2}(a)} \tag{3.3}$$

for all i_1, i_2. But if $a \in A'$ and $b \in B$ then

$$\chi_{i_1}(ba)\overline{\chi_{i_2}(ba)} = \chi_B(b)\chi_{i_1}(a)\overline{\chi_B(b)\chi_{i_2}(a)} = \chi_{i_1}(a)\overline{\chi_{i_2}(a)} \qquad \text{as } |\chi_B(b)| = 1,$$

which shows that the summand in (3.3) is constant on each coset of B in A'. Thus

$$\delta_{i_1,i_2}|A'| = |B| \sum_{j=1}^{l} \chi_{i_1}(a_j)\overline{\chi_{i_2}(a_j)}$$

which shows that the vectors $\big(\chi_i(a_1), \dots, \chi_i(a_l)\big)$ are orthogonal for $i = 1, \dots, l$. Thus the matrix $\big(\chi_i(a_j)\big)_{i,j=1}^{l}$ is $\sqrt{[A' : B]}$ times an orthogonal matrix, and is thus invertible. Multiplying the set of equations (3.2) by the inverse matrix, we obtain $\pi(a_j^{-1})(v_0) = 0$ for all j, which is impossible as $v_0 \neq 0$. □

Corollary 3.12. *Let G be a group of automorphisms of H which fix the elements of Z. Let (π, V) be an irreducible representation of H with central character χ_0. For each $g \in G$ there exists a nonzero linear map $\eta(g) : V \to V$, unique up to scalar multiples, such that*

$$\pi(g \cdot h)\eta(g) = \eta(g)\pi(h) \tag{3.4}$$

for all $h \in H$. The maps $\eta(g)$ are then invertible and their images in $PGL(V)$ form a projective representation ω of G.

Proof. Since G acts on H by automorphisms, $h \mapsto \pi(g \cdot h)$ is another irreducible representation of H on V for each $g \in G$. By Schur's lemma, there exists an intertwiner $\eta(g) : V \to V$, unique up to scalar multiples, such that (3.4) holds, and since the two representations are irreducible, $\eta(g)$ is invertible for all $g \in G$. For all $h \in H$ and $g_1, g_2 \in G$ we have

$$\eta(g_1)\eta(g_2)\pi(h) = \eta(g_1)\pi(g_2 \cdot h)\eta(g_2) = \pi\big(g_1 g_2 \cdot h\big)\eta(g_1)\eta(g_2),$$

so by the uniqueness of $\eta(g_1 g_2)$ we have $\eta(g_1)\eta(g_2)\mathbb{C}^\times = \eta(g_1 g_2)\mathbb{C}^\times$. This shows that the map $\omega : G \to PGL(V)$ with $\omega(g) := q\big(\eta(g)\big)$ (where $q : GL(V) \to PGL(V)$ is the quotient map) is a projective representation. □

3.2 A projective representation of $SL(2,F)$

Let F be a finite field of odd order q, V a vector space over F of finite dimension d, and B a nondegenerate symmetric bilinear form on V. Let H be the corresponding Heisenberg group with centre $Z = \{(0,0,x) \mid x \in F\}$. Let ψ be a fixed nontrivial additive character of F, and define the character χ_0 of Z by $\chi_0(0,0,x) = \psi(x)$. Let A be the subgroup $\{(u,0,x) \mid u \in V, x \in F\}$.

Proposition 3.13. *H is two-step nilpotent, χ_0 is generic and $\overline{A}$ is polarising.*

Proof. Each left coset of Z in H looks like $(u,v,x)Z = (u,v,F)$ with multiplication $(u,v,F)(u',v',F) = (u+u', v+v', F)$, so H/Z is isomorphic to the direct product $V \times V$ which is abelian. Thus H is two-step nilpotent.

Since ψ is nontrivial, there exists $y \in F$ with $\psi(y) \neq 1$.

It is useful to calculate $\langle (u,v,F), (u',v',F) \rangle$ for $u, v, u', v' \in V$. In fact,

$$\begin{aligned}\langle (u,v,F), (u',v',F) \rangle &= \chi_0\big((u,v,0)(u',v',0)(-u,-v,0)(-u',-v',0)\big) \\ &= \chi_0\big((u+u', v+v', x+x'+B(u,v')-B(v,u')) \\ &\qquad (-u-u', -v-v', -x-x'+B(u,v')-B(v,u'))\big) \\ &= \chi_0\big(0,0,2B(u,v')-2B(v,u')\big) \\ &= \psi\big(2B(u,v')-2B(v,u')\big),\end{aligned}$$

since the extra terms involving $u+u'$ and $v+v'$ cancel out.

Suppose there is some $(u,v,F) \in H/Z$ with $\langle (u,v,F), (u',v',F) \rangle = 1$ for all $(u',v',F) \in H/Z$. Then we have $\psi\big(2B(u,v') - 2B(v,u')\big) = 1$ for all $u', v' \in V$. If $u \neq 0$, we can find $w \in V$ with $B(u,w) \neq 0$ since B is nondegenerate. Then it follows that $\psi\big(2B\big(u, \frac{y}{2B(u,w)}w\big) - 2B(v,0)\big) = \psi(y) \neq 1$, a contradiction. Thus $u = 0$ and similarly $v = 0$. So χ_0 is generic.

Finally, for any elements $(u,0,x), (u',0,x') \in A$, we have $\langle (u,0,F), (u',0,F) \rangle = \psi\big(2B(u,0) - 2B(0,u')\big) = 1$, so $\overline{A} \subseteq \overline{A}^\perp$. Any element not in A is of the form (u,v,x) for $v \neq 0$; for that element, there exists $w \in V$ with $B(v,w) \neq 1$, so

$\langle (u,v,F), (-\frac{y}{2B(v,w)}w, 0, F) \rangle = \psi(y) \neq 1$, so $(u,v,F) \notin \overline{A}^{\perp}$. Thus $\overline{A} = \overline{A}^{\perp}$, which means that $\overline{A}$ is polarising. □

Theorem 3.11 shows us that there is a unique irreducible representation of H with central character χ_0, and that it can be modelled as the representation of H induced from the character $\chi_A : (u, 0, x) \mapsto \psi(x)$ of A. That is, we consider the space V_ρ of functions $\varphi : H \to \mathbb{C}$ satisfying $\varphi(ah) = \chi_A(a)\varphi(h)$ for $a \in A$, $h \in H$, and the representation ρ of H on V_ρ given by $\big(\rho(g)\varphi\big)(h) = \varphi(hg)$.

For notational convenience, we will use an equivalent representation motivated by the observation that $\{(0, v, 0) \mid v \in V\}$ is a set of right coset representatives for A in H. Let W be the space of all functions $\Phi : V \to \mathbb{C}$ and let $T : W \to V_\rho$ be the map satisfying $(T\Phi)(0, v, 0) = \Phi(v)$ for all $\Phi \in W$ and $v \in V$; note that the values of $T\Phi$ on the coset $A(0, v, 0)$ are completely determined by $(T\Phi)(0, v, 0)$. T is easily seen to be a vector space isomorphism, so we can define the representation π of H on W by $\pi(g) = T^{-1} \circ \rho(g) \circ T$. Then π is by definition equivalent to ρ. We want to forget about ρ and V_ρ, so we need to calculate the action of π on V explicitly. If $v \in V$ and $g = (u', v', x') \in H$ then

$$\begin{aligned}
\big(\pi(g)\Phi\big)(v) &= T^{-1}\big(\rho(g)(T\Phi)\big)(v) \\
&= \big(\rho(g)(T\Phi)\big)(0, v, 0) \\
&= (T\Phi)\big((0, v, 0)(u', v', x')\big) \\
&= (T\Phi)\big(u', v + v', x' - B(v, u')\big) \\
&= (T\Phi)\big((u', 0, x' - 2B(v, u') - B(u', v'))(0, v + v', 0)\big) \\
&= \psi\big(x' - B(2v + v', u')\big)\Phi(v + v'). \qquad (3.5)
\end{aligned}$$

We can now use Proposition 2.13 and Corollary 3.12 to obtain a projective representation of $SL(2, F)$ on W. Once again, we want to find explicit formulas for this representation, but this time there is a certain amount of guesswork involved, because (3.4) makes it easy to check whether η is correct but difficult to find η in the first place. It will suffice to find the action of a set of generators of $SL(2, F)$.

We first consider the action of $g = \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)$ on W. By (3.5),

$$\begin{aligned}
\big(\pi(u', v', x')\Phi\big)(v) &= \psi\big(x' - B(2v + v', u')\big)\Phi(v + v'), \\
\text{and} \quad \big(\pi(u' + bv', v', x')\Phi\big)(v) &= \psi\big(x' - B(2v + v', u' + bv')\big)\Phi(v + v') \\
&= \psi\big(x' - B(2v + v', u')\big) \\
&\quad \cdot \psi\big(bB(v, v) - bB(v + v', v + v')\big)\Phi(v + v'). \qquad (3.6)
\end{aligned}$$

This last equation strongly suggests an intertwiner from $\pi(u', v', x')$ to $\pi(u' + bv', v', x')$ must involve some factor like $\psi\big(bB(v, v)\big)$. If we make the guess $\big(\eta\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)\Phi\big)(v) = \psi(bB(v, v))\Phi(v)$ for $v \in V$, then the calculation (3.6) shows that (3.4) holds.

Now we consider $g = \left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right)$. By (3.5),

$$\begin{aligned} \big(\pi(u',v',x')\Phi\big)(v) &= \psi\big(x' - B(2v+v',u')\big)\Phi(v+v'), \\ \text{and} \quad \big(\pi(au',a^{-1}v',x')\Phi\big)(v) &= \psi\big(x' - B(2v+a^{-1}v',au')\big)\Phi(v+a^{-1}v') \\ &= \psi\big(x' - 2B(av,u') - B(v',u')\big)\Phi(v+a^{-1}v'). \end{aligned}$$

The term $-2B(av,u')$ suggests an intertwiner involving the replacement of v with av. If we make the guess $\big(\eta\left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right)\Phi\big)(v) = L_a\Phi(av)$ for $v \in V$, then (3.4) holds. (We include the arbitrary constant $L_a \neq 0$, to be chosen later, to make it easier to lift to a true representation.)

Finally we consider $g = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$. By (3.5),

$$\begin{aligned} \big(\pi(u',v',x')\Phi\big)(v) &= \psi\big(x' - 2B(v,u') - B(v',u')\big)\Phi(v+v'), \\ \text{and} \quad \big(\pi(v',-u',x')\Phi\big)(v) &= \psi\big(x' - 2B(v,v') + B(u',v')\big)\Phi(v-u'). \end{aligned}$$

Comparison of the terms within the ψ-factor is fruitless this time. However, we have the luxury of V being finite, so we consider the most general linear transformation and deduce the correct intertwiner. Let

$$\left(\eta\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\Phi\right)(v) = \sum_{u\in V} f(v,u)\Phi(u)$$

for some coefficients $f(v,u) \in \mathbb{C}$. We want (3.4) to hold, so we compute, remembering that $g = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ and $h = (u',v',x')$:

$$\begin{aligned} \Big(\pi(g\cdot h)\big(\eta(g)\Phi\big)\Big)(v) &= \psi\big(x' - 2B(v,v') + B(v',u')\big)\big(\eta(g)\Phi\big)(v-u') \\ &= \psi\big(x' - 2B(v,v') + B(v',u')\big)\sum_{u\in V} f(v-u',u)\Phi(u) \\ &= \sum_{u\in V}\psi\big(x' - 2B(v,v') + B(v',u')\big)f(v-u',u)\Phi(u), \end{aligned}$$

whereas

$$\begin{aligned} \Big(\eta(g)\big(\pi(h)\Phi\big)\Big)(v) &= \sum_{u\in V} f(v,u)\big(\pi(u',v',x')\Phi\big)(u) \\ &= \sum_{u\in V} f(v,u)\psi\big(x' - 2B(u,u') - B(u',v')\big)\Phi(u+v') \\ &= \sum_{u\in V} f(v,u-v')\psi\big(x' - 2B(u-v',u') - B(u',v')\big)\Phi(u) \\ &= \sum_{u\in V} f(v,u-v')\psi\big(x' - 2B(u,u') + B(u',v')\big)\Phi(u). \end{aligned}$$

Thus it will suffice that

$$\psi(-2B(v,v'))f(v-u',u) = \psi(-2B(u,u'))f(v,u-v')$$

for all $u, v, u', v' \in V$, and the obvious choice satisfying this equation is $f(u,v) := \psi(2B(u,v))$. For $\Phi \in W$ we define the FOURIER TRANSFORM $\widehat{\Phi}$ of Φ by

$$\widehat{\Phi}(v) = \sum_{u \in V} \psi(2B(u,v))\Phi(u)$$

for all $v \in V$. Then the map $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) : W \to W$ defined by $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)\Phi = K\widehat{\Phi}$ satisfies (3.4), where $K \neq 0$ is a constant yet to be determined.

3.3 A true representation of $SL(2,F)$

By Corollary 3.12, we have now found a projective representation ω of $SL(2,F)$ on V. We wish to lift this to a true representation of $SL(2,F)$. It will be enough to verify that η preserves the relations in Proposition 2.15, as that will show η is a true representation.

For neatness we abbreviate our notation by writing $t(a) \cdot \Phi$ and $n(b) \cdot \Phi$ instead of $\eta\big(\alpha(t(a))\big)\Phi$ and $\eta\big(\alpha(n(b))\big)\Phi$, and so on; we need to check this dot action satisfies the relations. The first three relations are easily checked, if we impose the condition that $L_{a_1}L_{a_2} = L_{a_1a_2}$ for $a_1, a_2 \in F^\times$:

$$\begin{aligned}
\big(t(a_1) \cdot t(a_2) \cdot \Phi\big)(v) &= L_{a_1}\big(t(a_2) \cdot \Phi\big)(a_1 v) \\
&= L_{a_1}L_{a_2}\Phi(a_2a_1v) = \big(t(a_1a_2) \cdot \Phi\big)(v); \\
\big(n(b_1) \cdot n(b_2) \cdot \Phi\big)(v) &= \psi\big(b_1B(v,v)\big)\big(n(b_2) \cdot \Phi\big)(v) \\
&= \psi\big((b_1+b_2)B(v,v)\big)\Phi(v) = \big(n(b_1+b_2) \cdot \Phi\big)(v); \\
\text{and} \quad \big(t(a) \cdot n(b) \cdot \Phi\big)(v) &= L_a\big(n(b) \cdot \Phi\big)(av) = L_a\psi\big(bB(av,av)\big)\Phi(av) \\
&= \psi\big(a^2bB(v,v)\big)\big(t(a) \cdot \Phi\big)(v) = \big(n(a^2b) \cdot t(a) \cdot \Phi\big)(v).
\end{aligned}$$

To verify the fourth relation, we calculate:

$$\begin{aligned}
\big(w \cdot t(a) \cdot w \cdot \Phi\big)(v) &= K\sum_{u \in V}\psi\big(2B(u,v)\big)\big(t(a) \cdot w \cdot \Phi\big)(v) \\
&= K\sum_{u \in V}\psi\big(2B(u,v)\big)L_a\left(K\sum_{w \in V}\psi\big(2B(w,au)\big)\Phi(w)\right) \\
&= L_aK^2\sum_{u,w \in V}\psi\big(2B(u,v+aw)\big)\Phi(w) \\
&= L_aK^2\sum_{w \in V}\left(\sum_{u \in V}\psi\big(2B(u,v+aw)\big)\right)\Phi(w).
\end{aligned}$$

We evaluate this using a standard character trick. Since ψ is nontrivial, there is some $f \in F$ with $\psi(f) \neq 1$. Since B is nondegenerate, if $v + aw \neq 0$ there exists some $u_0 \in V$ with $b_0 := B(u_0, v+aw) \neq 0$. If we let $u_1 := u_0 f/2b_0$ then $\psi\big(2B(u_1, v+aw)\big) = \psi(f) \neq 1$, and so

$$\sum_{u \in V}\psi\big(2B(u,v+aw)\big) = \sum_{u \in V}\psi\big(2B(u+u_1,v+aw)\big) = \psi(f)\sum_{u \in V}\psi\big(2B(u,v+aw)\big),$$

which shows that the sum is zero. On the other hand, if $v+aw=0$ the above sum is obviously $|V|=q^d$. Thus

$$\big(w\cdot t(a)\cdot w\cdot\Phi\big)(v)=L_aK^2q^d\Phi(-v/a)=L_aL_{-1/a}^{-1}K^2q^d\big(t(-1/a)\cdot\Phi\big)(v).$$

But we have already had to insist that $L_\bullet$ is a character on $F^\times$, so $L_aL_{-1/a}^{-1}=L_{-a^2}=L_{-1}L_{a^2}$. We want the fourth relation to be preserved regardless of the value of a, so we need L_{a^2} to be independent of a. Since $L_\bullet$ is a character, this means we need $L_{a^2}=1$ for all a. This means that $L_\bullet$ is either the quadratic character ξ on $F^\times$ or the trivial character. (Recall that $F^\times$ is cyclic and of even order $q-1$, so the squares form a subgroup of index two, and the quadratic character sends the squares to 1 and the nonsquares to -1.) It then follows, if we want the fourth relation to be preserved, that we need $L_{-1}K^2q^d=1$.

We now check the fifth and final relation, bearing in mind the constraints on K and $L_\bullet$ already determined. We calculate:

$$\begin{aligned}
&\big(n(-1/b)\cdot w\cdot n(-b)\cdot t(-b)\cdot\Phi\big)(v)\\
&\quad=\psi\big(-B(v,v)/b\big)\big(w\cdot n(-b)\cdot t(-b)\cdot\Phi\big)(v)\\
&\quad=K\psi\big(-B(v,v)/b\big)\sum_{u\in V}\psi\big(2B(u,v)\big)\big(n(-b)\cdot t(-b)\cdot\Phi\big)(u)\\
&\quad=KL_{-b}\sum_{u\in V}\psi\big(-B(v,v)/b+2B(u,v)-bB(u,u)\big)\Phi(-bu)\\
&\quad=KL_{-b}\sum_{u\in V}\psi\big(-B(v-bu,v-bu)/b\big)\Phi(-bu)\\
&\quad=KL_{-b}\sum_{u\in V}\psi\big(-B(u+v,u+v)/b\big)\Phi(u),
\end{aligned}$$

whereas

$$\begin{aligned}
&\big(w\cdot n(b)\cdot w\cdot\Phi\big)(v)\\
&\quad=K\sum_{u\in V}\psi\big(2B(u,v)\big)\big(n(b)\cdot w\cdot\Phi\big)(u)\\
&\quad=K\sum_{u\in V}\psi\big(2B(u,v)+bB(u,u)\big)\big(w\cdot\Phi\big)(u)\\
&\quad=K^2\sum_{u,w\in V}\psi\big(2B(u,v)+bB(u,u)+2B(w,u)\big)\Phi(w)\\
&\quad=K^2\sum_{u,w\in V}\begin{pmatrix}\psi\big(bB\big(u+(v+w)/b,u+(v+w)/b\big)\big)\\ \cdot\,\psi\big(-B(v+w,v+w)/b\big)\Phi(w)\end{pmatrix}\\
&\quad=K^2\sum_{w\in V}\left(\sum_{u\in V}\psi\big(bB(u,u)\big)\right)\psi\big(-B(v+w,v+w)/b\big)\Phi(w),
\end{aligned}$$

so the relation is preserved if $K\sum_{u\in V}\psi\big(bB(u,u)\big)=L_{-b}$ for all $b\in F^\times$. We need to evaluate the sum in this expression.

Lemma 3.14 (Diagonalisation of symmetric bilinear forms)**.** *Let F be a field whose characteristic is not 2. Let B be a nondegenerate symmetric bilinear form over a finite-dimensional F-vector space V. Then there exists a basis $b_1, \ldots, b_k$ of V such that $B(b_i, b_j) = 0$ if $i \neq j$ and, for each i, $B(b_i, b_i) \neq 0$.*

Proof. We proceed by induction on $\dim V$. When $\dim V = 0$, the statement is vacuously true (the empty basis works). Now assume the statement for all F-vector spaces of dimension less than d, and take V of dimension d. Since B is nonzero and $B(u,v) = \frac{1}{4}\big(B(u+v, u+v) - B(u-v, u-v)\big)$, there is some $b_1 \in V$ with $B(b_1, b_1) \neq 0$. Let $V' = (Fb_1)^{\perp}$; then $V = Fb_1 \oplus V'$, so $\dim V' = d-1$ and $B|_{V' \times V'}$ is nondegenerate (since if there is $v \in V'$ with $B(v,u) = 0$ for all $u \in V'$, then also $B(v, b_1) = 0$ and so $B(v,u) = 0$ for all $u \in V$, so $v = 0$). The inductive assumption then implies there is a basis $b_2, \ldots, b_d$ of the complement V' so that $b_1, \ldots, b_d$ is a suitable basis of V. □

Let $b_1, \ldots, b_d$ be a basis of V such that $B(b_i, b_j) = f_i \delta_{ij}$ for all i, j, where $f_i \in F^{\times}$. Then

$$\begin{aligned}\sum_{u \in V} \psi\big(bB(u,u)\big) &= \sum_{x_1,\ldots,x_d \in F} \psi\big(b(x_1^2 f_1 + \ldots + x_d^2 f_d)\big)\\ &= \sum_{x_1,\ldots,x_d \in F} \psi(bf_1x_1^2) \ldots \psi(bf_dx_d^2)\\ &= \prod_{i=1}^{d} \sum_{x \in F} \psi(bf_ix^2). \qquad (3.7)\end{aligned}$$

We want to evaluate these sums.

Lemma 3.15. *There exists $\varepsilon_0 \in \{\pm 1, \pm i\}$ such that $\varepsilon_0^2 = \xi(-1)$ and*

$$\sum_{x \in F} \psi(tx^2) = \varepsilon_0 \, \xi(t) \sqrt{q}$$

for all $t \in F^{\times}$.

Proof. First observe that $y \in F^{\times}$ can be written as a square in exactly two ways if $\xi(y) = 1$ and in exactly no ways if $\xi(y) = -1$, and 0 can be written as a square in exactly one way. Thus

$$\sum_{x \in F} \psi(tx^2) = 1 + \sum_{y \in F^{\times}} \big(1 + \xi(y)\big)\psi(ty) = \sum_{y \in F} \psi(ty) + \sum_{y \in F^{\times}} \xi(y)\psi(ty).$$

But $\sum_{y \in F} \psi(ty) = 0$ by the same character trick as before, since $t \neq 0$. We calculate the modulus of the second sum:

$$\begin{aligned}\left|\sum_{y\in F^\times}\xi(y)\psi(ty)\right| &= \sum_{y,z\in F^\times}\xi(y)\psi(ty)\overline{\xi(z)\psi(tz)}\\ &= \sum_{y,z\in F^\times}\xi(y/z)\psi\big(t(y-z)\big)\\ &= \sum_{r,z\in F^\times}\xi(r)\psi\big(tz(r-1)\big) \qquad \text{(letting } r=y/z)\\ &= \sum_{r\in F^\times}\left(\sum_{z\in F^\times}\psi\big(tz(r-1)\big)\right)\xi(r).\end{aligned}$$

A third application of the character trick yields

$$\sum_{x\in F^\times}\psi\big(tx(r-1)\big) = \sum_{x\in F}\psi\big(tx(r-1)\big) - 1 = \begin{cases} q-1 & \text{if } r=1;\\ -1 & \text{otherwise.}\end{cases}$$

Thus

$$\begin{aligned}\left|\sum_{y\in F^\times}\xi(y)\psi(ty)\right| &= (q-1)\xi(1) + \sum_{r\in F^\times\setminus\{1\}}(-1)\xi(r)\\ &= q - \sum_{r\in F^\times}\xi(r) = q\end{aligned}$$

by the character trick, applied this time to ξ. But

$$\begin{aligned}\overline{\sum_{y\in F^\times}\xi(y)\psi(ty)} &= \sum_{y\in F^\times}\xi(y)\psi(-ty) \qquad \text{as } \xi(y)\in\{\pm1\}\\ &= \sum_{y\in F^\times}\xi(-y)\psi(ty)\\ &= \xi(-1)\sum_{y\in F^\times}\xi(y)\psi(ty),\end{aligned}$$

so the sum is real if $\xi(-1)=1$ and purely imaginary if $\xi(-1)=-1$. Pick $\varepsilon_0\in\{\pm1,\pm i\}$ such that $\sum_{y\in F^\times}\xi(y)\psi(y)=\varepsilon_0\sqrt{q}$. Then $\varepsilon_0^2=\xi(-1)$ and

$$\sum_{y\in F^\times}\xi(y)\psi(ty) = \sum_{y\in F^\times}\xi(y/t)\psi(y) = \xi(t)\sum_{y\in F^\times}\xi(y)\psi(y) = \varepsilon_0\,\xi(t)\sqrt{q}$$

for all $t\in F^\times$. This proves the lemma. □

Each factor in the product (3.7) can therefore be written as $\varepsilon_0\,\xi(bf_i)\sqrt{q}$, so we have

$$\sum_{u\in V}\psi\big(bB(u,u)\big) = \prod_{i=1}^{d}\varepsilon_0\,\xi(bf_i)\sqrt{q} = \varepsilon\xi(b)^d q^{d/2}$$

where $\varepsilon = \varepsilon_0^d \xi(f_1 \dots f_d)$.

Returning to the relations that need to be checked, our previous calculations show the relations will be preserved if

$$L_{-1}K^2 q^d = 1 \tag{3.8}$$

$$\text{and} \qquad K\varepsilon\xi(b)^d q^{d/2} = L_{-b} \qquad \text{for all } b \in F^\times. \tag{3.9}$$

Recall that $L_\bullet$ is either the trivial or quadratic character on $F^\times$, and observe that the only dependence of the left hand side of (3.9) on b is in the factor $\xi(b)^d$. Thus, in order for the relations to hold, $L_b\xi(b)^{-d}$ must be a character of $F^\times$ and independent of b, so $L_b = \xi(b)^d$ for all $b \in F^\times$. Then (3.9) gives

$$K = \varepsilon^{-1}L_{-1}q^{-d/2} = \varepsilon^{-1}\xi(-1)^d q^{-d/2},$$

and fortunately then

$$\xi(-1)^d K^2 q^d = \xi(-1)^d \varepsilon^{-2} q^{-d} q^d = 1$$

as $\varepsilon^2 = \varepsilon_0^{2d} = \xi(-1)^d$, so (3.8) holds as well. In other words, η is a true representation of $SL(2,F)$, called the Weil representation for $SL(2,F)$, which is given by the following formulas:

$$\left(\eta\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\Phi\right)(v) = \psi\big(bB(v,v)\big)\Phi(v) \qquad \text{for } b \in F,$$

$$\left(\eta\begin{pmatrix}a & \\ & 1/a\end{pmatrix}\Phi\right)(v) = \xi(a)^d\,\Phi(av) \qquad \text{for } a \in F^\times,$$

$$\text{and} \qquad \left(\eta\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\Phi\right)(v) = \delta q^{-d/2}\sum_{u\in V}\psi\big(2B(u,v)\big)\Phi(u),$$

where $\delta = \varepsilon^{-1}\xi(-1)^d$.

3.4 A motivating example

Before embarking on the general case, we remark on an easily constructed class of representations which help to motivate the general construction. Let T be the subgroup of $GL(2,F)$ consisting of the diagonal matrices, and let B be the so-called Borel subgroup of $GL(2,F)$, consisting of the upper triangular matrices. Incidentally, T stands for *torus*, which means an algebraic group defined over F which is isomorphic over some extension of F to a direct product of copies of $F^\times$ (by analogy with the geometric torus $\mathbb{T}\times\mathbb{T}$). Tori provide an important way to parametrise the irreducible representations of $GL(2,F)$ (see [1], p. 401), but we will not examine this aspect further.

Let χ_1 and χ_2 be characters of $F^\times$ and let $\chi : T \to \mathbb{C}^\times : \left(\begin{smallmatrix}a_1 & \\ & a_2\end{smallmatrix}\right) \mapsto \chi_1(a_1)\chi_2(a_2)$ be the corresponding character of T. Clearly χ extends to a character of B (which we will also call χ) by

$$\chi\begin{pmatrix}a & b\\ & d\end{pmatrix} = \chi_1(a)\chi_2(d). \tag{3.10}$$

We let $\mathcal{B}(\chi_1, \chi_2)$ be the representation of $GL(2, F)$ induced from the character χ of B. Each such representation has dimension $[GL(2, F) : B] = (q^2 - 1)(q^2 - q)/q(q - 1)^2 = q + 1$. Mackey theory again comes in helpful to compute the dimensions of the spaces of intertwiners between these representations.

Lemma 3.16. *Let χ_1, χ_2, μ_1 and μ_2 be characters of $F^\times$. Then*

$$\dim \mathrm{Hom}_{GL(2,F)}\big(\mathcal{B}(\chi_1, \chi_2), \mathcal{B}(\mu_1, \mu_2)\big) = \begin{cases} 0 & \text{if } \{\chi_1, \chi_2\} \neq \{\mu_1, \mu_2\}; \\ 1 & \text{if } \{\chi_1, \chi_2\} = \{\mu_1, \mu_2\},\ \chi_1 \neq \chi_2; \\ 2 & \text{if } \chi_1 = \chi_2 = \mu_1 = \mu_2. \end{cases}$$

Proof. The Bruhat decomposition for $GL(2, F)$ states that $GL(2, F) = B \sqcup B\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)B$; this is true because $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -a/c \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B$ if $c \neq 0$. It will suffice to calculate the dimension of $\mathcal{D}$ as defined in Proposition 2.9.

Let χ and μ be the characters of B obtained by (3.10) from (χ_1, χ_2) and (μ_1, μ_2) respectively.

Take any $\Delta \in \mathcal{D}$. Then Δ takes its values in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$, which we may identify with $\mathbb{C}$ via $\varphi \leftrightarrow \varphi(1)$. Furthermore, Δ is determined by its values on the double coset representatives $\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$, so certainly $\dim \mathcal{D} \leq 2$. By definition of $\mathcal{D}$,

$$\begin{aligned} &\Delta\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) = \mu_1(x)\mu_2(y)\Delta\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)\chi_1(x)^{-1}\chi_2(y)^{-1} \\ \text{and} \qquad &\Delta\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right) = \mu_1(x)\mu_2(y)\Delta\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\chi_1(y)^{-1}\chi_2(x)^{-1} \end{aligned}$$

for all $x, y \in F^\times$. If $\chi_1 \neq \mu_1$, there is some $x \in F^\times$ such that $\chi_1(x) \neq \mu_1(x)$, which shows that $\Delta\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) = 0$. Similarly, if $\chi_2 \neq \mu_2$ then $\Delta\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) = 0$, and if $\chi_1 \neq \mu_2$ or $\chi_2 \neq \mu_1$ then $\Delta\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right) = 0$. In other words, $\dim \mathcal{D} = 0$ unless $\{\chi_1, \chi_2\} = \{\mu_1, \mu_2\}$, and $\dim \mathcal{D} \leq 1$ unless $\chi_1 = \chi_2 = \mu_1 = \mu_2$.

We now show the bounds above are met. If $(\chi_1, \chi_2) = (\mu_1, \mu_2)$ then the function $\Delta_1 : G \to \mathbb{C}$ given by

$$\Delta_1(g) = \begin{cases} \chi(g) & \text{if } g \in B, \\ 0 & \text{otherwise,} \end{cases}$$

is easily seen to be in $\mathcal{D}$. If $(\chi_1, \chi_2) = (\mu_2, \mu_1)$ then we tentatively define $\Delta_2 : G \to \mathbb{C}$ by

$$\Delta_2(g) = \begin{cases} \chi(b)\mu(b') & \text{if } g = b\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)b' \text{ for } b, b' \in B, \\ 0 & \text{otherwise.} \end{cases}$$

This is well-defined because $\chi(b)\mu(b') = 1$ if $b\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)b' = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$ for $b, b' \in B$, and Δ_2 is then clearly an element of $\mathcal{D}$. We have exhibited a nonzero element of $\mathcal{D}$ in the case $\{\chi_1, \chi_2\} = \{\mu_1, \mu_2\}$, $\chi_1 \neq \chi_2$, and two linearly independent elements of $\mathcal{D}$ in the case $\chi_1 = \chi_2 = \mu_1 = \mu_2$, so the lemma is proved. $\square$

Proposition 3.17. *The representations $\mathcal{B}(\chi_1, \chi_2)$ for $\chi_1 \neq \chi_2$ are irreducible and pairwise inequivalent, except that $\mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\chi_2, \chi_1)$. (They are known as the* PRINCIPAL SERIES REPRESENTATIONS.*)*

Proof. This follows from Lemma 3.16 and Corollaries 2.6 and 2.7. □

Proposition 3.18. *Each representation $\mathcal{B}(\chi_1, \chi_1)$ is the direct sum of the character $\widetilde{\chi_1} : g \mapsto \chi_1(\det g)$ of G with an irreducible representation of dimension q (known as a* SPECIAL REPRESENTATION*). These representations are all pairwise inequivalent.*

Proof. We observe that $\mathcal{B}(\chi_1, \chi_1)$ has a subrepresentation equivalent to the given character. To be specific, the function $f : GL(2, F) \to \mathbb{C}$ given by $f(g) = \widetilde{\chi_1}(g) = \chi_1(\det g)$ satisfies $f(bg) = \chi_1(\det b)\chi_1(\det g) = \chi(b)\chi_1(\det g)$ for all $b \in B$ and $g \in GL(2, F)$, where χ is the character of B obtained from (χ_1, χ_1) by (3.10). So f lies in the space of $\mathcal{B}(\chi_1, \chi_1)$, and $\mathbb{C}f$ is clearly an invariant subspace equivalent to $\widetilde{\chi_1}$. It follows from Lemma 3.16 and Corollary 2.6 that $\mathbb{C}f$ has a q-dimensional complement which is irreducible. It then follows from Corollary 2.7 that $\widetilde{\chi_1}$ and the special representation corresponding to χ_1 are inequivalent across choices of χ_1. □

3.5 Extending the representation to $GL(2, F)$

We would like to obtain a representation of $GL(2, F)$ from η. It turns out that the paradigm of section 3.4 is extended in a surprising way: in that section we considered multiplicative characters of the two-dimensional F-algebra of diagonal matrices (known as the *split case*); in general we may also consider characters of the quadratic field extension of F (which is unique up to isomorphism). Let E be either of these two F-algebras. In the split case, rather than dealing with diagonal matrices, we will consider the F-algebra $E = F \oplus F$, which is isomorphic but more convenient to write.

The parameters V and B of section 3.2 need to be specialised. First denote by $x \mapsto \overline{x}$ a nontrivial F-algebra automorphism of E; specifically, we will take $(a, b) \mapsto (b, a)$ in the split case and the (unique) nontrivial Galois automorphism $x \mapsto x^q$ in the case of a field extension. The TRACE and NORM maps on E are then given by $\mathrm{tr} : x \mapsto x + \overline{x}$ and $N : x \mapsto x\overline{x}$ respectively, regarding the images as elements of F (via $(x, x) \mapsto x$ in the split case). We then take $V = E$ (so $d = 2$ and all factors $\xi(\cdot)^d$ disappear) and $B(u, v) = \frac{1}{2}\,\mathrm{tr}(u\overline{v})$, which is clearly a nondegenerate symmetric bilinear form on V. Note that then $B(u, u) = \frac{1}{2}(u\overline{u} + \overline{u}u) = N(u)$. Let $E_1^\times$ be the set of elements of E of norm 1. The following lemma is very important:

Lemma 3.19. *The norm map $N : E \to F$ is surjective.*

Proof. In the split case, $N(a_1, a_2) = a_1a_2$, so $N(a_1, 1) = a_1$ for all $a_1 \in F$, so N is surjective. Otherwise, E is a field extension of F. Zero is clearly a norm, as $N(0) = 0\,\overline{0} = 0$. For the rest, we know $N(x) = x^{q+1}$ for $x \in E^\times$, so $N : E^\times \to F^\times$ is a group homomorphism. Moreover, the equation $x^{q+1} = 1$ can have at most $q + 1$ roots, since E is a field, so $|\ker N| \leq q + 1$. Thus $|N(E^\times)| = [E^\times : \ker N] \geq (q^2 - 1)/(q + 1) = q - 1 = |F^\times|$, so N is surjective. □

We need to find a way to extend the representation η of $SL(2,F)$ to $GL(2,F)$. We have not yet used any characters of $E^\times$, and the representation space of η has dimension q^2, which is rather large. Fix a character χ of $E^\times$ and consider the subspace

$$W(\chi) = \bigl\{\Phi : E \to \mathbb{C} \bigm| \Phi(tx) = \chi(t)^{-1}\Phi(x) \text{ for all } t \in E_1^\times\bigr\}.$$

This subspace is invariant under the action of η. To see this, we take $\Phi \in W(\chi)$ and $t \in E_1^\times$, and check each of the three families of generators in turn (where, as usual, $a \in F^\times$ and $b \in F$):

$$\psi\bigl(bN(tv)\bigr)\Phi(tv) = \psi\bigl(bN(t)N(v)\bigr)\chi(t)^{-1}\Phi(v) = \chi(t)^{-1}\psi\bigl(bN(v)\bigr)\Phi(v);$$
$$\Phi\bigl(a(tv)\bigr) = \Phi(tav) = \chi(t)^{-1}\Phi(av);$$

and

$$\begin{aligned}\sum_{u\in V}\psi\bigl(\mathrm{tr}(u\overline{tv})\bigr)\Phi(u) &= \sum_{u\in V}\psi\bigl(\mathrm{tr}(ut^{-1}\overline{v})\bigr)\Phi(u)\\ &= \sum_{u\in V}\psi\bigl(\mathrm{tr}(u\overline{v})\bigr)\Phi(tu) = \chi(t)^{-1}\sum_{u\in V}\psi\bigl(\mathrm{tr}(u\overline{v})\bigr)\Phi(u).\end{aligned}$$

We want to extend this representation to all of $GL(2,F)$. To do so, we need to understand in what circumstances a representation of a subgroup can be extended to a representation of the whole group. The following lemma gives an extension in circumstances that satisfy our needs:

Lemma 3.20. *Let G be a group. Let H be a subgroup and M a normal subgroup of G such that $MH = G$ and $M \cap H = 1$. Then $G = M \rtimes H$. Suppose in addition that σ is a representation of M and τ a representation of H, both on the space V, and that*

$$\tau(h)\sigma(m)\tau(h)^{-1} = \sigma(hmh^{-1})$$

for all $m \in M$ and $h \in H$. Then the map

$$\pi : G \to GL(V) : mh \mapsto \sigma(m)\tau(h)$$

is a representation of G.

Proof. If $m_1h_1 = m_2h_2$ for $m_1, m_2 \in M$ and $h_1, h_2 \in H$, then $h_1h_2^{-1} = m_1^{-1}m_2$, which equals 1 since $M\cap H = 1$. So every element of G can be written uniquely as mh for $m \in M$, $h \in H$. Then the multiplication is given by $m_1h_1m_2h_2 = m_1(h_1m_2h_1^{-1})h_1h_2$, which shows that G is the semidirect product $M \rtimes H$, where H acts on M by conjugation.

To check π is a representation, we need only check that it is a homomorphism, in other words that

$$\sigma(m_1)\tau(h_1)\sigma(m_2)\tau(h_2) = \sigma\bigl(m_1(h_1m_2h_1^{-1})\bigr)\tau(h_1h_2)$$

for all $m_1, m_2 \in M$ and $h_1, h_2 \in H$. Since σ and τ are homomorphisms, this is equivalent to

$$\tau(h_1)\sigma(m_2)\tau(h_1)^{-1} = \sigma(h_1m_2h_1^{-1}),$$

which is what we have assumed. □

Let $G = GL(2,F)$, $M = SL(2,F)$ and $H = \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \;\middle|\; a \in F^\times \right\}$ in Lemma 3.20. Obviously $M \trianglelefteq G$, $M \cap H = 1$ and $MH = G$ by the simple decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a}{ad-bc} & b \\ \frac{c}{ad-bc} & d \end{pmatrix} \begin{pmatrix} ad-bc & \\ & 1 \end{pmatrix}.$$

Thus $G = M \rtimes H$, and if we can find compatible representations on G and H we will have a representation of the whole of G. We have already found a representation of M: for all $g \in SL(2,F)$, denote by $\sigma(g)$ the restriction of $\eta(g)$ to $W(\chi)$, where for neatness we suppress from our notation the dependence of σ on the character χ. Then σ is a representation of M on $W(\chi)$.

It is less obvious what representation τ of H we should consider, but it turns out that the following choice works. We define

$$\left(\tau\begin{pmatrix} a & \\ & 1 \end{pmatrix}\Phi\right)(v) = \chi(c)\Phi(cx)$$

for any $c \in E^\times$ with $N(c) = a$. Such a c exists by Lemma 3.19. We need to check that the right hand side is well-defined. Indeed, if $N(c) = N(c') = a$ then $c/c' \in E_1^\times$, so $\chi(c)^{-1}\Phi(cx) = \chi(c)\chi(c/c')^{-1}\Phi(c'x) = \chi(c')\Phi(c'x)$, so the right-hand side is well-defined. If $\Phi \in W(\chi)$ then $\chi(c')\Phi(c'tx) = \chi(t)^{-1}\chi(c')\Phi(c'x)$ for $t \in E_1^\times$, so $\theta(\chi)\begin{pmatrix} a & \\ & 1 \end{pmatrix}\Phi \in W(\chi)$, which shows $\tau\begin{pmatrix} a & \\ & 1 \end{pmatrix}$ maps $W(\chi)$ into itself. Finally, if $a, a' \in F^\times$ and $c, c' \in E^\times$ with $N(c) = a$ and $N(c') = a'$, then $N(cc') = aa'$; thus

$$\left(\tau\begin{pmatrix} a & \\ & 1 \end{pmatrix}\tau\begin{pmatrix} a' & \\ & 1 \end{pmatrix}\Phi\right)(x) = \chi(c')\chi(c)\Phi(cc'x) = \chi(cc')\Phi(cc'x) = \left(\tau\begin{pmatrix} aa' & \\ & 1 \end{pmatrix}\Phi\right)(x),$$

so τ is in fact a representation of H on $W(\chi)$.

By Lemma 3.20, all that remains to be checked is that

$$\tau\begin{pmatrix} a & \\ & 1 \end{pmatrix}\sigma(n)\,\tau\begin{pmatrix} 1/a & \\ & 1 \end{pmatrix} = \sigma\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} n \begin{pmatrix} 1/a & \\ & 1 \end{pmatrix}\right)$$

for all $n \in SL(2,F)$, and it will suffice to check this just in the case where n is one of our generators of $SL(2,F)$, since the conjugate of a product is the product of the conjugates. We will again denote the actions of τ and σ by a centred dot, and we introduce the notation $s(a)$ for the matrix $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$ in addition to the the notations from Proposition 2.15 for generators of $SL(2,F)$. Then we calculate that

$$\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} 1/a & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & ab \\ & 1 \end{pmatrix},$$

$$\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} A & \\ & 1/A \end{pmatrix}\begin{pmatrix} 1/a & \\ & 1 \end{pmatrix} = \begin{pmatrix} A & \\ & 1/A \end{pmatrix}$$

and $$\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1/a & \\ & 1 \end{pmatrix} = \begin{pmatrix} & a \\ -1/a & \end{pmatrix} = \begin{pmatrix} a & \\ & 1/a \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

whereas, if $N(c) = a$:

$$\begin{aligned}
\big(s(a)\cdot n(b)\cdot s(1/a)\cdot\Phi\big)(v) &= \chi(c)\psi\big(bN(cv)\big)\big(s(1/a)\cdot\Phi\big)(cv)\\
&= \chi(c)\psi\big(abN(v)\big)\chi(1/c)\Phi(v) = \big(n(ab)\cdot\Phi\big)(v);\\
\big(s(a)\cdot t(A)\cdot s(1/a)\cdot\Phi\big)(v) &= \chi(c)\chi(1/c)\Phi(cAc^{-1}v) = \big(t(A)\cdot\Phi\big)(v); \text{ and}\\
\big(s(a)\cdot w\cdot s(1/a)\cdot\Phi\big)(v) &= \chi(c)\,\delta q^{-1}\sum_{u\in E}\psi\big(\mathrm{tr}(u\overline{cv})\big)\chi(1/c)\Phi(u/c)\\
&= \delta q^{-1}\sum_{u\in E}\psi\big(\mathrm{tr}(u\overline{av})\big)\Phi(u) = \big(t(a)\cdot w\cdot\Phi\big)(v),
\end{aligned}$$

where in the last line we replaced u with cu in the summation. We have now shown the map $\theta(\chi) : GL(2,F)\to GL\big(W(\chi)\big)$ defined by

$$\theta(\chi)\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \sigma\begin{pmatrix} \frac{a}{ad-bc} & b\\ \frac{c}{ad-bc} & d\end{pmatrix}\tau\begin{pmatrix} ad-bc & \\ & 1\end{pmatrix}$$

is a representation. To be explicit, this representation is given on generators of $GL(2,F)$ by

$$\begin{aligned}
\left(\theta(\chi)\begin{pmatrix} 1 & b\\ & 1\end{pmatrix}\Phi\right)(v) &= \psi\big(bN(v)\big)\Phi(v) \qquad \text{for } b\in F,\\
\left(\theta(\chi)\begin{pmatrix} a & \\ & 1/a\end{pmatrix}\Phi\right)(v) &= \Phi(av) \qquad \text{for } a\in F^\times,\\
\left(\theta(\chi)\begin{pmatrix} a & \\ & 1\end{pmatrix}\Phi\right)(v) &= \chi(c)\Phi(cv) \qquad \text{for } a\in F^\times, \text{ where } N(c)=a,\\
\text{and} \qquad \left(\theta(\chi)\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\Phi\right)(v) &= \delta q^{-1}\sum_{u\in E}\psi\big(\mathrm{tr}(u\overline{v})\big)\Phi(u),
\end{aligned}$$

for all $\Phi\in W(\chi)$ and all $v\in E$. Here $\delta = \varepsilon_0^{-2}\xi^{-1}(f_1f_2) = \xi(-f_1f_2)$, where f_1 and f_2 are constants obtained from diagonalising N according to Lemma 3.14. In the split case, $N(a,b) = ab = \big(\frac{a+b}{2}\big)^2 - \big(\frac{a-b}{2}\big)^2$, so $\delta = \xi\big(-1(-1)\big) = 1$. In the field extension case, $E = F(\sqrt{d})$ for some nonsquare $d\in F^\times$ and $N\big(a+b\sqrt{d}\big) = a^2 - b^2d$, so $\delta = \xi\big(-1(-d)\big) = \xi(d) = -1$.

Remark 3.21. Throughout the last two sections, we have assumed F was of odd order. This was necessary in order to use Lemma 3.14 to diagonalise a bilinear form, among other results. However, it turns out that the representation defined by the equations above is a true representation of $GL(2,F)$ even when F is of even order. This is easily checked by verifying that the representing transformations satisfy the relations required by Proposition 2.15 and Lemma 3.20. We will omit these calculations.

3.6 Cuspidality and irreducibility

In the split case, it transpires that the representation we have found is equivalent to a principal series representation.

Proposition 3.22. *Let $E = F \oplus F$ and let $\chi : E^\times \to \mathbb{C}^\times$ be a multiplicative character of E which is nontrivial on $E_1^\times$. Construct the representation $\big(\theta(\chi), W(\chi)\big)$ as in section 3.5. Let χ_1 and χ_2 be the multiplicative characters of F such that $\chi(a,b) = \chi_1(a)\chi_2(b)$ for all $a, b \in F^\times$. Then $W(\chi) \cong \mathcal{B}(\chi_1, \chi_2)$.*

Proof. It is easy to see that $E_1^\times = \{(t, 1/t) \mid t \in F^\times\}$. Since $\chi(E_1^\times) \neq 1$, we know that $\chi_1(x)\chi_2(1/x) \neq 1$ for some $x \in F^\times$, so $\chi_1 \neq \chi_2$. Proposition 3.17 then tells us that $\mathcal{B}(\chi_1, \chi_2)$ is irreducible.

Observe that $\dim \mathcal{B}(\chi_1, \chi_2) = [GL(2, F) : B] = q + 1$. To find the dimension of $W(\chi)$, observe that the values of $\Phi \in W(\chi)$ on $E^\times$ are determined by the values on the coset representatives of $E_1^\times$ in $E^\times$. Furthermore, $\Phi(0,0) = \chi(t, 1/t)\Phi(0,0)$ for all $t \in E_1^\times$, which means that $\Phi(0,0) = 0$ as χ is nontrivial on $E_1^\times$. Finally, $\Phi(0,t)$ and $\Phi(t,0)$ for $t \in F^\times$ are determined by $\Phi(0,1)$ and $\Phi(1,0)$ respectively. Thus $\dim W(\chi) = [E^\times : E_1^\times] + 2 = (q-1)^2/(q-1) + 2 = q + 1$.

It will suffice to exhibit a nonzero intertwiner $T : W(\chi) \to \mathcal{B}(\chi_1, \chi_2)$, as then $W(\chi)$ must have an irreducible invariant subspace equivalent to $\mathcal{B}(\chi_1, \chi_2)$ by Proposition 2.5, and that subspace must be the whole space $W(\chi)$ since $\dim W(\chi) = \dim \mathcal{B}(\chi_1, \chi_2)$.

Define T by

$$(T\Phi)(g) = \big(\theta(\chi)(g)\Phi\big)(0,1)$$

for $\Phi \in W(\chi)$. To check that $T\Phi \in \mathcal{B}(\chi_1, \chi_2)$, we need to check that $(T\Phi)(bg) = \chi(b)\,(T\Phi)(g)$ for all $g \in GL(2, F)$ and $b \in B$. By taking $\Psi = \theta(\chi)(g)\Phi$, it suffices to check that $\big(\theta(\chi)(b)\Psi\big)(0,1) = \chi(b)\,\Psi(0,1)$ for all $b \in B$ and $\Psi \in W(\chi)$. Indeed,

$$\begin{aligned}
\left(\theta(\chi)\begin{pmatrix} a & b \\ & d \end{pmatrix}\Psi\right)(0,1) &= \left(\sigma\begin{pmatrix} 1/d & \\ & d \end{pmatrix}\sigma\begin{pmatrix} 1 & db \\ & 1 \end{pmatrix}\tau\begin{pmatrix} ad & \\ & 1 \end{pmatrix}\Psi\right)(0,1) \\
&= \chi(ad, 1)\Psi(0, 1/d) \\
&= \chi(a, d)\Psi(0,1)
\end{aligned}$$

for all $\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \in B$ and $\Psi \in W(\chi)$. It is clear from the definition of T that

$$\big(T\,\theta(\chi)(h)\Phi\big)(g) = (T\Phi)(gh) = \big(\pi(h)T\Phi\big)(g)$$

for all $\Phi \in W(\chi)$ and $g, h \in H$ (where π denotes the representation $\mathcal{B}(\chi_1, \chi_2)$), so T intertwines $\theta(\chi)$ and $\mathcal{B}(\chi_1, \chi_2)$. Finally, T is nonzero because $(T\Phi)(1) = \Phi(0,1) \neq 0$ for some $\Phi \in W(\chi)$, so we are done. □

In the case where E is a field, $\theta(\chi)$ is a new representation not seen before. To analyse it, we consider the subgroup U of $GL(2, F)$ defined by

$$U = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \;\middle|\; b \in F \right\}.$$

We can get useful information about a representation π of $GL(2, F)$ by considering the restriction of π to U. If F is considered as an additive group, then $U \cong F$ via the map $\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right) \mapsto b$. Thus the irreducible representations of U are in bijection with the additive characters of F. We need the following important lemma:

Lemma 3.23. *If ψ is a nontrivial additive character of F, then every additive character of F is of the form $x \mapsto \psi(ax)$ for some $a \in F$, and all these characters are distinct.*

Proof. Certainly $x \mapsto \psi(ax)$ is an additive character of F for $a \in F$. We know there is some c with $\psi(c) \neq 1$. If $a \neq b$ then for $x = c/(a-b)$ we have $\psi\big((a-b)x\big) \neq 1$, so $\psi(ax) \neq \psi(bx)$. Thus the characters $x \mapsto \psi(ax)$ for $a \in F$ are all distinct. Since there are exactly q additive characters of F, they must all be of this form. □

By Maschke's theorem, any representation (π, V) of $GL(2,F)$, when considered as a representation of U, breaks down as a direct sum of irreducible U-invariant subspaces, each of which is one-dimensional since U is abelian. On each of these subspaces there is some $a \in F$ such that the action of U on that subspace is given by $\pi\left(\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}\right)(v) = \psi(ab)v$. For any $a \in F$, the a-ISOTYPIC SUBSPACE is defined as

$$V_a = \left\{ v \in V \;\middle|\; \pi\begin{pmatrix}1 & b\\ & 1\end{pmatrix}(v) = \psi(ab)v \text{ for all } b \in F \right\},$$

which is the union of some of the U-invariant subspaces previously mentioned; it follows that

$$V = \bigoplus_{a \in F} V_a.$$

However, the extra structure obtained from π being a representation of the whole of $GL(2,F)$ makes for an interesting action on the set of isotypic subspaces of V. To be precise, pick any $r \in F^\times$. If $a \in F$ and $v \in V$ is such that $\pi\left(\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}\right)(v) = \psi(ab)v$ for all $b \in B$, then

$$\pi\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\pi\begin{pmatrix}r & \\ & 1\end{pmatrix}(v) = \pi\begin{pmatrix}r & \\ & 1\end{pmatrix}\pi\begin{pmatrix}1 & b/r\\ & 1\end{pmatrix}(v) = \psi(ab/r)\,\pi\begin{pmatrix}r & \\ & 1\end{pmatrix}(v)$$

for all $b \in B$ and hence $\pi\left(\begin{smallmatrix}r & \\ & 1\end{smallmatrix}\right)v \in V_{a/r}$, and it is easy to see that $\pi\left(\begin{smallmatrix}r & \\ & 1\end{smallmatrix}\right)$ is bijective from V_a to $V_{a/r}$. Since, for all $r \in F^\times$, $\pi\left(\begin{smallmatrix}r & \\ & 1\end{smallmatrix}\right)$ is a vector space isomorphism which preserves the isotypic subspaces (as sets), there is an action of $F^\times$ on the set of isotypic subspaces such that $r \cdot V_a = V_{a/r}$.

This action is almost transitive—almost, because it fixes V_0. There is a special class of representations for which this problem goes away:

Definition 3.24. A representation (π, V) of $GL(2,F)$ is CUSPIDAL if $V_0 = 0$. Equivalently, (π, V) is cuspidal if and only if there is no $v \in V$ with $\pi\left(\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}\right)(v) = v$ for all $b \in F$.

In particular, if (π, V) is cuspidal, then it is the direct sum of $q-1$ isomorphic subspaces, so $q - 1 \mid \dim V$.

Proposition 3.25. *If E is a field and χ is a character of $E^\times$ such that $\chi(E_1^\times)$ is not trivial, then the representation $\theta(\chi)$ constructed in section 3.5 is cuspidal.*

Proof. We can find $c \in E_1^\times$ such that $\chi(c) \neq 1$. Let $\Phi \in W(\chi)$ and observe that $\Phi(0) = \chi(c)^{-1}\Phi(0)$, so $\Phi(0) = 0$. Suppose there exists $e \in E^\times$ such that $\Phi(e) \neq 0$. Then $N(e) \neq 0$, so $\psi\big(bN(e)\big) \neq 1$ for some $b \in F^\times$ since ψ is nontrivial. Thus $\big(\theta(\chi)\big(\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}\big)\Phi\big)(e) = \psi\big(bN(e)\big)\Phi(e) \neq \Phi(e)$, so $\Phi \notin W(\chi)_0$. In other words, $W(\chi)_0 = 0$. □

Corollary 3.26. *Under the same assumptions, the representation $\theta(\chi)$ is irreducible.*

Proof. As remarked in Proposition 3.25, $\Phi(0) = 0$ for all $\Phi \in W(\chi)$, and if $e \in E^\times$ then $\Phi(e)$ determines the values of Φ on the coset $eE_1^\times$, by definition of $W(\chi)$. Thus $\dim W(\chi) = [E^\times : E_1^\times] = |F^\times| = q - 1$, since $\ker N = E_1^\times$ and $\operatorname{im} N = F^\times$ (considering N as a homomorphism $E^\times \to F^\times$). It is clear from the second characterisation in Definition 3.24 that a subrepresentation of a cuspidal representation is also cuspidal, so any subrepresentation of $\theta(\chi)$ must have dimension 0 or $q - 1$, which means that $\theta(\chi)$ is irreducible. □

3.7 Inequivalence of cuspidal representations

We saw in Proposition 3.17 that the principal series representations $\mathcal{B}(\chi_1, \chi_2)$ are pairwise inequivalent, up to swapping χ_1 and χ_2. We would like to prove a similar result for the cuspidal representations $\theta(\chi)$ studied in section 3.6. Unfortunately, Mackey theory is of no use here, because the representations $\theta(\chi)$ are not constructed as induced representations. We take instead a more elementary approach, considering the conditions that a nonzero intertwining map must satisfy.

Proposition 3.27. *Let χ_1 and χ_2 be characters of $E^\times$ such that neither $\chi_1(E_1^\times)$ nor $\chi_2(E_1^\times)$ is trivial. Construct the representations $\big(\theta(\chi_k), W(\chi_k)\big)$ for $k = 1, 2$ as in section 3.5. Suppose that $T : W(\chi_1) \to W(\chi_2)$ is a nonzero intertwiner from $\theta(\chi_1)$ to $\theta(\chi_2)$. Then either $\chi_2 = \chi_1$ or $\chi_2 = \chi_1'$, where $\chi_1'(u) = \chi_1(\overline{u})$ for all $u \in E^\times$.*

Proof. In order to make calculations with T, it is easiest to use explicit bases of $W(\chi_1)$ and $W(\chi_2)$. Recall our earlier remarks that if $\Phi \in W(\chi_k)$ ($k = 1$ or 2), then $\Phi(0) = 0$; furthermore, the values of Φ on the coset $cE_1^\times$ are completely determined by $\Phi(c)$, for $c \in E^\times$, and are independent of the values of Φ on the other cosets of $E_1^\times$. Thus a set of $q-1$ functions Φ, each supported on a different coset of $E_1^\times$, form a basis for $W(\chi_k)$.

To be precise, fix $c_1, c_2, \ldots, c_{q-1} \in E^\times$, all in different cosets of $E_1^\times$ (in other words, all with different norms). Then, for $k = 1, 2$, we define a basis $\{\Phi_1^k, \ldots, \Phi_{q-1}^k\}$ of $W(\chi_k)$ by

$$\Phi_i^k(c_j) = \delta_{ij}$$

for $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta symbol. (Note that $\Phi_i^1 \neq \Phi_i^2$, in general, because of the different conditions $W(\chi_1)$ and $W(\chi_2)$ place on their members.) This definition has the advantage that the coefficient of Φ_i^k in the function $\Phi \in W(\chi_k)$ is easily calculated to be $\Phi(c_i)$.

For example, we can calculate

$$\left(\theta(\chi_k)\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\Phi_i^k\right)(c_j) = \psi\big(bN(c_j)\big)\Phi_i^k(c_j) = \psi\big(bN(c_j)\big)\delta_{ij},$$

and so $\theta(\chi_k)\left(\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}\right)\Phi_i^k = \psi\big(bN(c_i)\big)\Phi_i^k$ for $1 \le i \le q-1$ and $k = 1, 2$. Similarly,

$$\left(\theta(\chi_k)\begin{pmatrix}a & \\ & 1/a\end{pmatrix}\Phi_i^k\right)(c_j) = \Phi_i^k(ac_j) = \begin{cases}\chi_k(c_i/ac_j) & \text{if } N(ac_j) = N(c_i),\\ 0 & \text{otherwise,}\end{cases}$$

which means that $\theta(\chi_k)\left(\begin{smallmatrix}a & \\ & 1/a\end{smallmatrix}\right)\Phi_i^k = \chi_k(c_i/ac_j)\Phi_j^k$, where j is chosen such that $N(c_j) = N(c_i/a)$. Another easy calculation is that if $a \in F^\times$ and $N(c) = a$, then

$$\left(\theta(\chi_k)\begin{pmatrix}a & \\ & 1\end{pmatrix}\Phi_i^k\right)(c_j) = \chi_k(c)\Phi_i^k(cc_j) = \begin{cases}\chi_k(c_i/c_j) & \text{if } N(cc_j) = N(c_i),\\ 0 & \text{otherwise,}\end{cases}$$

so $\theta(\chi_k)\left(\begin{smallmatrix}a & \\ & 1\end{smallmatrix}\right)\Phi_i^k = \chi_k(c_i/c_j)\Phi_j^k$, where j is chosen such that $N(c_j) = N(c_i)/a$. Finally, we consider w:

$$\begin{aligned}\left(\theta(\chi_k)\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\Phi_i^k\right)(c_j) &= \delta q^{-1}\sum_{u\in V}\psi\big(\operatorname{tr}(u\overline{c_j})\big)\Phi_i^k(u)\\ &= \delta q^{-1}\sum_{l\in E_1^\times}\psi\big(\operatorname{tr}(lc_i\overline{c_j})\big)\Phi_i^k(lc_i)\\ &= \delta q^{-1}\sum_{l\in E_1^\times}\psi\big(\operatorname{tr}(lc_i\overline{c_j})\big)\chi_k(l)^{-1},\end{aligned}$$

so

$$\theta(\chi_k)\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\Phi_i^k = \delta q^{-1}\sum_{j=1}^{q-1}\left(\sum_{l\in E_1^\times}\psi\big(\operatorname{tr}(lc_i\overline{c_j})\big)\chi_k(l)^{-1}\right)\Phi_j^k.$$

We want to find the matrix of T with respect to these bases. Let $t_{ij} \in \mathbb{C}$ be such that

$$T\Phi_i^1 = \sum_{j=1}^{q-1} t_{ij}\Phi_j^2.$$

for $1 \le i, j \le q-1$. Since T is an intertwiner, we have

$$T \circ \theta(\chi_1)(g) = \theta(\chi_2)(g) \circ T \tag{3.11}$$

for all $g \in GL(2, F)$. We first apply (3.11) to $g = \left(\begin{smallmatrix}1 & b\\ & 1\end{smallmatrix}\right)$ and evaluate at Φ_i^1, obtaining

$$\psi\big(bN(c_i)\big)\sum_{j=1}^{q-1} t_{ij}\Phi_j^2 = \sum_{j=1}^{q-1} t_{ij}\psi\big(bN(c_j)\big)\Phi_j^2$$

for all $b \in F$ and all i. Since the Φ_j^2 are linearly independent, this implies that

$$t_{ij}\psi\big(bN(c_i)\big) = t_{ij}\psi\big(bN(c_j)\big)$$

for all $b \in F$ and all i, j. If $i \neq j$ then $N(c_i) \neq N(c_j)$, so by Lemma 3.23 $\psi\big(bN(c_i)\big) \neq \psi\big(bN(c_j)\big)$ for some $b \in F$, and so $t_{ij} = 0$. On the other hand, no condition on t_{ii} is implied.

Now we apply (3.11) to $g = \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ for $a \in F^\times$, obtaining

$$t_{jj}\chi_1\left(\frac{c_i}{c_j}\right) = t_{ii}\chi_2\left(\frac{c_i}{c_j}\right) \tag{3.12}$$

whenever $N(c_j) = N(c_i)/a$. Since a is also arbitrary, this means (3.12) holds for all i. Since $T \neq 0$, this means that $t_{ii} = \chi_1(c_i)/\chi_2(c_i)$ for all i (up to scaling T by a nonzero constant, which is irrelevant to us). But (3.11) with $g = \left(\begin{smallmatrix} a & \\ & 1/a \end{smallmatrix}\right)$ yields

$$t_{jj}\chi_1\left(\frac{c_i}{ac_j}\right) = t_{ii}\chi_2\left(\frac{c_i}{ac_j}\right),$$

if $a \in F^\times$, where j is chosen such that $N(c_j) = N(c_i/a)$. Combined with our knowledge of t_{ii} and t_{jj}, this implies that $\chi_1(a) = \chi_2(a)$ for all $a \in F^\times$, which is at least a step in the right direction.

However, there are many characters which extend $\chi_1|_{F^\times}$ (by the argument in Proposition 3.7). To obtain the conclusion we want, we need to apply (3.11) to $g = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$. Taking the coefficient of Φ_j^k on both sides, this yields:

$$\frac{\chi_1(c_j)}{\chi_2(c_j)}\sum_{l\in E_1^\times}\psi\big(\mathrm{tr}(lc_i\overline{c_j})\big)\chi_1(l)^{-1} = \frac{\chi_1(c_i)}{\chi_2(c_i)}\sum_{l\in E_1^\times}\psi\big(\mathrm{tr}(lc_i\overline{c_j})\big)\chi_2(l)^{-1}$$

for all i, j. Grouping the characters χ_k and observing that $l^{-1} = \bar{l}$ since $N(l) = 1$, we obtain

$$\sum_{l\in E_1^\times}\psi\big(\mathrm{tr}(lc_i\overline{c_j})\big)\left(\chi_1\left(\frac{c_j\bar{l}}{c_i}\right) - \chi_2\left(\frac{c_j\bar{l}}{c_i}\right)\right) = 0$$

for all i, j. Multiplying by $\chi_1(c_i\overline{c_i})$ (which equals $\chi_2(c_i\overline{c_i})$ since $c_i\overline{c_i} \in F^\times$), we see that

$$\sum_{l\in E_1^\times}\psi\big(\mathrm{tr}(lc_i\overline{c_j})\big)\big(\chi_1(c_j\overline{c_i l}) - \chi_2(c_j\overline{c_i l})\big) = 0,$$

and summing over l instead of $\bar{l}$, this becomes

$$\sum_{l\in E_1^\times}\psi\big(\mathrm{tr}(lc_j\overline{c_i})\big)\big(\chi_1(lc_j\overline{c_i}) - \chi_2(lc_j\overline{c_i})\big) = 0$$

for all i, j. Since $N(c_j\overline{c_i})$ takes each value in $F^\times$ for some i, j, our equation simply says that

$$\sum_{\substack{m\in E^\times\\ N(m)=f}}\psi\big(\mathrm{tr}(m)\big)\big(\chi_1(m) - \chi_2(m)\big) = 0$$

for all $f \in F^\times$. The substitution $m = am'$ for $a \in F^\times$ yields

$$\sum_{\substack{m' \in E^\times \\ N(m')=f}} \psi\bigl(a \operatorname{tr}(m')\bigr)\bigl(\chi_1(m') - \chi_2(m')\bigr) = 0 \tag{3.13}$$

for all $a, f \in F^\times$, after dividing by $\chi_1(a)$ (which equals $\chi_2(a)$).

It's not immediately obvious what the next step should be, since we have no real information about χ_1, χ_2 and ψ. In previous sections, when evaluating sums involving characters, we have either changed the summation index or used orthogonality relations. It is not clear that changing the summation index any further would reveal anything new, but using orthogonality of characters appears promising. If we could say the additive characters $a \mapsto \psi\bigl(a \operatorname{tr}(m')\bigr)$ were orthogonal for different m' (such that $N(m') = f$) then we might be able to conclude that $\chi_1(m') = \chi_2(m')$ for all relevant m'. However, the characters $a \mapsto \psi\bigl(a \operatorname{tr}(m')\bigr)$ are not even all distinct, because if $m' \notin F^\times$ then m' and $\overline{m'}$ are distinct elements with the same norm and trace. (Of course, that method could never have worked, because the conclusion that $\chi_1(m') = \chi_2(m')$ is too strong.)

A better idea along the same lines is to consider all the possible values of $\operatorname{tr}(m')$, and make that into the summation index. Because the multiplicative group of every finite field is cyclic, we can choose a generator γ of $E^\times$, and it will suffice to prove the claim for $\chi_2(\gamma)$, since that determines the whole of χ_2. Let

$$S = \bigl\{p \in F \bigm| p = \operatorname{tr}(l\gamma) \text{ for some } l \in E_1^\times\bigr\}.$$

Luckily, this choice simplifies the summation considerably, because of the following fact: if $p \in S$ then there are exactly two $l \in E_1^\times$ such that $\operatorname{tr}(l\gamma) = p$. To see this, observe that $\operatorname{tr}(l\gamma) = p$ if and only if

$$\begin{aligned} & l\gamma + l^{-1}\overline{\gamma} = p \qquad\qquad \text{(remembering that } \bar{l} = l^{-1}\text{)} \\ \iff \quad & \gamma l^2 - pl + \overline{\gamma} = 0. \end{aligned}$$

The discriminant of this quadratic equation in l is $p^2 - 4\gamma\overline{\gamma}$, which is nonzero for the following reason: $\gamma\overline{\gamma} = \gamma^{q+1}$ is a generator of $F^\times$ and hence not a square in $F^\times$, because $|F^\times| = q-1$ is even. Therefore there are either zero or two solutions for l, and so there must be two solutions if $p \in S$; the other solution is $\overline{\gamma}/\gamma l$, which is also in $E_1^\times$.

For each $p \in S$ choose any $l(p)$ such that $\operatorname{tr}\bigl(\gamma l(p)\bigr) = p$. The two terms in the summation in (3.13) whose ψ factor is $\psi(ap)$ are those with $l = l(p)$ and $l = \overline{\gamma}/\gamma l(p) = \overline{\gamma l(p)}/\gamma$. Thus we can rewrite (3.13) as

$$\sum_{p \in S} \psi(ap)\Bigl(\chi_1\bigl(l(p)\gamma\bigr) - \chi_2\bigl(l(p)\gamma\bigr) + \chi_1\bigl(\overline{l(p)\gamma}\bigr) - \chi_2\bigl(\overline{l(p)\gamma}\bigr)\Bigr) = 0 \tag{3.14}$$

for all $a \in F^\times$. But χ_1 and χ_2 are nontrivial on $E_1^\times$, so

$$\sum_{p \in S}\Bigl(\chi_k\bigl(l(p)\gamma\bigr) + \chi_k\bigl(\overline{l(p)\gamma}\bigr)\Bigr) = \sum_{l \in E_1^\times} \chi_k(l\gamma) = 0$$

for $k = 1, 2$, by the same character trick we used before. Thus (3.14) is also true in the case $a = 0$, so it is true for all $a \in F$. Now we know the characters $a \mapsto \chi(ap)$ for $p \in S$ are distinct, hence orthogonal, hence linearly independent. It follows that the coefficients of $\psi(ap)$ in (3.14) are all zero. In particular, when $p = \mathrm{tr}(\gamma)$, we may choose $l(p) = 1$ and hence

$$\chi_1(\gamma) + \chi_1(\overline{\gamma}) - \chi_2(\gamma) - \chi_2(\overline{\gamma}) = 0.$$

But we also have $\chi_1(\gamma)\chi_1(\overline{\gamma}) = \chi_2(\gamma)\chi_2(\overline{\gamma})$, since χ_1 and χ_2 agree on $F^\times$. So the pairs $\{\chi_1(\gamma), \chi_1(\overline{\gamma})\}$ and $\{\chi_2(\gamma), \chi_2(\overline{\gamma})\}$ are both the solution set of the same quadratic equation, so either $\chi_2(\gamma) = \chi_1(\gamma)$ or $\chi_2(\gamma) = \chi_1(\overline{\gamma})$. Since γ generates $E^\times$, we now know the relationship: in the first case $\chi_2 = \chi_1$, and in the second case $\chi_2 = \chi_1'$. □

The above proposition shows that the representations $\theta(\chi)$ (where χ is a character of $E^\times$ such that $\chi(E_1^\times)$ is nontrivial) are all inequivalent, except for the "conjugate pairs" of characters χ and χ' described above.

For completeness, we should check that $\theta(\chi)$ is equivalent to $\theta(\chi')$, where $\chi'(x) = \chi(\overline{x})$ for all $x \in E^\times$. It is easily checked that the map $T : W(\chi) \to W(\chi')$ as $(T\Phi)(u) = \Phi(\overline{u})$ for all $u \in E$ is an intertwiner from $\theta(\chi)$ to $\theta(\chi')$. However, in the following paragraph this check is rendered unnecessary, because if $\theta(\chi)$ were not equivalent to $\theta(\chi')$, there would be too many irreducible representations of $GL(2, F)$.

The natural question now is whether there are any more irreducible representations of $GL(2, F)$. Recalling a lemma from finite group representation theory, it will be enough to check that the sum of the squares of the dimensions of the inequivalent irreducible representations we know about is equal to the order of the group. Here are the irreducible representations we know about, in summary form:

Type	Number	Dimension	$\sum$ Dimension^2
Character	$q-1$	1	$q-1$
Special	$q-1$	q	$q^3 - q^2$
Principal series	$\frac{1}{2}(q-1)(q-2)$	$q+1$	$\frac{1}{2}(q^4 - q^3 - 3q^2 + q + 2)$
Cuspidal	$\frac{1}{2}q(q-1)$	$q-1$	$\frac{1}{2}(q^4 - 3q^3 + 3q^2 - q)$
			$q^4 - q^3 - q^2 + q$

To find the order of $GL(2, F)$, we need only observe that, in order to obtain an invertible matrix, the first column can be any of $q^2 - 1$ nonzero vectors, and once the first column is chosen, the second column can be any of $q^2 - q$ vectors which are not collinear with the first column. So $|GL(2, F)| = (q^2 - 1)(q^2 - q) = q^4 - q^3 - q^2 + q$, which shows that we have found all the irreducible representations of $GL(2, F)$.

Chapter 4

Representations over local fields

In this chapter, we modify our previous techniques in order to find representations of $GL(2, F)$, where F is a non-Archimedean local field with odd residue characteristic. We first outline the definition and properties of local fields, introduce the appropriate notion of *admissible representations*, and give some background on the Fourier transform over local fields. We then quote a theorem of Segal, Shale and Weil, which provides a family of intertwiners corresponding to elements of $SL(2, F)$ in the same way as the Finite Stone–von Neumann Theorem did in the finite field case. The proof of the theorem is beyond the scope of this essay. We construct a projective representation of $SL(2, F)$, and lift it to a true representation of $SL(2, F)$ called the Weil representation. We obtain a model of this representation which operates very similarly to the finite field case.

Finally, by specialising the parameters of our representation, we obtain a family of representations of $GL(2, F)_+$, where $GL(2, F)_+$ is a certain subgroup of $GL(2, F)$ of index two. We then use the induced representation to find representations of $GL(2, F)$. The reason we cannot obtain a representation of $GL(2, F)$ directly is that the norm map from a quadratic extension of F is no longer surjective in the local field case. The representations we obtain are known as the *dihedral supercuspidal* representations.

4.1 Local fields

There are multiple definitions of a local field; we will choose one of the simpler definitions, and piece together the most important properties.

Definition 4.1. A field F, together with a topology $\mathcal{T}$ on F, is called a TOPOLOGICAL FIELD if the maps

$$\begin{aligned} & F \times F \to F : (x, y) \mapsto x + y, \\ & F \times F \to F : (x, y) \mapsto xy, \\ \text{and} \qquad & F \setminus \{0\} \to F : \qquad x \mapsto x^{-1} \end{aligned}$$

are continuous (with respect to the product topology on $F \times F$). The topological fields F and G are TOPOLOGICALLY ISOMORPHIC if there exists a field isomorphism $F \to G$

which is also a homeomorphism.

If F is Hausdorff, locally compact and not discrete, then we call F a LOCAL FIELD.

While the topological definition is very general, it turns out that local fields can be analysed in terms of the more concrete concept of an *absolute value*.

Definition 4.2. An ABSOLUTE VALUE on a field F is a map $\varphi : F \to \mathbb{R}$ which satisfies the following conditions:

(a) $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in F$;

(b) $\varphi(a) \geq 0$ for all $a \in F$, with equality if and only if $a = 0$; and

(c) $\varphi(a + b) \leq \varphi(a) + \varphi(b)$ for all $a, b \in F$.

Clearly an absolute value φ on F induces a metric d on F given by $d(a, b) = \varphi(a-b)$, and hence a topology on F.

Now we can state the classification theorem for local fields.

Theorem 4.3. *Every local field is topologically isomorphic to one of the following:*

- $\mathbb{R}$ *or* $\mathbb{C}$, *with the topology induced by the standard absolute value;*
- *the* p*-adic numbers* $\mathbb{Q}_p$ *for some prime* p*, with the topology induced by the standard* p*-adic absolute value, or a finite extension of* $\mathbb{Q}_p$*, with the product topology as a vector space over* $\mathbb{Q}_p$*; or*
- *the field* $\mathbb{F}_q(\!(t)\!)$ *of formal Laurent series (that is, formal power series in* t *and* t^{-1} *with finitely many terms of negative degree) over some finite field* $\mathbb{F}_q$*, with the topology induced by the absolute value* $\varphi\big(\sum_{n\in\mathbb{Z}} a_n t^n\big) = q^{-k}$*, where* k *is the smallest integer such that* $a_k \neq 0$*.*

We prove this theorem through a number of lemmas. We first need a slight weakening of the conditions on an absolute value.

Lemma 4.4. *Suppose that* $\psi : F \to \mathbb{R}$ *satisfies (a) and (b) above, and that in addition*

(d) there exists some $C \in \mathbb{R}$ *such that* $\psi(1 + a) \leq C$ *whenever* $\psi(a) \leq 1$*.*

Then there is some $\lambda > 0$ *such that* ψ^λ *is an absolute value on* F*.*

Proof. See [2], Lemmas 1.1 and 1.2 on p. 13. □

Lemma 4.5. *Let* F *be a local field with topology* $\mathcal{T}$*. There exists a continuous map* $\psi : F \to \mathbb{R}$ *which satisfies (a), (b) and (d) above. Moreover,* $S = \{a \in F \mid \psi(a) \leq 1\}$ *is compact. In addition,* $\mathcal{T} = \mathcal{T}_\varphi$*, where* φ *is any absolute value obtained from* ψ *by Lemma 4.4 and* $\mathcal{T}_\varphi$ *is the topology on* F *induced by* φ*. In other words, the topology on* F *is induced by an absolute value.*

Proof. See [11], Lemmas 1, 2 and 3 on pp. 369–70. The essential step is to take a Haar measure μ on the Borel σ-algebra of the additive group F. Then for all $a \in F^\times$ the measure μ_a, defined by $\mu_a(X) = \mu(aX)$ for Borel sets X, is also an additive Haar measure on F. Since the Haar measure is unique up to scalar multiple, there is some $\psi(a) \in \mathbb{R}$ such that $\mu(aX) = \psi(a)\mu(X)$ for all Borel sets X and all $a \in F^\times$. Setting $\psi(0) = 0$, it can then be checked that ψ is continuous and satisfies (a), (b) and (d).

For the last claim, what is proved in [11] is that $\mathcal{T} = \mathcal{T}_\psi$, where $\mathcal{T}_\psi$ denotes the topology given by the base of neighbourhoods $B(x, r) = \{y \in F \mid \psi(x - y) < r\}$ for all $x \in F$ and $r \in \mathbb{R}^+$. Since $\varphi = \psi^\lambda$ for some $\lambda > 0$, we have $\mathcal{T}_\psi = \mathcal{T}_\varphi$. □

The next lemma allows us to associate to each local field a unique absolute value, up to *equivalence*, which in this context means raising the absolute value to a positive power.

Lemma 4.6. *If φ_1 and φ_2 are two absolute values on a local field F which both induce the same topology, then $\varphi_1 = \varphi_2^\lambda$ for some $\lambda > 0$.*

Proof. See [2], Lemma 3.2 on p. 20. □

The essential feature determining the type of a local field is whether or not its absolute value φ satisfies the ULTRAMETRIC INEQUALITY, which states that $\varphi(u+v) \leq \max\{\varphi(u), \varphi(v)\}$ for all $u, v \in F$. It is easy to see that this is a property of the local field itself, since the ultrametric inequality is preserved among equivalent absolute values. In reference to the Archimedean property of $\mathbb{R}$ (namely, that for every $r \in \mathbb{R}$ there is an integer n with $|n| > |r|$), local fields where the ultrametric inequality fails are called ARCHIMEDEAN. All other local fields are called NON-ARCHIMEDEAN.

Lemma 4.7. *Let F be a local field with topology $\mathcal{T}$. Then F is complete with respect to any absolute value which induces the topology $\mathcal{T}$.*

Proof. Let φ be an absolute value on F which induces the topology $\mathcal{T}$. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in F with respect to φ. Since this sequence is Cauchy, it is bounded, so there exists some $M > 0$ such that $\varphi(x_n) < M$ for all $n \in \mathbb{N}$. Picking any $b \in F$ such that $\varphi(b) > M$, we see that $x_n \in \{a \in F \mid \varphi(a) \leq \varphi(b)\} = bS$ for all $n \in \mathbb{N}$, and bS is compact since S is compact (by Lemma 4.5) and multiplication is continuous. Since bS is compact, it is complete with respect to φ, so x_n converges to some $x \in bS$. □

Theorem 4.8 (Ostrowski)**.** *Let F be an Archimedean local field. Then F is topologically isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.*

Proof. See [2], Theorem 1.1 on p. 33. We first need to apply Lemma 4.7 above to see that F is complete with respect to the absolute value we use on it. □

From this point on, we will consider only non-Archimedean local fields.

Definition 4.9. Let F be a non-Archimedean local field with absolute value φ. Then the definition of an absolute value and the ultrametric inequality imply that

$$\mathfrak{o} := \{x \in F \mid \varphi(x) \leq 1\}$$

is a subring of F, called the RING OF INTEGERS OF F. Since $\varphi(a)\varphi(1/a) = 1$ for all $a \in F^\times$, the elements of $\mathfrak{o}$ of absolute value 1 are invertible in $\mathfrak{o}$. It follows that

$$\mathfrak{p} := \{x \in F \mid \varphi(x) < 1\}$$

is a maximal ideal of F, and $\mathfrak{o}/\mathfrak{p}$ is a field, called the RESIDUE CLASS FIELD of F. The characteristic of $\mathfrak{o}/\mathfrak{p}$ is known as the RESIDUE CHARACTERISTIC of F.

Remark 4.10. Since the elements of absolute value 1 are invertible in $\mathfrak{o}$, any proper ideal of $\mathfrak{o}$ is contained in $\mathfrak{p}$. Thus $\mathfrak{p}$ is the unique maximal ideal of $\mathfrak{o}$.

Proposition 4.11. *Let F, φ, $\mathfrak{o}$ and $\mathfrak{p}$ be as above. Then $\mathfrak{o}$ and $\mathfrak{p}$ are open and compact, and $\mathfrak{o}/\mathfrak{p}$ is finite. Moreover, there exists $\varpi \in \mathfrak{p}$ such that $\mathfrak{p} = \varpi\mathfrak{o}$, $\varphi(\varpi) \in (0,1)$ and $\operatorname{im}\varphi = \{0\} \cup \{\varphi(\varpi)^n \mid n \in \mathbb{Z}\}$.*

Proof. By Lemma 4.5, $\mathfrak{o}$ is compact. Since φ is continuous and $\mathfrak{p}$ is the preimage of the open set $(-\infty, 1)$ under φ, $\mathfrak{p}$ is open. We see that

$$F \setminus \mathfrak{p} = \bigcup_{x \in F\setminus\mathfrak{p}} (x + \mathfrak{p}) \qquad \text{and} \qquad \mathfrak{o} = \bigsqcup_{x+\mathfrak{p} \in \mathfrak{o}/\mathfrak{p}} (x + \mathfrak{p})$$

which shows that $\mathfrak{p}$ is closed, hence compact, and $\mathfrak{o}$ is open. Moreover, since the union is disjoint, all the sets $x + \mathfrak{p}$ are needed to cover $\mathfrak{o}$. Since $\mathfrak{o}$ is compact, this means that $\mathfrak{o}/\mathfrak{p}$ is finite. Let $q = |\mathfrak{o}/\mathfrak{p}|$.

Observe that $\varphi(\mathfrak{p})$ is the continuous image of a compact set and is therefore compact, but also that $\varphi(\mathfrak{p}) \subseteq [0,1)$. Let $\alpha = \sup_{p\in\mathfrak{p}} \varphi(p)$. Since $\mathfrak{p}$ is compact, there exists some $\varpi \in \mathfrak{p}$ such that $\varphi(\varpi) = \alpha$, so $\alpha < 1$. But $\alpha \neq 0$ (otherwise $\varphi(p) = 0$ for all $p \in \mathfrak{p}$, which would mean that $\mathfrak{p} = \{0\}$ and thus that F were discrete, contrary to assumption). So $\alpha \in (0,1)$.

For any $x \in F^\times$, let $k = \min\{n \in \mathbb{Z} \mid \varpi^n x \in \mathfrak{o}\}$. Then $\varphi(\varpi^k x) \in (\alpha, 1]$ by the minimality of k, so $\varphi(\varpi^k x) = 1$ since φ takes no values in the range $(\alpha, 1)$. Thus $\varphi(x) = \varphi(\varpi^{-k}) = \alpha^{-k}$. Since also $\varphi(0) = 0$, we have $\operatorname{im}\varphi = \{0\} \cup \{\alpha^n \mid n \in \mathbb{Z}\}$. □

Remark 4.12. Every non-Archimedean local field is totally disconnected. To see this, using the notations above, suppose $x, y \in F$ and $x \neq y$. If k is such that $\varphi(x-y) = \alpha^k$, then $x + \varpi^{k+1}\mathfrak{o}$ is an open and closed set which contains x but not y.

Definition 4.13. Let F, $\mathfrak{o}$, $\mathfrak{p}$ and ϖ be as above, and let $q = |\mathfrak{o}/\mathfrak{p}|$. The STANDARD ABSOLUTE VALUE on F is the absolute value φ satisfying $\varphi(\varpi) = 1/q$.

Henceforth we will always use the standard absolute value on each non-Archimedean local field, and we will denote it by $x \mapsto |x|$. One reason why this is a natural choice is that the additive Haar measure μ on F then satisfies

$$\mu(aS) = |a|\,\mu(S) \tag{4.1}$$

for any Borel subset S of F.

Lemma 4.14. *Let F, $\mathfrak{o}$, $\mathfrak{p}$ and ϖ be as above, and let S be a complete set of coset representatives for $\mathfrak{p}$ in $\mathfrak{o}$, including 0. Then every $x \in F$ can be written uniquely as a convergent series of the form*

$$x = \sum_{n\in\mathbb{Z}} s_n \varpi^n$$

for some coefficients $s_n \in S$ such that $s_n = 0$ for all but finitely many negative n.

Proof. By multiplying by a suitable power of ϖ, it will suffice to prove that every $x \in \mathfrak{o}$ can be written uniquely in the above form. We prove by induction on k that there exists a unique choice of $s_0, s_1, \dots, s_k \in S$ such that $x - \sum_{n=0}^{k} s_n\varpi^n \in \varpi^{k+1}\mathfrak{o}$ for all $k \geq -1$. For the $k = -1$ case, we already have $x \in \mathfrak{o}$. Now, supposing the result true for k, by definition of S there exists a unique $s_{k+1} \in S$ such that $x - \sum_{n=0}^{k+1} s_n\varpi^n \in \varpi^{k+1}\mathfrak{o}$. □

Clearly the residue characteristic p of F always divides the characteristic of F, so $\operatorname{char} F = 0$ or $\operatorname{char} F = p$. The proof of Theorem 4.3 is completed by the following proposition.

Proposition 4.15. *Let F be a non-Archimedean local field with residue characteristic p, and let $\mathfrak{o}$, $\mathfrak{p}$ and q be as above. If $\operatorname{char} F = p$, then F is isomorphic to $\mathbb{F}_q(\!(t)\!)$. If $\operatorname{char} F = 0$, then F is isomorphic to a finite extension of $\mathbb{Q}_p$.*

Proof. If $\operatorname{char} F = p$, then $\mathfrak{o}$ is isomorphic to the ring $\mathbb{F}_q[\![t]\!]$ of formal power series over the finite field of order q—for the proof, see [8], Theorem 2 on p. 33. F is clearly then isomorphic to the field of fractions of $\mathbb{F}_q[\![t]\!]$, namely $\mathbb{F}_q(\!(t)\!)$.

If $\operatorname{char} F = 0$, then there is a natural injective ring homomorphism $i : \mathbb{Z} \to \mathfrak{o}$ determined by $1 \mapsto 1$. We note that $i(p) + \mathfrak{p} = 0$ in $\mathfrak{o}/\mathfrak{p}$, which means that $|i(p)| < 1$. In addition, $|i(a) - i(b)| = |i(p)|^k$ if $|a - b|_p = p^{-k}$, which means that Cauchy sequences in $\mathbb{Z}$ with respect to the p-adic metric are mapped to Cauchy sequences in $\mathfrak{o}$. Since $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ under the p-adic metric, there exists a unique continuous map $j : \mathbb{Z}_p \to \mathfrak{o}$ extending i, and j is clearly an injective ring homomorphism. Since F is a field, there exists a unique embedding $\hat{\jmath} : \mathbb{Q}_p \to F$ extending j.

We know $q = p^k$ for some $k \in \mathbb{N}$. Pick any k-tuple $(b_1 = 1, b_2, \cdots, b_k) \subseteq \mathfrak{o}^k$ such that $(b_1 + \mathfrak{p}, \cdots, b_k + \mathfrak{p})$ forms a basis for $\mathfrak{o}/\mathfrak{p}$ over its prime subfield. Then $S = \big\{ j(a_1)b_1 + \cdots + j(a_k)b_k \;\big|\; a_i \in \{0, \dots, p-1\} \big\}$ forms a complete set of coset representatives for $\mathfrak{p}$ in $\mathfrak{o}$. Since $|i(p)| < 1$, we can find $l \geq 1$ and $s \in S$ such that $i(p) - \varpi^l s \in \varpi^{l+1}\mathfrak{o}$. By Lemma 4.14, every $x \in F$ can be written uniquely in the form $x = \sum_{n\in\mathbb{Z}} s_n\varpi^n$ for some coefficients $s_n \in S$. This can be uniquely decomposed in the form $x = \sum_{n=0}^{l-1} \sum_{i=1}^{k} \hat{\jmath}(a_{in}) b_i \varpi^n$ for some $a_{in} \in \mathbb{Q}_p$. Thus $[F : \hat{\jmath}(\mathbb{Q}_p)] = lk$, so F is isomorphic to a finite extension of $\mathbb{Q}_p$. □

The representation theory of Lie groups like $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$ uses techniques from Lie theory (see [1], Chapter 2, and [9], Chapter V for $SL(2,\mathbb{R})$). By contrast, the representation theory of $GL(2,F)$, where F is a non-Archimedean local field, has strong parallels to the finite field theory, and that is what we will study in this chapter.

4.2 Smooth and admissible representations

It is not generally practical to study arbitrary representations of infinite topological groups, because the topology on the group is of no use in that case. Very often, attention is restricted to *unitary representations*, which are homomorphisms from a group into the group of unitary operators on a Hilbert space which satisfy a certain strong continuity property (see [9], Chapter I). We will, however, find it more convenient to operate within the framework of *smooth* and *admissible* representations, which have the advantage of requiring no topology on the representation space.

We first state a lemma, useful in its own right, which illustrates how we can define a useful substitute for continuity, called *smoothness*, without using any topology on the target vector space.

Lemma 4.16. *Let G be a totally disconnected locally compact group and let ψ be a character of G. Then ψ is locally constant; that is, every $g \in G$ has a neighbourhood on which ψ is constant.*

Proof. It will suffice to prove that ψ is constant on an open neighbourhood of the identity, because ψ is then constant on any translation of that neighbourhood. By the general theory of totally disconnected groups, any open neighbourhood of the identity in G contains a compact open subgroup. Consider the open neighbourhood $\psi^{-1}\{x \in \mathbb{C}^\times \mid \operatorname{Re} x > 0\}$ of the identity. It contains a compact open subgroup U. Then $\psi(U)$ is a subgroup of $\mathbb{T}$ all of whose elements have positive real parts; thus it is the trivial subgroup, and so ψ is constant on U as claimed. □

It is too much to ask that a representation itself be locally constant, but we instead ask that the group's action on each individual vector in the representation space be locally constant.

Definition 4.17. Let G be a totally disconnected locally compact group, V a complex vector space and $\pi : G \to GL(V)$ a homomorphism. Then π is a SMOOTH REPRESENTATION if for any $v \in V$ the stabiliser $G_v = \{g \in G \mid \pi(g)v = v\}$ is open. If in addition the subspace $\{v \in V \mid \pi(h)v = v \text{ for all } h \in H\}$ is finite dimensional for every open subgroup $H \leq G$, then π is an ADMISSIBLE REPRESENTATION.

Occasionally we will need to use a UNITARY REPRESENTATION, which means a representation $\rho : G \to \mathcal{U}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, where $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$.

We will primarily be considering admissible representations of $GL(2,F)$.

4.3 Fourier analysis on local fields

In order to find representations of $GL(2, F)$, we will need to understand the properties of Fourier transforms over local fields. The theory is developed similarly to Fourier theory on $\mathbb{R}$, except that some of the arguments are a little simpler in the local field case.

Let F be a non-Archimedean local field, $\mathfrak{o}$ the ring of integers of F, $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{o}$, ϖ any generator of $\mathfrak{p}$, and $|\cdot|$ the standard absolute value on F. Let ψ be a fixed nontrivial additive character of F. By Lemma 4.16, there exists some $n \in \mathbb{Z}$ such that ψ is trivial on $\varpi^n\mathfrak{o}$. Let $c \in \mathbb{Z}$ be the least such n; $\varpi^c\mathfrak{o}$ is then called the CONDUCTOR of ψ.

Let μ be an additive Haar measure on F, normalised so that $\mu(\mathfrak{o}) = 1$.

Lemma 4.18. *For any $n \in \mathbb{Z}$ and $t \in F$,*

$$\int_{\varpi^n\mathfrak{o}} \psi(tx)\, d\mu(x) = \begin{cases} q^{-n} & \text{if } |t| \leq q^{n-c}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $|t| \leq q^{n-c}$ then $\psi(tx) = 1$ for all $x \in \varpi^n\mathfrak{o}$. Otherwise, take any $y \in \varpi^{c-1}\mathfrak{o}$ such that $\psi(y) \neq 1$. Then observe that $y/t \in \varpi^n\mathfrak{o}$, so by the translation-invariance of μ,

$$\int_{\varpi^n\mathfrak{o}} \psi(tx)\, d\mu(x) = \int_{\varpi^n\mathfrak{o}} \psi\big(t(x + y/t)\big)\, d\mu(x) = \psi(y) \int_{\varpi^n\mathfrak{o}} \psi(tx)\, d\mu(x). \qquad \square$$

Definition 4.19. Let V be a finite-dimensional vector space over F. We define the SCHWARTZ SPACE $\mathcal{S}(V)$ of V to be the complex vector space of locally constant functions $f : V \to \mathbb{C}$ of compact support.

Definition 4.20. Let $f \in \mathcal{S}(F)$. The FOURIER TRANSFORM $\hat{f}$ of f is defined by

$$\hat{f}(t) = \int_F f(x)\psi(tx)\, d\mu(x)$$

for all $t \in F$.

If $f \in \mathcal{S}(F)$, there exist $m, n \in \mathbb{Z}$ such that $\operatorname{supp} f \subseteq \varpi^m\mathfrak{o}$ and f is constant on cosets of $\varpi^n\mathfrak{o}$ in F. Thus f is bounded. It is easily seen from Lemma 4.18 that $\hat{f}(t) = \|f\|_1$ for sufficiently small t and $\hat{f}(t) = 0$ for sufficiently large t. Thus $\hat{f} \in \mathcal{S}(F)$.

Lemma 4.21. *Let $f, g \in \mathcal{S}(F)$. Then*

$$\int_F \hat{f}(x)g(x)\, d\mu(x) = \int_F f(y)\hat{g}(y)\, d\mu(y).$$

Proof. For all $x, y \in F$, $|f(y)g(x)\psi(xy)| \leq \|f\|_\infty \|g\|_\infty$, so

$$\int_F \left(\int_F |f(y)g(x)\psi(xy)| d\mu(y) \right) d\mu(x) \leq \mu(\operatorname{supp} f)\mu(\operatorname{supp} g)\|f\|_\infty\|g\|_\infty < \infty.$$

Thus by Fubini's theorem

$$\int_F \left(\int_F f(y)\psi(xy)\, d\mu(y) \right) g(x)\, d\mu(x) = \int_F f(y) \left(\int_F g(x)\psi(xy)\, d\mu(x) \right) d\mu(x). \qquad \square$$

Lemma 4.22. *For all $n \in \mathbb{Z}$ we have $\widehat{\mathbf{1}_{\varpi^n \mathfrak{o}}} = q^{-n}\mathbf{1}_{\varpi^{c-n}\mathfrak{o}}$.*

Proof. This is a direct consequence of Lemma 4.18. $\square$

For any $f : F \to \mathbb{C}$ and $y \in F$ the TRANSLATE f_y is defined by $f_y(x) = f(x-y)$ for all $x \in F$. From the definition of the Fourier transform, we obtain $\widehat{f_y}(t) = \psi(ty)\hat{f}(t)$.

Proposition 4.23 (Fourier inversion formula)**.** *Let $f \in \mathcal{S}(F)$. Then*

$$\hat{\hat{f}}(x) = q^{-c} f(-x)$$

for all $x \in F$.

Proof. The following integrals estimate $\hat{\hat{f}}(x)$: if $n \in \mathbb{Z}$,

$$\begin{aligned} \int_F \hat{f}(y)\psi(xy)\mathbf{1}_{\varpi^{-n}\mathfrak{o}}(y)\, d\mu(y) &= \int_F \widehat{f_x}(y)\mathbf{1}_{\varpi^{-n}\mathfrak{o}}(y)\, d\mu(y) \\ &= \int_F f_x(y)\widehat{\mathbf{1}_{\varpi^{-n}\mathfrak{o}}}(y)\, d\mu(y) \quad \text{by Lemma 4.21} \\ &= q^n \int_{\varpi^{c+n}\mathfrak{o}} f(y-x)\, d\mu(y) \quad \text{by Lemma 4.22} \\ &= q^{-c} f(-x) \quad \text{for sufficiently large } n. \end{aligned} \tag{4.2}$$

But the integrand on the left of (4.2) is dominated for all $n \in \mathbb{Z}$ by $|\hat{f}|$, which is integrable. By the Dominated Convergence Theorem, as $n \to \infty$ we have

$$\int_F \hat{f}(y)\psi(xy)\mathbf{1}_{\varpi^{-n}\mathfrak{o}}(y)\, d\mu(y) \to \int_F \hat{f}(y)\psi(xy)\, d\mu(y) = \hat{\hat{f}}(x). \qquad \square$$

Proposition 4.24. *If $f \in \mathcal{S}(F)$, then $\|\hat{f}\|_2^2 = q^{-c}\|f\|_2^2$.*

Proof. Denote by $\overline{f}$ the function $x \mapsto \overline{f(x)}$. It follows from the definition of the Fourier transform that $\overline{\hat{f}(x)} = \hat{\overline{f}}(-x)$ for all $x \in F$. Thus

$$\begin{aligned} \|\hat{f}\|_2^2 = \int_F \hat{f}(x)\overline{\hat{f}(x)}\, d\mu(x) &= \int_F \hat{f}(x)\hat{\overline{f}}(-x)\, d\mu(x) \\ &= \int_F f(x)\hat{\hat{\overline{f}}}(-x)\, d\mu(x) = q^{-c}\int_F f(x)\overline{f(x)} = q^{-c}\|f\|_2^2. \qquad \square \end{aligned}$$

We need to consider Fourier transforms on larger spaces than the Schwartz space. We will focus on L^2 spaces.

Definition 4.25. Let V be a finite-dimensional vector space over F and let ν be a Haar measure on V. For any measurable functions $f, g : V \to \mathbb{C}$ we define the CONVOLUTION $f * g$ of f and g with respect to the measure ν by

$$(f * g)(x) = \int_V f(x - y)g(y)\, d\nu(y)$$

for all x such that the integral is defined.

Proposition 4.26. *The Schwartz space $\mathcal{S}(F)$ is dense in $L^2(F)$. For any finite-dimensional vector space V over F, the Schwartz space $\mathcal{S}(V)$ is dense in $L^2(V)$.*

Proof. Let $f \in L^2(F)$. Then $f \cdot \mathbf{1}_K \in L^1$ for all compact $K \subseteq F$ by the Cauchy-Schwarz inequality. For each $n \in \mathbb{N}$ let $f_n = \mathbf{1}_{\varpi^{-n}\mathfrak{o}} \cdot (q^n \mathbf{1}_{\varpi^n \mathfrak{o}} * f)$. We note the convolution $(q^n \mathbf{1}_{\varpi^n \mathfrak{o}} * f)(x)$ is the integral of f over a compact subset of F, so $f_n(x)$ is defined for all x. Since $\mathbf{1}_{\varpi^n\mathfrak{o}}(x + t) = \mathbf{1}_{\varpi^n\mathfrak{o}}(x)$ for all $x \in F$ and $t \in \varpi^n\mathfrak{o}$, we have $f_n \in \mathcal{S}(F)$. We claim $f_n \to f$ in $L^2(F)$ as $n \to \infty$. Indeed, fix $\varepsilon > 0$. By the continuity of translation ([4], Proposition 2.41), there exists $n_0 \in \mathbb{N}$ such that $\|f - f_y\|_2^2 < \varepsilon/2$ whenever $y \in \varpi^{n_0}\mathfrak{o}$. By the Dominated Convergence Theorem, there exists $m_0 \in \mathbb{N}$ such that $\int_{F \setminus \varpi^{-n}\mathfrak{o}} |f|^2\, d\mu < \varepsilon/2$ for all $n > m_0$. Thus for all $n > \max\{n_0, m_0\}$ we have

$$\begin{aligned}
\int_F |f - f_n|^2\, d\mu &= \int_{F\setminus\varpi^{-n}\mathfrak{o}} |f|^2\, d\mu + \int_{\varpi^{-n}\mathfrak{o}} \left| f(x) - q^n \int_{\varpi^n\mathfrak{o}} f(x - y)\, d\mu(y) \right|^2 d\mu(x) \\
&< \frac{\varepsilon}{2} + q^{2n} \int_{\varpi^{-n}\mathfrak{o}} \left| \int_{\varpi^n\mathfrak{o}} \big(f(x) - f(x - y)\big)\, d\mu(y) \right|^2 d\mu(x) \\
&\overset{1}{\leq} \frac{\varepsilon}{2} + q^{2n} \int_{\varpi^{-n}\mathfrak{o}} q^{-n} \int_{\varpi^n\mathfrak{o}} \big|f(x) - f(x - y)\big|^2\, d\mu(y)\, d\mu(x) \\
&\overset{2}{=} \frac{\varepsilon}{2} + q^{n} \int_{\varpi^{n}\mathfrak{o}} \int_{\varpi^{-n}\mathfrak{o}} \big|f(x) - f_y(x)\big|^2\, d\mu(x)\, d\mu(y) \\
&\leq \frac{\varepsilon}{2} + q^n \int_{\varpi^n\mathfrak{o}} \frac{\varepsilon}{2}\, d\mu(y) = \varepsilon,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality at step 1 and Fubini's theorem at step 2. The statement for $\mathcal{S}(V)$ is proved by a very similar argument in multiple dimensions, replacing $\varpi^n\mathfrak{o}$ by a small compact open subgroup of V. □

4.4 A projective representation of $SL(2, F)$

For the remainder of this chapter, we will take F to be a non-Archimedean local field with odd residue characteristic. Let $\mathfrak{o}$ be the ring of integers of F, $\mathfrak{p}$ the unique maximal ideal of $\mathfrak{o}$, ϖ any generator of $\mathfrak{p}$, and $|\cdot|$ the standard absolute value on F. As in the finite field case, let V be a vector space over F of finite dimension d, and B a nondegenerate symmetric bilinear form on V. Let H be the corresponding Heisenberg group, as defined in Definition 2.11. Let ψ be a fixed nontrivial additive character of F.

Our aim is to obtain representations of $GL(2,F)$ by roughly the same steps as we used in the finite field case: first obtain a projective representation of $SL(2,F)$, lift to a true representation of $SL(2,F)$, then extend that representation to a representation of $GL(2,F)$. There are a few more obstacles along this path in the local field case, but the broad outline and conclusions will be very similar.

The first obstacle we encounter is that the finite Stone–von Neumann Theorem of Section 3.1 only applies to finite groups. We will state a theorem of Segal, Shale and Weil, which provides a similar result for a special class of locally compact abelian groups. (The proof of the theorem is beyond the scope of this essay.) This special class does not contain the Heisenberg group H, but it does contain a closely related group, $A(V)$, which we will define next.

Definition 4.27. Let $A(V)$ be the set $V \times V \times \mathbb{T}$, equipped with the multiplication

$$(u,v,t)(u',v',t') = \big(u+u', v+v', tt'\psi(-2B(v,u'))\big).$$

We can check this is a group: associativity follows from the fact that

$$\psi\big(-2B(v,u')\big)\psi\big(-2B(v+v',u'')\big) = \psi\big(-2B(v,u'+u'')\big)\psi\big(-2B(v',u'')\big)$$

for all $u,u',u'',v,v',v'' \in V$; the identity element is $(0,0,1)$; and the inverse of (u,v,t) is $\big(-u,-v,t^{-1}\psi(-2B(v,u))\big)$.

Remark 4.28. The group $A(G)$ can actually be defined for any locally compact abelian group G, by taking the underlying set to be $G^* \times G \times \mathbb{T}$, where G^* is the group of characters of G, and replacing $\psi(-2B(v,u'))$ in the multiplication rule with $u'(v)$. We will not need this level of generality.

There is a unitary representation ρ of $A(V)$ on $L^2(V)$ given by

$$\big(\rho(u',v',t')\Phi\big)(v) = t'\psi\big(-2B(v,u')\big)\Phi(v+v')$$

for all $(u',v',t') \in A(G)$ and $v \in V$. That this is in fact a representation reduces to the same identity as in Definition 4.27. It is easily checked that the representation is unitary: the operators $\rho(u',v',t')$ are invertible and

$$\int_V \big|t'\psi\big(-2B(v,u')\big)\Phi(v+v')\big|^2 d\nu(v) = \int_V \big|\Phi(v+v')\big|^2 d\nu(v) = \int_V \big|\Phi(v)\big|^2 d\nu(v)$$

for all $(u',v',t') \in A(V)$ and $\Phi \in L^2(V)$, by the translation-invariance of ν.

By an argument similar to that in Remark 2.12, the centre of $A(V)$ is $\{(0,0,t) \mid t \in \mathbb{T}\}$. Let B_0 be the group of automorphisms of $A(V)$ which fix every element of its centre.

Theorem 4.29 (Segal, Shale and Weil)**.** *The representation ρ has no closed invariant subspaces except for 0 and $L^2(V)$ itself. For each $\sigma \in B_0(V)$ there exists a unitary operator $\omega(\sigma)$ on $L^2(V)$, unique up to scalar multiple, such that*

$$\rho(\sigma \cdot h) = \omega(\sigma)\rho(h)\omega(\sigma)^{-1} \tag{4.3}$$

for all $h \in A(V)$.

Proof. See [1], Theorem 4.8.3 on p. 531, which follows Weil's proof in [10]. □

We would like to transfer this result to H. The first step is to find a homomorphism $\tau : H \to A(V)$. Because of the multiplication laws in H and $A(V)$ are defined slightly differently, we need a correction factor in the third coordinate; we define τ by

$$\tau(u,v,x) = \big(u,v,\psi(x - B(u,v))\big)$$

for all $(u,v,x) \in H$. This is indeed a homomorphism, because

$$\begin{aligned}\tau(u,v,x)\tau(u',v',x') &= \big(u,v,\psi(x - B(u,v))\big)\big(u',v',\psi(x' - B(u',v'))\big)\\ &= \big(u+u', v+v', \psi(x+x' - B(u,v) - B(u',v') - 2B(u',v)\big)\\ &= \tau\big(u+u', v+v', x+x' + B(u,v') - B(u',v)\big)\\ &= \tau\big((u,v,x)(u',v',x')\big)\end{aligned}$$

for all $(u,v,x),(u',v',x') \in H$.

Consequently, we have a representation $\pi = \rho\circ\tau$ of H on $L^2(V)$. This representation is given by the formula

$$\begin{aligned}\big(\pi(u',v',x')\Phi\big)(v) &= \big(\rho\big(u',v',\psi(x' - B(u',v'))\big)\Phi\big)(v)\\ &= \psi\big(x' - B(2v+v',u')\big)\Phi(v+v') \qquad (4.4)\end{aligned}$$

for $(u',v',x') \in H$, $\Phi \in L^2(V)$ and $v \in V$, which is identical to the representation given by (3.5) that we considered in the finite field case.

Recall the action of $SL(2,F)$ on H given in of Proposition 2.13. We would like to use Theorem 4.29 in conjunction with Proposition 2.13 to obtain a projective representation of $SL(2,F)$. In other words, we would like an action of $SL(2,F)$ on $A(V)$ by automorphisms which satisfies

$$g \cdot \tau(h) = \tau(g \cdot h)$$

for all $g \in SL(2,F)$ and $h \in H$, which would mean that

$$\begin{aligned}\begin{pmatrix} a & b\\ c & d\end{pmatrix} \cdot \big(u,v,\psi(x - B(u,v))\big) &= \tau(au+bv, cu+dv, x)\\ &= \big(au+bv, cu+dv, \psi(x - B(au+bv, cu+dv))\big)\end{aligned}$$

for all $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL(2,F)$ and $(u,v,x) \in H$. This motivates the following action of $SL(2,F)$ on $A(V)$:

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} \cdot (u,v,t) := \big(au+bv, cu+dv, t\psi(B(u,v) - B(au+bv,cu+dv))\big).$$

This is indeed an action, because the factor $\psi(-B(au+bv,cu+dv))$ in the third coordinate cancels out when we evaluate $\left(\begin{smallmatrix} a' & b'\\ c' & d'\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \cdot (u,v,x)$. It is also an action by group automorphisms, because

$$\big(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \cdot (u,v,t)\big)\big(\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \cdot (u',v',t')\big) = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \cdot \big(u+u', v+v', tt'\psi(-2B(v,u'))\big)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,F)$ and $(u,v,t),(u',v',t') \in A(V)$; checking this relation comes down to the fact that

$$\psi\big(B(u,v') - B(v,u') - B(au+bv, cu'+dv') + B(au'+bv', cu+dv)\big) \\ = \psi\big((1-ad+bc)B(u,v') + (-1-bc+ad)B(v,u')\big) = 1.$$

Note that these automorphisms fix the centre of $A(V)$, so $\sigma_g \in B_0(V)$ for all $g \in SL(2,F)$, where σ_g denotes the automorphism of $A(V)$ determined by g.

We can now use our action to study the representation π of H on $L^2(V)$. Indeed, from (4.3) there exists a unitary transformation $\eta(g) = \omega(\sigma_g)$ of $L^2(V)$, unique up to scalar multiple, such that

$$\rho\big(\tau(g\cdot h)\big) = \rho\big(g\cdot\tau(h)\big) = \eta(g)\rho\big(\tau(h)\big)\eta(g)^{-1}$$

for all $g \in SL(2,F)$ and $h \in H$, or in other words

$$\pi(g\cdot h)\eta(g) = \eta(g)\pi(h). \tag{4.5}$$

By the same argument as in Corollary 3.12, the images of $\eta(g)$ in $PGL\big(L^2(V)\big)$ for $g \in SL(2,F)$ form a projective representation of $SL(2,F)$.

The representation π of H on $L^2(V)$ is given by (4.4), which is the same equation as (3.5) in Section 3.2. Therefore, the reasoning of that section applies unchanged in this section to calculate the intertwiners $\eta(g)$ for $g \in SL(2,F)$, except that we must verify the resulting intertwiners are unitary. Consider any character $L_\bullet : a \mapsto L_a$ of $F^\times$, to be specified later. If we define

$$\left(\eta\begin{pmatrix} 1 & b \\ & 1\end{pmatrix}\Phi\right)(v) = \psi\big(bB(v,v)\big)\Phi(v)$$

$$\text{and}\quad \left(\eta\begin{pmatrix} a & \\ & 1/a\end{pmatrix}\Phi\right)(v) = L_a|a|^{d/2}\Phi(av),$$

for $\Phi \in L^2(V)$ and $v \in V$, then we claim that $\eta\left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right)$ and $\eta\left(\begin{smallmatrix} a & \\ & 1/a\end{smallmatrix}\right)$ are unitary operators on $L^2(V)$ satisfying (4.5). They are certainly invertible, since $\eta\left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right)$ and $\eta\left(\begin{smallmatrix} 1 & -b \\ & 1\end{smallmatrix}\right)$ are mutually inverse, and so are $\eta\left(\begin{smallmatrix} a & \\ & 1/a\end{smallmatrix}\right)$ and $\eta\left(\begin{smallmatrix} 1/a & \\ & a\end{smallmatrix}\right)$. It is clear that the results of applying these operators to $\Phi \in L^2(V)$ are themselves $\mathcal{A}$-measurable and square-integrable. That $\eta\left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right)$ is unitary is clear from the fact that $\big|\psi\big(bB(v,v)\big)\big| = 1$ for all $b \in F$ and $v \in V$. That $\eta\left(\begin{smallmatrix} a & \\ & 1/a\end{smallmatrix}\right)$ is unitary requires a calculation:

$$\int_V \big|L_a|a|^{d/2}\Phi(av)\big|^2 d\nu(v) = \int_V |\Phi(av)|^2|a|^d d\nu(v) = \int_V |\Phi(u)|^2 d\nu(u),$$

since $\nu(aS) = |a|^d\nu(S)$ for all Borel subsets S of V, where ν is the Haar measure on$V$; this follows from (4.1) and the fact that the product measure of μ in each of the d dimensions of V gives the Haar measure on V. So $\eta\left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right)$ and $\eta\left(\begin{smallmatrix} a & \\ & 1/a\end{smallmatrix}\right)$ (for $a \in F^\times$ and $b \in F$) are the unitary operators determined by (4.5), up to scalars.

Finding $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ is not quite as easy as in the finite field case, since we cannot simply consider every possible operator on the space $L^2(V)$. In the finite field case, we found that $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ was a finite Fourier transform operator. In the local field case, it turns out that the operator $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ is essentially the Fourier transform on $L^2(V)$.

For any $\Phi \in \mathcal{S}(V)$, we define the Fourier transform $\widehat{\Phi} : V \to V$ of Φ by

$$\widehat{\Phi}(v) = \int_V \psi\big(2B(u,v)\big)\Phi(u)\,d\nu(u) \tag{4.6}$$

for all $v \in V$. One can easily check that $\widehat{\Phi} \in \mathcal{S}(V)$. We will occasionally denote $\widehat{\Phi}$ by $\mathcal{F}\Phi$. We recall that we have not yet fixed the normalisation of the Haar measure ν, so we now specify ν to be normalised such that the Fourier inversion formula holds in the form $\big(\mathcal{F}\widehat{\Phi}\big)(v) = \Phi(-v)$ for all $\Phi \in \mathcal{S}(V)$. The measure ν is then called the SELF-DUAL Haar measure with respect to the pairing $(u,v) \mapsto \psi\big(2B(u,v)\big)$, and the Plancherel theorem holds in the form $\|\Phi\|_2 = \|\widehat{\Phi}\|_2$ for all $\Phi \in \mathcal{S}(V)$. The Plancherel theorem implies that this map can be extended uniquely to an isometric isomorphism $\mathcal{F} : L^2(V) \to L^2(V) : \Phi \mapsto \widehat{\Phi}$. See [4], Chapter 4, for details of this standard construction.

We set

$$\eta\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\Phi = K\widehat{\Phi} \tag{4.7}$$

for all $\Phi \in L^2(V)$, where $K \in \mathbb{T}$ is a constant yet to be determined. Since $\mathcal{F}$ is unitary, $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ is a unitary operator on $L^2(V)$.

To prove $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ satisfies (4.5), we note that for all $(u',v',x') \in H$, all $\Phi \in \mathcal{S}(V)$ and all $v \in V$ we have

$$\begin{aligned}
\Big(\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)\pi(u',v',x')\Phi\Big)(v) &= \int_V \psi\big(2B(u,v)\big)\psi\big(x' - B(2u+v',u')\big)\Phi(u+v')\,d\nu(u) \\
&\overset{1}{=} \int_V \psi\big(2B(u-v',v)\big)\psi\big(x' - B(2u-v',u')\big)\Phi(u)\,d\nu(u) \\
&= \psi\big(x' - B(2v-u',v')\big)\int_V \psi\big(2B(u,v-u')\big)\Phi(u)\,d\nu(u) \\
&= \Big(\pi\Big(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)\cdot(u',v',x')\Big)\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)\Phi\Big)(v),
\end{aligned}$$

where we translated the integrand by $-v'$ at step 1. Since the above equation holds for all Φ in $\mathcal{S}(V)$, since $\mathcal{S}(V)$ is dense in $L^2(V)$, and since $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ and $\pi(h)$ (for $h \in H$) are continuous, it follows that $\pi\Big(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)\cdot h\Big)\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)\Phi = \eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)\pi(h)\Phi$ for all $h \in H$ and all $\Phi \in L^2(V)$. We conclude that $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ is the operator determined up to scalars by (4.5).

4.5 A true representation of $SL(2,F)$

We must again check that the values of the constants in the definition of η can be chosen such that η becomes a true representation of $SL(2,F)$. It turns out that this can only be done if the dimension d of V is even, so we will henceforth assume that d

is even. We will again use the presentation in Proposition 2.15 to check whether η is a true representation.

Since the representing operators $\eta(g)$ for $g \in SL(2,F)$ are all continuous and since $\mathcal{S}(V)$ is a dense subspace of $L^2(V)$, it will suffice to prove the relations hold when applied to elements of $\mathcal{S}(V)$. This allows us to calculate Fourier transforms directly as integrals.

By the same calculations as in Section 3.3, the first three relations of (2.2) are satisfied by the images of $t(a)$ and $n(b)$ under η (for $a \in F^\times$ and $b \in F$), since we have already insisted that $L_\bullet : a \mapsto L_a$ be a character of $F^\times$. Thus we need only check the fourth and fifth relations, taking into account the new definition of $\eta\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ as a Fourier transform.

The fourth relation can be checked in a similar fashion to the finite field case: for all $\Phi \in \mathcal{S}(V)$ and $v \in V$,

$$\begin{aligned}\big(w \cdot t(a) \cdot w \cdot \Phi\big)(v) &= K \int_V L_a |a|^{d/2} K \psi\big(2B(u,v)\big) \widehat{\Phi}(au)\, d\nu(u) \\ &= K^2 L_a |a|^{-d/2} \int_V \psi\big(2B(au, v/a)\big) \widehat{\Phi}(au) |a|^d\, d\nu(u) \\ &= K^2 L_a |a|^{-d/2} \int_V \psi\big(2B(u', v/a)\big) \widehat{\Phi}(u')\, d\nu(u') \\ &= K^2 L_a |a|^{-d/2} \Phi(-v/a),\end{aligned}$$

by the Fourier inversion formula, whereas

$$\big(t(-1/a) \cdot \Phi\big)(v) = L_{-1/a} |a|^{-d/2} \Phi(-v/a).$$

Thus the fourth relation is satisfied if and only if $K^2 L_a = L_{-1/a}$ for all $a \in F^\times$. Since $L_\bullet$ is a character of $F^\times$, this is equivalent to requiring that $L_{a^2} = 1$ for all $a \in F^\times$ and $K^2 = L_{-1}$.

Finally, we consider the fifth relation. The approach we used in the finite field case fails here, because the function $u \mapsto \psi\big(bB(u,u)\big)$ (for $b \in F$) is not integrable over V.

Before we can state the key proposition for this section, we need two results about squares in $F^\times$. We follow the approach suggested in [5] (Chapter 2, §§1.6–7).

Proposition 4.30. *Let F be a non-Archimedean local field with odd residue characteristic. Let $F^{\times 2}$ denote the subgroup $\{x^2 \mid x \in F^\times\}$ of $F^\times$. Then $[F^\times : F^{\times 2}] = 4$.*

Proof. Each $x \in F^\times$ can be written uniquely in the form $x = \varpi^k y$ for some $k \in \mathbb{Z}$ and $y \in \mathfrak{o} \setminus \mathfrak{p}$. So all squares in $F^\times$ can be written in the form $x = \varpi^{2k} y^2$ for $k \in \mathbb{Z}$ and $y \in \mathfrak{o} \setminus \mathfrak{p}$, and clearly then $y^2 + \mathfrak{p}$ is a square in $(\mathfrak{o}/\mathfrak{p})^\times$. Conversely, if $y' \in \mathfrak{o} \setminus \mathfrak{p}$ and $y' + \mathfrak{p}$ is a square in $(\mathfrak{o}/\mathfrak{p})^\times$, then there exists $y \in \mathfrak{o}$ such that $y^2 = y'$ by Hensel's lemma (see [8], Proposition 7 on p. 34). It follows that the squares in $F^\times$ are exactly those $x = \varpi^k y$ such that k is even and $y + \mathfrak{p}$ is a square in $(\mathfrak{o}/\mathfrak{p})^\times$. Since $(\mathfrak{o}/\mathfrak{p})^\times$ is cyclic and of even order, its squares form a subgroup of index two. $\square$

Corollary 4.31. *Maintaining the assumptions of Proposition 4.30, let E be a quadratic extension of F, and denote by $N : E \to F$ the norm map corresponding to this extension. Then $[F^\times : N(E^\times)] = 2$.*

Proof. We can write $E = F(\sqrt{c})$ for some nonsquare $c \in F$; then the norm is given by $N(a + b\sqrt{c}) = a^2 - b^2 c$. It is then clear that $F^{\times 2} \le N(E^\times) \le F^\times$, so it will suffice to show that $F^{\times 2} \ne N(E^\times)$ and $N(E^\times) \ne F^\times$.

Let $\gamma \in \mathfrak{o} \setminus \mathfrak{p}$ be any element such that $\gamma + \mathfrak{p}$ generates the cyclic group $(\mathfrak{o}/\mathfrak{p})^\times$; then γ, ϖ and $\varpi\gamma$ are all nonsquare in $F^\times$. By Proposition 4.30, they are a complete set of coset representatives for $F^{\times 2}$ in $F^\times$. It suffices to consider the cases $E = F(\sqrt{c})$ for $c = \gamma$, ϖ and $\varpi\gamma$.

If $c = \varpi$ or $\varpi\gamma$, then $N(\sqrt{c}) = -c$ is not a square from $F^\times$. In the case where $c = \gamma$, we observe that $\gamma + \mathfrak{p}$ is not square in $\mathfrak{o}/\mathfrak{p}$, so there is an extension $\mathcal{E} = (\mathfrak{o}/\mathfrak{p})(\sqrt{\gamma + \mathfrak{p}})$ of $\mathfrak{o}/\mathfrak{p}$. By Lemma 3.19, there exist $x, y \in \mathfrak{o}$ such that $x^2 - y^2\gamma + \mathfrak{p} = \gamma + \mathfrak{p}$, which is not a square in $\mathfrak{o}/\mathfrak{p}$. In both cases, $F^{\times 2} \ne N(E^\times)$.

Suppose $c = \varpi$ or $\varpi\gamma$ and suppose $a^2 - b^2 c = \gamma$ for some $a, b \in F$. Since $|b^2 c| = q^{-2k-1}$ and $|a^2| = q^{-2l}$ for some $k, l \in \mathbb{Z}$, we must have $k \ge 0$. Then $a^2 + \mathfrak{p} = \gamma + \mathfrak{p}$, which is impossible. Now suppose $c = \gamma$ and suppose $a^2 - b^2\gamma = \varpi$ for some $a, b \in F$. Since $\mathfrak{p} = a^2 - b^2\gamma + \mathfrak{p}$ is a norm from $\mathcal{E}$, we must have $a, b \in \mathfrak{p}$, which means that $a^2 - b^2\gamma \in \varpi^2\mathfrak{o}$, a contradiction. In both cases, $N(E^\times) \ne F^\times$. □

By virtue of Corollary 4.31, for each quadratic extension E of F there exists a unique nontrivial character of $F^\times$ which is trivial on $N(E^\times)$, called the QUADRATIC CHARACTER of $F^\times$ attached to the extension E.

By Lemma 3.14, we can find a basis $b_1, \dots, b_d$ of V and constants $c_1, \dots, c_d \in F^\times$ such that $B(x_1 b_1 + \cdots + x_d b_d, y_1 b_1 + \cdots + y_d b_d) = c_1 x_1 y_1 + \cdots + c_d x_d y_d$ for all $x_i, y_i \in F$. Let $\Delta = (-1)^{d/2} c_1 \dots c_d$. Let ξ be the trivial character of $F^\times$ if Δ is a square in F, and the quadratic character of $F^\times$ attached to the extension $F(\sqrt{\Delta})$ otherwise.

For notational convenience, for all $b \in F$ we define $F_b : V \to \mathbb{C}$ by $F_b(v) = \psi\big(bB(v,v)\big)$ for all $v \in V$. The action of $\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)$ under η can then be written as $\eta\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)\Phi = F_b\Phi$, where juxtaposition of functions $V \to \mathbb{C}$ denotes pointwise multiplication.

Proposition 4.32. *For all $\Phi \in \mathcal{S}(V)$ and $b \in F$, the convolution $F_b * \Phi$ lies in $\mathcal{S}(V)$. In addition, there exists a constant $\delta \in \{\pm 1, \pm i\}$ satisfying $\delta^2 = \xi(-1)$, such that*

$$\widehat{F_b * \Phi} = \delta \xi(b) |b|^{-d/2} F_{-1/b} \widehat{\Phi}$$

for all $b \in F^\times$ and $\Phi \in \mathcal{S}(V)$.

Proof. Let V' be a finite-dimensional vector space over F, let B' be a nondegenerate symmetric bilinear form on V' and let ν' be the Haar measure on V' which is self-dual with respect to the pairing $\mathcal{P}' : (u, v) \mapsto \psi\big(2B'(u,v)\big)$. Denote by $\Phi *' \Psi$ the convolution of the functions Φ and Ψ with respect to ν', and denote by $\mathcal{F}'$ the Fourier transform on $\mathcal{S}(V')$ with respect to $\mathcal{P}'$. Let $f_{B'} : V' \to \mathbb{C}$ be the function given by

$f_{B'}(v) = \psi\big(B'(v,v)\big)$ for all $v \in V'$. (In particular, $F_b = f_{bB}$ for all $b \in F^\times$.) Then, for all $\Phi \in \mathcal{S}(V')$ and $v \in V'$,

$$\begin{aligned}(f_{B'} *' \Phi)(v) &= \int_V \psi\big(B'(v-u, v-u)\big)\Phi(u)\, d\nu'(u) \\ &= \psi\big(B'(v,v)\big) \int_V \psi\big(2B(u,-v) + B(u,u)\big)\Phi(u)\, d\nu'(u) = f_{B'}(v)\, \mathcal{F}'(f_{B'}\Phi)(-v). \quad (4.8)\end{aligned}$$

Now $f_{B'}$ and Φ are both locally constant functions of compact support, and $\mathcal{F}'$ preserves $\mathcal{S}(V')$, so $f_{B'} *' \Phi \in \mathcal{S}(V')$.

For each such choice of V' and B' we define a number $\gamma(B') \in \mathbb{T}$ using the projective representation η' of $SL(2,F)$ on $L^2(V')$, which is defined as in Section 4.4. The constants used in that representation are unimportant, but we will denote them by K' and L'_a (for $a \in F^\times$) to acknowledge that they depend on V' and B'. In our presentation for $SL(2,F)$ (Proposition 2.15), it is clear from the fourth and fifth relations that $wn(-1)w = n(1)wn(1)$ and hence that $n(-1)w = wn(1)t(-1)wn(1)$. We will apply η' to both sides of this equation and write the action of η' on $\mathcal{S}(V')$ by a raised dot. Since η' is a unitary projective representation, there exists $\gamma(B') \in \mathbb{T}$ such that

$$\begin{aligned}\mathcal{F}'(f_{B'} *' \Phi) &\overset{1}{=} \mathcal{F}'\big(K'^{-1} L'^{-1}_{-1}\, n(1) \cdot t(-1) \cdot w \cdot n(1) \cdot \Phi\big) \\ &= K'^{-2} L'^{-1}_{-1}\, w \cdot n(1) \cdot w \cdot t(-1) \cdot n(1) \cdot \Phi \\ &= \gamma(B')\, K'^{-1}\, n(-1) \cdot w \cdot \Phi \\ &= \gamma(B')\, f_{-B'}\, \mathcal{F}'\Phi \qquad (4.9)\end{aligned}$$

for all $\Phi \in \mathcal{S}(V')$, where step 1 follows from (4.8).

For any $b \in F^\times$, we can consider the special case where $V' = V$ and $B' = bB$. We have $\mathcal{F}'(f_{bB} *' \Phi) = \gamma(bB)\, f_{-bB}\, \mathcal{F}'\Phi$ for all $\Phi \in \mathcal{S}(V)$. To determine ν', note that

$$\begin{aligned}&\int_V \psi\big(2bB(u,v)\big)\left(\int_V \psi\big(2bB(w,u)\big)\Phi(w)\, d\nu(w)\right) d\nu(u) \\ &\overset{2}{=} |b|^{-d} \int_V \psi\big(2B(u',v)\big)\left(\int_V \psi\big(2B(w,u')\big)\Phi(w)\, d\nu(w)\right) d\nu(u') = |b|^{-d}\Phi(-v)\end{aligned}$$

for all $\Phi \in \mathcal{S}(V)$ and $v \in V$ by the Fourier inversion formula, where we substituted $u' = bu$ at step 2. Thus we have $\nu' = |b|^{d/2}\nu$, since ν' is self-dual with respect to $\mathcal{P}'$. Thus $|b|^{d/2}\mathcal{F}'(f_{bB} * \Phi) = \gamma(bB)\, f_{-bB}\, \mathcal{F}'\Phi$ for all $\Phi \in \mathcal{S}(V)$, where $*$ denotes convolution with respect to ν. Evaluating this equation at v/b, we obtain $\mathcal{F}(f_{bB} * \Phi)(v) = |b|^{-d/2}\gamma(bB)\, f_{-b^{-1}B}(v)\, \mathcal{F}\Phi(v)$ for all $v \in V$. Let $\delta = \gamma(B)$. It now suffices to prove that $\gamma(bB) = \xi(b)\gamma(B)$ for all $b \in F^\times$.

We first need some properties of γ. For $i = 1, 2$, let V_i be a finite-dimensional F-vector space, B_i a nondegenerate symmetric bilinear form on V_i, ν_i the corresponding self-dual Haar measure on V_i and $\mathcal{F}_i$ the corresponding Fourier transform on $\mathcal{S}(V_i)$. Define the symmetric bilinear form B' on $V' := V_1 \oplus V_2$ by $B'\big((u_1,u_2),(v_1,v_2)\big) = B_1(u_1,v_1) + B_2(u_2,v_2)$, and let $\mathcal{F}'$ be the corresponding Fourier transform. It is easy to check that B' is nondegenerate and $\nu' = \nu_1 \times \nu_2$ is the self-dual Haar measure on V' with respect to B'. For any functions $\Phi_i : V_i \to \mathbb{C}$ (for $i = 1, 2$), consider the function $\Phi_1 \otimes \Phi_2 : V' \to \mathbb{C}$ defined by $(\Phi_1 \otimes \Phi_2)(v_1, v_2) = \Phi_1(v_1)\Phi_2(v_2)$. We easily see that $f_{B'} = f_{B_1} \otimes f_{B_2}$. By Fubini's theorem,

$$\mathcal{F}'\big(f_{B'} * (\Phi_1 \otimes \Phi_2)\big) = \mathcal{F}'\big((f_{B_1} * \Phi_1) \otimes (f_{B_2} * \Phi_2)\big) = \mathcal{F}_1(f_{B_1} * \Phi_1) \otimes \mathcal{F}_2(f_{B_2} * \Phi_2)$$

for all $\Phi_1 \in \mathcal{S}(V_1)$ and $\Phi_2 \in \mathcal{S}(V_2)$. Therefore, by (4.9),

$$\gamma(B') = \gamma(B_1)\gamma(B_2). \tag{4.10}$$

Also note that $\overline{f_{B'}(v)} = f_{-B'}(v) = f_{-B'}(-v)$ and $\overline{\mathcal{F}'\Phi(v)} = \mathcal{F}'\Phi(-v)$ for all $\Phi \in \mathcal{S}(V')$ and $v \in V'$, so taking conjugates of (4.9) shows that $\gamma(B')^{-1} = \overline{\gamma(B')} = \gamma(-B')$.

We introduce the notation $\mathrm{QF}(c_1, \ldots, c_k)$ for a k-dimensional vector space over F, together with a symmetric bilinear form which can be diagonalised to the matrix $\mathrm{diag}(c_1, \ldots, c_k)$.

We also introduce the HILBERT SYMBOL $(\cdot, \cdot)$ on $F^\times$. For $a, b \in F^\times$, we set $(a, b) = 1$ if there exist $x, y, z, w \in F$, not all zero, such that $x^2 - ay^2 - bz^2 + abw^2 = 0$; otherwise we set $(a, b) = -1$. The Hilbert symbol is clearly symmetric in its arguments. If b is a square, then $(a, b) = 1$ for all $a \in F^\times$. If b is not a square, then $(a, b) = 1$ if and only if a is a norm from $F(\sqrt{b})$, because if x, y, z, w are not all zero then $x^2 - ay^2 - bz^2 + abw^2 = 0$ is equivalent to $a = N\big((x + z\sqrt{b})/(y + w\sqrt{b})\big)$. As a special case, we see that $(a, \Delta) = \xi(a)$ for all $a \in F^\times$. For all $b \in F^\times$, $a \mapsto (a, b)$ is a character of $F^\times$, so the Hilbert symbol is bilinear. If $b = -a$ then $(x, y, z, w) = (0, 1, 1, 0)$ is a solution, so $(a, -a) = 1$.

We claim that

$$\gamma\big(\mathrm{QF}(1, -a, -b, ab)\big) = (a, b) \tag{4.11}$$

for all $a, b \in F^\times$. Suppose that $(a, b) = 1$. If b is a square, then a change of basis gives $\mathrm{QF}(1, -a, -b, ab) \cong \mathrm{QF}(1, -a, -1, a)$, so (4.10) gives

$$\gamma\big(\mathrm{QF}(1, -a, -b, ab)\big) = \gamma\big(\mathrm{QF}(1, -a)\big)\gamma\big(\mathrm{QF}(-1, a)\big) = 1.$$

Otherwise a is a norm from $F(\sqrt{b})$, so we can find $x_0, z_0 \in F$, not both zero, such that $a = x_0^2 - bz_0^2$. Then

$$\begin{aligned} x^2 - ay^2 - bz^2 + cw^2 &= x^2 - bz^2 - (x_0^2 - bz_0^2)(y^2 - bw^2) \\ &= x^2 - bz^2 - (x_0 y + bz_0 w)^2 + b(x_0 w + z_0 y)^2, \end{aligned}$$

so a change of basis gives $\mathrm{QF}(1, -a, -b, ab) \cong \mathrm{QF}(1, -b, -1, b)$. Thus, once again,

$$\gamma\big(\mathrm{QF}(1, -a, -b, ab)\big) = \gamma\big(\mathrm{QF}(1, -b)\big)\gamma\big(\mathrm{QF}(-1, b)\big) = 1.$$

Suppose now that $(a, b) = -1$. Consider the QUATERNION ALGEBRA Q, which is defined to be an F-vector space with basis $\{1, i, j, k\}$, subject to the multiplication rules $1x = x1 = x$ for $x = i, j, k$, as well as $i^2 = a$, $j^2 = b$, $k^2 = -ab$, $ij = -ji = k$, $jk = -kj = -bi$, and $ki = -ik = -aj$. This is an associative algebra with 1. Let

Q have the product topology of four copies of F. If we define the conjugate of an element $x+yi+zj+wk$ to be $\overline{x+yi+zj+wk} = x-yi-zj-wk$, it is clear from the definition that $\alpha\overline{\alpha} = \overline{\alpha}\alpha = x^2-ay^2-bz^2+abw^2$ for all $\alpha = x+yi+zj+wk \in Q$. The value $\alpha\overline{\alpha}$ can be regarded as an element of F; it is known as the REDUCED NORM of α, and is denoted $\operatorname{Nrd}(\alpha)$. It is easy to check that $\operatorname{Nrd}(\alpha\beta) = \operatorname{Nrd}(\alpha)\operatorname{Nrd}(\beta)$ for all $\alpha, \beta \in Q$. There is an obvious nondegenerate symmetric bilinear form B' on Q such that $B'(\alpha,\alpha) = \operatorname{Nrd}(\alpha)$; under that bilinear form, $Q \cong \operatorname{QF}(1,-a,-b,ab)$. Since $(a,b) = -1$, nonzero elements have nonzero reduced norms, so every nonzero element α of Q has a left and right inverse $\operatorname{Nrd}(\alpha)^{-1}\overline{\alpha}$. Thus Q is in fact a division algebra. Let $U = \operatorname{Nrd}^{-1}(\varpi^{-m}\mathfrak{o})$, where m is an integer that will be specified later, and let

$$U' = \{v \in Q \mid \psi\big(2B'(u,v)\big) = 1 \text{ for all } u \in U\}.$$

Let $x_0 \in F$ such that $\psi(x_0) \neq 1$. For $t = 1, i, j, k$ we can find $a_t \in E$ such that $2B'(t, a_t) = x_0$, since B' is nondegenerate. Note that $2B'(x_1+x_i i+x_j j+x_k k, a_t x_t^{-1} t) = x_0$ for all $t \in \{1,i,j,k\}$ and $x_1, x_i, x_j, x_k \in F$. Thus if $x_1 + x_i i + x_j j + x_k k \in U'$ then $a_t x_t^{-1} t \notin U$ for all $t \in \{1,i,j,k\}$. In particular, $a_t^2 x_t^{-2}\operatorname{Nrd}(t) = \operatorname{Nrd}(a_t x_t^{-1} t) \notin \varpi^{-m}\mathfrak{o}$ and hence $|x_t| < q^{-m/2}|a_t||\operatorname{Nrd}(t)|^{1/2}$ for all $t \in \{1,i,j,k\}$. Therefore we may choose m large enough that $\psi\big(\operatorname{Nrd}(v)\big) = 1$ for all $v \in U'$. Now

$$\begin{aligned} f_{B'} *' \mathbf{1}_{U'}(v) &= \int_{U'} f_{B'}(v-u)\,d\nu'(u) = f_{B'}(v)\int_{U'} \psi\big(-2B(u,v)\big) f_{B'}(u)\,d\nu'(u) \\ &\overset{1}{=} f_{B'}(v)\int_{U'} \psi\big(-2B(u,v)\big)\,d\nu'(u) \overset{2}{=} \nu'(U') f_{B'}(v)\mathbf{1}_U(v) \end{aligned}$$

for all $v \in Q$, where step 1 follows because $f_{B'}(u) = 1$ for all $u \in U'$, and step 2 follows by definition of U'. Now evaluating (4.9) at zero with $\Phi = \mathbf{1}_{U'}$ and dividing by $\nu'(U')$ yields $\int_U f_{B'}(v)\,d\nu'(v) = \gamma(B')$. This will allow us to calculate $\gamma(B')$. It is easy to check that left-multiplication by $x+yi+zj+wk$ in Q corresponds to multiplication by a matrix over F whose determinant is $(x^2-ay^2-bz^2+abw^2)^2$. It follows that the measure λ on $Q^\times$ defined by $\lambda(S) = \int_S |\operatorname{Nrd}(v)|^{-2}\,d\nu'(v)$ is invariant under left-translation by elements of $Q^\times$; in other words, it is a left multiplicative Haar measure on $Q^\times$. So we can calculate that

$$\gamma(B') = \int_U f_{B'}(v)|\operatorname{Nrd}(v)|^2\,d\lambda(v) \overset{1}{=} C\int_{\varpi^{-m}\mathfrak{o}} \psi(x)|x|^2\,d\widetilde{\mu}(x) = C\int_{\varpi^{-m}\mathfrak{o}} \psi(x)|x|\,d\mu(x)$$

for some positive constant C, where $\widetilde{\mu}$ is the multiplicative Haar measure on $F^\times$ given by $\widetilde{\mu}(S) = \int_S |x|^{-1}\,d\mu(x)$. Step 1 is justified by the fact that $\operatorname{Nrd} : Q^\times \to F^\times$ is a group homomorphism and the integrand is constant on the cosets of its kernel in $Q^\times$. We evaluate the above integral by subdividing the domain into subsets on which $|x|$ is constant. Let c be the smallest integer such that $\psi(\varpi^c\mathfrak{o}) = 1$. By Lemma 4.18, the integral $I_n = \int_{\varpi^n\mathfrak{o}} \psi(x)\,d\mu(x)$ is equal to q^{-n} if $n \geq c$ and zero otherwise. Thus

$$\int_{\varpi^n\mathfrak{o}\setminus\varpi^{n+1}\mathfrak{o}} \psi(x)|x|\,d\mu(x) = |\varpi^n|(I_n - I_{n+1}) = \begin{cases} q^{-n}(q^{-n}-q^{-n-1}) & \text{if } n \geq c; \\ q^{-c+1}(-q^{-c}) & \text{if } n = c-1; \text{ and} \\ 0 & \text{if } n \leq c-2. \end{cases}$$

We may assume m was chosen large enough that $-m \le c-1$. Then

$$\gamma(B') = C\Big(-q^{-2c+1} + (1-q^{-1})\sum_{n=c}^{\infty} q^{-2n}\Big) = C\Big(-q^{-2c+1} + \frac{q^{-2c}}{1+q^{-1}}\Big) = -C\frac{q^{-2c+1}}{1+q^{-1}},$$

which is negative. Since $|\gamma(B')| = 1$, we conclude that $\gamma(B') = -1$, so (4.11) is proved.

Now (4.11) implies that $\gamma\big(\mathrm{QF}(1,ab)\big) = (a,b)\gamma\big(\mathrm{QF}(a,b)\big)$ for $a,b \in F^\times$. To prove that $\gamma(bB) = (b,\Delta)\gamma(B)$, by writing V as a direct sum of two-dimensional subspaces it will suffice to prove that $\gamma\big(\mathrm{QF}(bc_1,bc_2)\big) = (b,-c_1c_2)\gamma\big(\mathrm{QF}(c_1,c_2)\big)$ for $b, c_1, c_2 \in F^\times$. Indeed,

$$\begin{aligned}\gamma\big(\mathrm{QF}(bc_1,bc_2)\big) &= (bc_1,bc_2)\gamma\big(\mathrm{QF}(1,b^2c_1c_2)\big)\\ &= (b,-b)(b,-c_1)(b,c_2)(c_1,c_2)\gamma\big(\mathrm{QF}(1,c_1c_2)\big)\\ &= (b,-c_1c_2)\gamma\big(\mathrm{QF}(c_1,c_2)\big).\end{aligned}$$

Finally, we must prove that $\delta^2 = \xi(-1)$, or in other words that $\gamma(B)^2 = (\Delta,-1)$. Again decomposing V into two-dimensional subspaces, it will suffice to prove that $\gamma\big(\mathrm{QF}(c_1,c_2)\big)^2 = (-c_1c_2,-1)$. By (4.11),

$$\gamma\big(\mathrm{QF}(c_1,c_2)\big)^2 = \gamma\big(\mathrm{QF}(1,c_1c_2)\big)^2 = \gamma\big(\mathrm{QF}(1,1,c_1c_2,c_1c_2)\big) = (-1,-c_1c_2). \qquad \square$$

If we make the definitions $L_a := \xi(a)$ and $K := \delta$, then our necessary and sufficient criteria for the first four relations to hold, namely that $K^2 = L_{-1}$ and $L_{a^2} = 1$ for all $a \in F^\times$, are satisfied. Furthermore, the fifth relation holds with these definitions, as we can readily verify, for any $\Phi \in \mathcal{S}(V)$, $v \in V$ and $b \in F^\times$:

$$\begin{aligned}\big(w\cdot n(b)\cdot w\cdot \Phi\big)(v) &= \big(w\cdot(\delta F_b\widehat{\Phi})\big)(v)\\ &= \big(w\cdot \xi(-b)|b|^{-d/2}(F_{-1/b}*\Phi)\widehat{\ }\big)(v)\\ &= \delta\xi(-b)|b|^{-d/2}(F_{-1/b}*\Phi)(-v),\end{aligned}$$

by Proposition 4.32 and the Fourier inversion formula, whereas

$$\begin{aligned}&\big(t(-1/b)\cdot n(-b)\cdot w\cdot n(-1/b)\cdot\Phi\big)(v)\\ &\quad= \xi(-b)|b|^{-d/2}F_{-b}(-v/b)\delta\widehat{F_{-1/b}\Phi}(-v/b)\\ &\quad= \delta\xi(-b)|b|^{-d/2}\psi\big(-B(v,v)/b\big)\int_V \psi\big(2B(u,-v/b)-B(u,u)/b\big)\Phi(u)\,d\nu(u)\\ &\quad= \delta\xi(-b)|b|^{-d/2}\int_V \psi\big(-B(-v-u,-v-u)/b\big)\Phi(u)\,d\nu(u)\\ &\quad= \delta\xi(-b)|b|^{-d/2}(F_{-1/b}*\Phi)(-v).\end{aligned}$$

Using the fact that $t(-1/b)\cdot n(-b)\cdot w\cdot n(-1/b)\cdot\Phi = n(-1/b)\cdot w\cdot n(-b)\cdot t(-b)\cdot\Phi$ for all $b \in F^\times$ and $\Phi \in \mathcal{S}(V)$, which follows from the earlier relations, we conclude that the fifth relation holds. So the map η specified above on generators of $SL(2,F)$ extends to a unique representation of $SL(2,F)$ on $L^2(V)$.

4.6 Extending the representation to $GL(2, F)$

By Proposition 4.30, the norm map of a quadratic extension of F is not surjective. Thus is not possible to construct a representation of $GL(2, F)$ directly in the same way as in Section 3.5. We instead extend the representation η obtained in the previous section to the subgroup of $GL(2, F)$ consisting of matrices whose determinants *are* norms, and then use the induced representation.

Let E be a quadratic extension of F. Denote by $a \mapsto \overline{a}$ the nontrivial Galois automorphism of E fixing F, and denote the corresponding trace and norm maps by $\operatorname{tr}(x) = x + \overline{x}$ and $N(x) = x\overline{x}$. Let $E_1^\times$ be the set of elements of E of norm 1. Let χ be a quasicharacter of $E^\times$ such that $\chi(E_1^\times) \neq 1$. We specialise η to the case where $V = E$ and $B(u, v) = \frac{1}{2}\operatorname{tr}(u\overline{v})$. The corresponding Fourier transform $\widehat{\Phi}$ of $\Phi \in \mathcal{S}(E)$ is given by $\widehat{\Phi}(v) = \int_E \psi\big(\operatorname{tr}(u\overline{v})\big)\Phi(u)\, d\nu(u)$ for all $v \in E$, where ν is the self-dual measure on E with respect to the pairing $(u, v) \mapsto \psi\big(\operatorname{tr}(u\overline{v})\big)$.

We can write $E = F(\sqrt{\Delta'})$ for some nonsquare $\Delta' \in F$; then $N(a + b\sqrt{\Delta'}) = a^2 - b^2\Delta'$, so in the terminology of Section 4.5 we have $\Delta = -1(-\Delta') = \Delta'$. It follows that ξ (as defined in that section) is in fact the quadratic character of $F^\times$ attached to the extension E.

Let $GL(2, F)_+$ be the subgroup of $GL(2, F)$ (of index two) consisting of all the matrices whose determinants lie in $N(E^\times)$.

Consider the subspace U_χ of $\mathcal{S}(E)$ defined by

$$U_\chi = \big\{\Phi \in \mathcal{S}(E) \;\big|\; \Phi(tx) = \chi(t)^{-1}\Phi(x) \text{ for all } t \in E_1^\times\big\}.$$

Theorem 4.33. *There exists an admissible representation θ_χ of $GL(2, F)_+$ on U_χ such that, for all $\Phi \in U_\chi$ and $v \in E$,*

$$\left(\theta_\chi\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\Phi\right)(v) = \psi\big(bN(v)\big)\Phi(v) \qquad \text{for } b \in F,$$

$$\left(\theta_\chi\begin{pmatrix}a & \\ & 1/a\end{pmatrix}\Phi\right)(v) = |a|\xi(a)\Phi(av) \qquad \text{for } a \in F^\times,$$

$$\left(\theta_\chi\begin{pmatrix}a & \\ & 1\end{pmatrix}\Phi\right)(v) = |a|^{1/2}\chi(c)\Phi(cv) \qquad \begin{array}{c}\text{for } a \in N(E^\times),\\ \text{where } N(c) = a,\end{array}$$

$$\text{and} \qquad \theta_\chi\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\Phi = \delta\widehat{\Phi},$$

where $\delta \in \mathbb{T}$ is the constant determined by Proposition 4.32. The representation $\theta_\chi^{GL(2,F)}$ of $GL(2, F)$ induced from θ_χ is also admissible.

Proof. By essentially the same calculations as in the finite field case, U_χ is invariant under the action of η.

We apply Lemma 3.20 once again to determine whether the representations of $SL(2, F)$ and $P := \big\{\left(\begin{smallmatrix}a & \\ & 1\end{smallmatrix}\right) \;\big|\; a \in N(E^\times)\big\}$ on U_χ are compatible. If we once again denote the matrix $\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ by $c(a)$, and denote the actions of $SL(2,F)$ and P both by a raised dot, then we easily calculate that, if $N(c)=a$,

$$\begin{aligned}
\big(s(a)\cdot n(b)\cdot s(1/a)\cdot\Phi\big)(v) &= |a|^{1/2}\chi(c)\psi\big(bN(cv)\big)\big(s(1/a)\cdot\Phi\big)(cv)\\
&= |a|^{1/2}\chi(c)\psi\big(abN(v)\big)|a|^{-1/2}\chi(1/c)\Phi(v) = \big(n(ab)\cdot\Phi\big)(v);\\
\big(s(a)\cdot t(A)\cdot s(1/a)\cdot\Phi\big)(v) &= |a|^{1/2}\chi(c)|A|\xi(A)|a|^{-1/2}\chi(1/c)\Phi(cAc^{-1}v)\\
&= \big(t(A)\cdot\Phi\big)(v);\ \text{and}\\
\big(s(a)\cdot w\cdot s(1/a)\cdot\Phi\big)(v) &= |a|^{1/2}\chi(c)\,\delta\int_E\psi\big(\operatorname{tr}(u\overline{cv})\big)|a|^{-1/2}\chi(1/c)\Phi(u/c)\,d\nu(u)\\
&\overset{1}{=}\delta\int_E\psi\big(\operatorname{tr}(u'\overline{av})\big)\Phi(u')\,|N(c)|d\nu(u')\\
&= |a|\xi(a)\,\delta\int_E\psi\big(\operatorname{tr}(u'\overline{av})\big)\Phi(u')\,d\nu(u') = \big(t(a)\cdot w\cdot\Phi\big)(v),
\end{aligned}$$

where we substituted $u=cu'$ at the step labelled 1. The constant obtained in this step requires some explanation. When we substitute $u=cu'$ we are performing a linear transformation of the F-vector space E. We can write $c=C+D\sqrt{\Delta}$ for some $C,D\in F$, not both zero. The matrix of the transformation $u'\mapsto cu'$ with respect to the basis $(1,\sqrt{\Delta})$ of E is then $\left(\begin{smallmatrix} C & D\Delta \\ D & C \end{smallmatrix}\right)$, which has determinant $C^2-D^2\Delta=N(c)$. Thus the integral transforms according to the rule $d\nu(cu')=|N(c)|d\nu(u')$. By Lemma 3.20, there exists a unique representation θ_χ satisfying the stated equations.

We now check that θ_χ is smooth. Let $\Phi\in U_\chi$. It will suffice to find a neighbourhood U of the identity matrix in $GL(2,F)$ such that $\theta_\chi(g)\Phi=\Phi$ for all $g\in U$, because then $US\subseteq S$, where S denotes the stabiliser of Φ in $GL(2,F)$, and this proves that S is open. This prompts us to consider the matrix decomposition

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \\ & d\end{pmatrix}\begin{pmatrix} 1 & bd\\ & 1\end{pmatrix}\begin{pmatrix} ad-bc & \\ & 1\end{pmatrix}\begin{pmatrix} -1 & \\ & -1\end{pmatrix}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix} 1 & -\frac{c}{d}\\ & 1\end{pmatrix}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}, \tag{4.12}$$

which holds for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in GL(2,F)$ with $d\neq 0$. Since $\widehat{\Phi}\in\mathcal{S}(E)$, $\operatorname{supp}\widehat{\Phi}$ is bounded, and hence there exists $k\in\mathbb{N}$ such that $\psi\big(-\frac{c}{d}N(v)\big)=1$ whenever $c\in\varpi^k\mathfrak{o}$, $d\in\mathfrak{o}$ and $v\in\operatorname{supp}\widehat{\Phi}$, since ψ is locally constant. Thus for all $c\in\varpi^k\mathfrak{o}$ and $d\in\mathfrak{o}$ we have $\mathcal{F}\big(F_{-c/d}\widehat{\Phi}\big)=\mathcal{F}\big(\widehat{\Phi}\big)$, and hence

$$t(-1)\cdot w\cdot n(-c/d)\cdot w\cdot\Phi = \delta^2\xi(-1)\Phi = \Phi \tag{4.13}$$

by the Fourier inversion formula.

Since χ is locally constant, there exists $k'\in\mathbb{N}$ such that $\chi(c')=1$ whenever $N(c')\in 1+\varpi^{k'}\mathfrak{o}$. In addition, since Φ is locally constant and of compact support, there exists a finite set I and constants $x_i\in E$ and $n_i\in\mathbb{N}$ for $i\in I$ such that Φ is constant on each of the sets $U_i=\{x\in E\mid N(x-x_i)\in\varpi^{n_i}\mathfrak{o}\}$ and such that $\operatorname{supp}\Phi\subseteq\bigcup_{i\in I}U_i$. Let m be an integer such that $N(\operatorname{supp}\Phi)\subseteq\varpi^m\mathfrak{o}$. Now if $k''\in\mathbb{N}$ and $c'\in 1+\varpi^{k''}\mathfrak{o}$ then $N(c'v-v)=N(v)N(c'-1)\in\varpi^{m+k''}\mathfrak{o}$ for all $v\in\operatorname{supp}\Phi$. Setting $k''=\max_{i\in I}n_i-m$, we see that if $c\in 1+\varpi^{k''}\mathfrak{o}$ and $v\in U_i$ then

$$|N(c'v-x_i)|\leq\max\{|N(c'v-v)|,|N(v-x_i)|\}\leq q^{-n_i}$$

by the ultrametric inequality, and therefore $\Phi(c'v) = \Phi(x_i) = \Phi(v)$. Finally, there exists $k''' \in \mathbb{N}$ such that $\psi\big(bdN(v)\big) = 1$ whenever $b \in \varpi^{k''}\mathfrak{o}$ and $d \in \mathfrak{o}$.

Let $n = \max\{k, k', k'', k'''\}$. Now suppose $a, d \in 1 + \varpi^n\mathfrak{o}$ and $b, c \in \varpi^n\mathfrak{o}$. We have $|ad - bc| = 1$, $|1/d| = 1$, $ad - bc \in 1 + \varpi^n\mathfrak{o}$ and $N(a) = a^2 \in 1 + \varpi^n\mathfrak{o}$. Also, $1/d + \mathfrak{o} = 1 + \mathfrak{o}$, so by Hensel's lemma $1/d$ is a square in $F^\times$, so $\xi(1/d) = 1$. Putting all the above facts together, we conclude that $\big(t(1/d) \cdot n(bd) \cdot s(ad - bc) \cdot \Phi\big)(v) = \Phi(v)$ for all $v \in \operatorname{supp}\Phi$. Combining this with (4.12) and (4.13), we conclude that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \Phi = \Phi$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, F)$ such that $\left\|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) - I\right\|_\infty \leq q^{-n}$. Thus θ_χ is smooth.

Now we show that θ_χ is admissible. Let H be an open subgroup of $GL(2, F)$, and let $W = \{\Phi \in U_\chi \mid \theta_\chi(h)\Phi = \Phi \text{ for all } h \in H\}$. Since ψ is nontrivial and locally constant, we can fix some $x_0 \in F$ such that $\psi(x_0) \neq 1$ and some nonzero $x_1 \in F$ such that $\psi(x) = 1$ whenever $|x| < |x_1|$. Since H is open, there exists $n \in \mathbb{N}$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H$ whenever $a, d \in 1 + \varpi^n\mathfrak{o}$ and $b, c \in \varpi^n\mathfrak{o}$. Pick any $\Phi \in W$. On the one hand, $n(b) \cdot \Phi = \theta_\chi\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)\Phi = \Phi$ for all $b \in \varpi^n\mathfrak{o}$, which means that $\psi\big(bN(v)\big) = 1$ for all $v \in \operatorname{supp}\Phi$ and $b \in \varpi^n\mathfrak{o}$. Since $\psi(x_0) \neq 1$, it follows that $N(v) \neq x_0/b$ for all $v \in \operatorname{supp}\Phi$ and $b \in \varpi^n\mathfrak{o}$, and so $|N(v)| < q^n|x_0|$ for all $v \in \operatorname{supp}\Phi$. Thus there is a closed and bounded set $K \subseteq E$, independent of Φ, such that $\operatorname{supp}\Phi \subseteq K$ for all $\Phi \in W$. On the other hand, $t(-1) \cdot w \cdot n(b) \cdot w \cdot \Phi = \theta_\chi\left(\begin{smallmatrix} 1 & \\ -b & 1 \end{smallmatrix}\right)\Phi = \Phi$ for all $b \in \varpi^n\mathfrak{o}$. Acting on both sides by w, we see that $n(b) \cdot w \cdot \Phi = w \cdot \Phi$ for all $b \in \varpi^n\mathfrak{o}$, which shows that $\operatorname{supp}\widehat{\Phi} \subseteq K$ for all $\Phi \in W$, by the same argument that showed $\operatorname{supp}\Phi \subseteq K$. Note that if $u = x + y\sqrt{\Delta}$ and $v = z + w\sqrt{\Delta}$ then $tr(u\overline{v}) = xz - yw\Delta$, and so $|tr(u\overline{v})| \leq \max\{|x||z|, |y||w||\Delta|\}$. Since K is bounded, we can find an open neighbourhood U of 0 in E such that $|\operatorname{tr}(u\overline{v})| \leq |x_1|$ whenever $u \in K$ and $v \in U$, so in particular $\psi\big(\operatorname{tr}(u\overline{v})\big) = 1$ whenever $u \in K$ and $v \in U$. Now, for all $\Phi \in W$, the Fourier inversion formula implies that

$$\Phi(-v - v') = \int_K \psi\big(\operatorname{tr}(u\overline{v})\big)\psi\big(\operatorname{tr}(u\overline{v'})\big)\widehat{\Phi}(u)\,d\nu(u) = \int_K \psi\big(\operatorname{tr}(u\overline{v})\big)\widehat{\Phi}(u)\,d\nu(u) = \Phi(-v)$$

for all $\Phi \in W$, $v \in E$ and $v' \in U$. For each $v \in K$ let $E_v = \{u \in E \mid u - v \in U\}$; these sets are open and cover K. Since K is closed and bounded, it is a closed subset of the compact set $\varpi^{k_0}\mathfrak{o}$ for some $k_0 \in \mathbb{Z}$, so K is compact. Thus we can find a finite set $K' \subseteq K$ such that the E_v for $v \in K'$ cover K. But, for all $\Phi \in W$, Φ is constant on each E_v and has support in K, so Φ is determined by the values $\Phi(v)$ for v in the finite set K'. Thus $\dim W < \infty$. We have now proved that θ_χ is admissible.

Finally, it is easy to check, from the definition of the induced representation, that the representation induced from an admissible representation of an index two subgroup is itself admissible. Thus $\theta_\chi^{GL(2,F)}$ is admissible. □

The representation $\theta_\chi^{GL(2,F)}$ is called a *dihedral* representation. With further work, it can be shown that the dihedral representations have a property called *supercuspidality* analogous to Definition 3.24, and that they are irreducible—see [1], Theorem 4.8.6 on pp. 541–2. These representations are remarkably close analogues of the cuspidal representations $\theta(\chi)$ discussed in Section 3.6.

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