Reversed radial SLE and the Brownian loop measure

Laurence S. Field* Gregory F. Lawler†

February 28, 2013

Abstract

The Brownian loop measure is a conformally invariant measure on loops in the plane that arises when studying the Schramm–Loewner evolution (SLE). When an SLE curve in a domain evolves from an interior point, it is natural to consider the loops that hit the curve and leave the domain, but their measure is infinite. We show that there is a related normalized quantity that is finite and invariant under Möbius transformations of the plane. We estimate this quantity when the curve is small and the domain simply connected. We then use this estimate to prove a formula for the Radon–Nikodym derivative of reversed radial SLE with respect to whole-plane SLE.

1 Introduction

Oded Schramm introduced the Schramm–Loewner evolution (SLE) in [11] as a one-parameter family of random curves defined in simply connected complex domains. The parameter $\kappa > 0$ determines the local behaviour of the curve. Schramm considered three types of SLE: *chordal* SLE, which connects two boundary points, *radial* SLE, which connects a boundary point to an interior point, and *whole-plane* SLE, which connects two points on the Riemann sphere. These random curves have two defining properties. First, they are invariant under conformal transformations of the domain. Second,

*University of Chicago.
†University of Chicago; research supported by NSF grant DMS-0907143.

they satisfy the domain Markov property: given an initial segment of the curve, the remainder of the curve follows the law of SLE in the slit domain. If $0 < \kappa \leq 4$, which is what we consider in this paper, the curves are simple.

As in [3, 6], we will consider SLE_κ in a domain D, when $\kappa \leq 4$, as a measure $\mu_D(z, w)$ on simple curves in D connecting z and w. Here z, w can be interior or boundary points, but if z or w is a boundary point, we assume that the boundary is locally analytic there. The measures are conformally covariant; that is, if $f : D \to f(D)$ is a conformal transformation, then

$$f \circ \mu_D(z, w) = |f'(z)|^{b_z} \, |f'(w)|^{b_w} \, \mu_{f(D)}(f(z), f(w)), \tag{1.1}$$

where b_z and b_w are the *boundary scaling exponent*

$$b := \frac{6 - \kappa}{2\kappa},$$

or the *interior scaling exponent*

$$\tilde{b} := \frac{(\kappa - 2)\, b}{4},$$

depending on which kind of point z and w are. We write the total mass of the measure $\mu_D(z, w)$ as $\Psi_D(z, w)$ and call it the SLE *partition function*. See also Dubédat's work [1] for a slightly different notion of SLE partition function in the context of the Gaussian free field.

Indeed, in many examples one can obtain the SLE partition function (at least conjecturally) as a normalized limit of partition functions of discrete measures. It is known that if $\kappa \leq 8/3$, or if D is simply connected or doubly connected, then $\Psi_D(z, w) < \infty$. It is conjectured that this is true for all D for $\kappa \leq 4$. If $\Psi_D(z, w) < \infty$, then we define $\mu_D^{\#}(z, w)$ to be the corresponding probability measure obtained by normalizing. The measure $\mu_D^{\#}(z, w)$ is conformally *invariant* and hence can be defined for nonsmooth boundary points provided that there exists a conformal transformation $f : D \to f(D)$ such that $f(z), f(w)$ are smooth boundary points and $\Psi_{f(D)}(f(z), f(w)) < \infty$.

Suppose $z \in \partial D$ is a smooth boundary point, and suppose that D_1 is a subdomain of D that agrees with it in a neighborhood of z. Let us compare $\mu := \mu_D(z, w)$ with $\mu_1 := \mu_{D_1}(z, w_1)$. Here w, w_1 can be either boundary or interior points. Let t be a stopping time for the SLE paths such that $\gamma_t := \gamma(0, t]$ lies in D_1. Then μ and μ_1 considered as measures on initial segments γ_t are mutually absolutely continuous with Radon-Nikodym derivative

$$\frac{d\mu_1}{d\mu}\,(\gamma_t) = \frac{\Psi_{D_1 \smallsetminus \gamma_t}(\gamma(t), w_1)}{\Psi_{D \smallsetminus \gamma_t}(\gamma(t), w)} \, \exp\Big\{\frac{\mathbf{c}}{2} \Lambda(\gamma_t, D \smallsetminus D_1; D)\Big\}. \tag{1.2}$$

We now explain the terms here. First, the partition functions $\Psi_{D_1\setminus\gamma_t}(\gamma(t),\tilde{w})$ and $\Psi_{D\setminus\gamma_t}(\gamma(t),w)$ do not exist because $D\setminus\gamma_t$ is not locally analytic at $\gamma(t)$. However, the ratio of partition functions is well defined using the rule (1.1),

$$\frac{\Psi_{D_1\setminus\gamma_t}(\gamma(t),\tilde{w})}{\Psi_{D\setminus\gamma_t}(\gamma(t),w)}=\frac{|f'(w_1)|^{b_{w_1}}\,\Psi_{f(D_1\setminus\gamma_t)}(f(\gamma(t)),f(w_1))}{|f'(w)|^{b_w}\,\Psi_{f(D\setminus\gamma_t)}(f(\gamma(t)),f(w))},\tag{1.3}$$

where $f:D\setminus\gamma_t\to f(D\setminus\gamma_t)$ is a conformal transformation. The parameter $\mathbf{c}=(3\kappa-8)b$ is the *central charge* (a parameter from conformal field theory) and $\Lambda(\gamma_t,D\setminus D_1;D)$ is a geometric quantity, namely, the Brownian loop measure (introduced in [7]) of the set of loops in D which intersect both γ_t and $D\setminus D_1$. For chordal SLE in simply connected domains we could equivalently consider Werner's $\mathrm{SLE}_{8/3}$ loop measure [12] for Λ, which amounts to considering the outer boundaries of the Brownian loops. For reversed radial SLE as presented in this paper, and for SLE in multiply connected domains [6], one sees topologically that it is no longer equivalent to consider the $\mathrm{SLE}_{8/3}$ loop measure for Λ; the Brownian loop measure must be used instead.

If D,D_1 are simply connected, $w_1=w\in\partial D$, and ∂D and ∂D_1 agree in a neighborhood of w, then we can let $t=\infty$ and see that $\mu_1\ll\mu$ with

$$\frac{d\mu_1}{d\mu}(\gamma)=1\{\gamma\subset D_1\}\exp\Big\{\frac{\mathbf{c}}{2}\Lambda(\gamma,D\setminus D_1;D)\Big\}.\tag{1.4}$$

This formula inspired the definition of SLE_κ in multiply connected domains that appeared in [6]. Essentially, for general domains D with $z\in D$ and $w\in D$ or w a smooth boundary point, one defines $\mu_D(z,w)$ so that (1.4) and the conformal covariance rule (1.1) hold across all domains. The consistency of this definition is easy to check. The definition does not immediately establish that it is a finite measure, but this has been proved for $\kappa\le 8/3$ (in which case $\mathbf{c}\le 0$) and for simply and doubly connected domains if $8/3<\kappa\le 4$.

To prove these results, one considers the case $D=\mathbb{H}$, $z=0$, $w=\infty$ with (by normalization) $\Psi_{\mathbb{H}}(0,\infty)=1$. Then, given an initial segment γ_t, let $g_t:\mathbb{H}\setminus\gamma_t\to\mathbb{H}$ be a conformal transformation with $g_t(z)=z+o(1)$ as $z\to\infty$. Then, $g'(\infty)=1$ and the ratio in (1.3) becomes

$$K_t:=\frac{\Psi_{D_1\setminus\gamma_t}(\gamma_t,w_1)}{\Psi_{\mathbb{H}\setminus\gamma_t}(0,\infty)}=|g_t'(w_1)|^{b_{w_1}}\,\Psi_{g_t(D_t)}(U_t,f(w_1)),$$

where $U_t=g_t(\gamma(t))$. The probability measure $\mu^{\#}_{D_1}(0,w_1)$ is obtained by weighting by K_t using the Girsanov theorem. Since K_t is not a local martingale, one must first multiply by a compensator, and computing this gives the Brownian loop term. That is to say, the term in (1.2) considered as a function of t is a local martingale. When one uses the Girsanov theorem, one finds that one can define the probability measure $\mu^{\#}_{D_1}(0, w_1)$ as a solution to the Loewner equation with a driving function U_t that has a drift. These new processes are sometimes called SLE(κ, ρ) processes. The method described in this paragraph was introduced in [8] in a slightly different form.

Another reason to consider SLE from the partition function point of view is to compare it to discrete models at criticality. One expects the discrete models that converge to SLE to have partition functions which converge to the SLE partition functions when one normalizes by a power of the lattice spacing. This agrees with the power-law conformal covariance rule for SLE partition functions. Moreover, the Radon–Nikodym derivative (1.2) of two SLE measures on an initial segment is entirely analogous to what occurs in families of finite measures on paths in a discrete lattice such as self-avoiding walk, loop-erased walk, λ–self-avoiding walk and the percolation exploration process. Each of these families has a weight function W such that its measures $\mu_D(z, w)$ are related by the first-step decomposition

$$\mu_D(z, w) = \sum W(D, [z, \zeta])\ [z, \zeta] \oplus \mu_{D \smallsetminus [z,\zeta]}(\zeta, w),$$

in which the sum runs over all vertices ζ that adjoin the starting point z. In such families we have the Radon–Nikodym derivative

$$\frac{d\mu_{D_1}(z, w_1)}{d\mu_D(z, w)}(\gamma_t) = \frac{W(D_1, \gamma_t)}{W(D, \gamma_t)}\ \frac{\Psi_{D_1 \smallsetminus \gamma_t}(\gamma(t), w_1)}{\Psi_{D \smallsetminus \gamma_t}(\gamma(t), w)},$$

where

$$W(D, \gamma_t) = \prod_{s=0}^{t-1} W(D \smallsetminus \gamma_s, [\gamma(s), \gamma(s+1)]).$$

In fact, in the discrete models mentioned above, this last quantity is a functional of the random walk loop measure.

If z, w are boundary points and D is simply connected, Dapeng Zhan showed in [13] that $\mu_D(w, z)$ can be obtained from $\mu_D(z, w)$ by reversing the paths. The argument also shows that in the probability measure $\mu_D(z, w)$, the conditional distribution given both an initial segment and a terminal segment is chordal SLE in the slit domain connecting the interior endpoints of the paths. If $z \in D$ and $w \in \partial D$, it was suggested in [3] to define $\mu_D(w, z)$ to be the reversal of radial $\mu_D(z, w)$. This definition was validated by Zhan [14], who constructed a probability measure on curves connecting boundary points on an annulus. These measures satisfy the condition that, given an initial and a terminal segment, radial SLE is distributed like annulus SLE in the remaining domain. In [6] it was shown that this measure is the same as the probability measure defined in [3]. In particular, it was shown that the annulus partition function is finite.

In this paper, we take a different, but as we show equivalent, approach to defining reversed radial SLE by giving its Radon-Nikodym derivative with respect to whole-plane SLE. Here we are using whole-plane SLE as the natural base measure for paths starting at an interior point in the same way that chordal SLE in $\mathbb{H}$ is the base measure for SLE starting at a boundary point. Motivated by formulas such as (1.2), we would like to be able to write the Radon-Nikodym derivative of reversed radial SLE with respect to whole-plane SLE as

$$\frac{d\mu_D(0,w)}{d\mu_{\mathbb{C}}(0,\infty)}(\gamma_T) = \exp\Big\{\frac{\mathbf{c}}{2}\Lambda(\gamma_T, \mathbb{C}\smallsetminus D; \mathbb{C})\Big\}\frac{\Psi_{D\smallsetminus\gamma_T}(\gamma(T),w)}{\Psi_{\mathbb{C}\smallsetminus\gamma_T}(\gamma(T),\infty)}. \tag{1.5}$$

This is false as written, because $\Lambda(\gamma_T, \mathbb{C}\smallsetminus D; \mathbb{C})$ is infinite.

To make sense of this Radon–Nikodym derivative, we introduce a finite normalized quantity $\Lambda^*(\gamma_T, \mathbb{C}\smallsetminus D)$ based on the loop measure, which has many of the properties we want. This is similar in spirit to "Wick products".

Theorem 1.1. *If V_1, V_2 are disjoint nonpolar closed subsets of the Riemann sphere, then the limit*

$$\Lambda^*(V_1,V_2) = \lim_{r\downarrow 0}[\Lambda(V_1,V_2;\mathcal{O}_r) - \log\log(1/r)], \tag{1.6}$$

exists where

$$\mathcal{O}_r = \{z\in\mathbb{C} : |z| > r\}.$$

Moreover, if f is a Möbius transformation of the Riemann sphere,

$$\Lambda^*(f(V_1), f(V_2)) = \Lambda^*(V_1,V_2).$$

We could write the assumption "disjoint nonpolar closed subsets of the Riemann sphere" as "disjoint closed subsets of $\mathbb{C}$, at least one of which is compact, such that Brownian motion hits both subsets at some positive time".

Roughly stated, the Brownian loop measure is infinite both because of short loops and because of long loops. We remark that the loop measure term

$\Lambda(V_1, V_2; \mathcal{O}_r)$ is necessarily finite in (1.6): intuitively, having V_1, V_2 disjoint prevents short loops, and long loops are very likely also to leave $\mathcal{O}_r$ at some point and hence not contribute to the loop measure term.

By Möbius invariance, $\Lambda^*(V_1, V_2)$ could equally well be defined by shrinking down around a point other than the origin, or by replacing $\Lambda(V_1, V_2; \mathcal{O}_r)$ with the mass of loops hitting V_1 and V_2 that stay in a disk of large radius $1/r$ as $r \downarrow 0$. Indeed, this is how we prove the Möbius invariance in Sect. 4.

Having introduced Λ^*, we can reformulate (1.5) correctly.

Theorem 1.2. *Let $\kappa \leq 4$. Let D be a simply connected domain containing 0 and $w \in \partial D$ a smooth boundary point. Let T be a stopping time for whole-plane SLE$_\kappa$ from 0 to ∞ such that γ does not leave D by time T. Then the Radon-Nikodym derivative of reversed radial SLE$_\kappa$ with respect to whole-plane SLE$_\kappa$ up to time T is*

$$\frac{d\mu_D(0,w)}{d\mu_{\mathbb{C}}(0,\infty)}(\gamma_T) = c_1 \exp\Big\{\frac{\mathbf{c}}{2}\Lambda^*(\gamma_T, \mathbb{C} \smallsetminus D)\Big\} \frac{\Psi_{D \smallsetminus \gamma_T}(\gamma(T), w)}{\Psi_{\mathbb{C} \smallsetminus \gamma_T}(\gamma(T), \infty)}, \tag{1.7}$$

where c_1 is a constant depending only on κ.

This theorem is also relevant to boundary/bulk SLE, the natural generalization of radial SLE to non–simply connected domains.

Corollary 1.3. *Let D be a complex domain containing 0, not necessarily simply connected, and $w \in \partial D$ a smooth boundary point. Then the measure $\mu_D(0, w)$ defined by (1.7) is the reversal of boundary/bulk SLE $\mu_D(w, 0)$, as defined in [6].*

We now describe the structure of the paper. In Sect. 2 we introduce some notation and present a number of preliminary results about Brownian motion, conformal mapping, and SLE, many of which have been proved elsewhere. In Sect. 3 we deal with reversed radial SLE and prove Theorem 1.2. In Sect. 4 we deal with the normalized loop measure independently of any SLE notions and prove Theorem 1.1. Proposition 3.3, which is an estimate for the normalized loop measure of loops hitting both boundary components of a conformal annulus, is proved in Sect. 4.4. This estimate is more precise than Theorem 1.2's proof requires, and may be of independent interest.

2 Preliminary results

We will use the following notation:

$$\mathbb{D}_r = \{z : |z| < r\}, \qquad \mathbb{D} = \mathbb{D}_1, \qquad \mathbb{H} = \{z : \operatorname{Im} z > 0\},$$
$$\mathcal{O}_r = \{z : |z| > r\}, \qquad \mathcal{O} = \mathcal{O}_1, \qquad \mathcal{O}_r(w) = w + \mathcal{O}_r,$$
$$A_{r,R} = \mathbb{D}_R \cap \mathcal{O}_r = \{z : r < |z| < R\}, \qquad A_R = A_{1,R},$$
$$C_r = \partial \mathbb{D}_r = \partial \mathcal{O}_r = \{z : |z| = r\}, \qquad C_r(w) = \partial \mathcal{O}_r(w) = \{z : |w - z| = r\}.$$

For $S \subset \mathbb{C}$, we denote the complement $\mathbb{C} \setminus S$ by S^c.

The implicit constants in all $O(\cdot)$ terms are universal unless otherwise stated. The constants in a $O_r(\cdot)$ term may depend on r but not on any other quantity. The notation $x \asymp y$ means that there is a universal constant $c > 0$ such that $c^{-1} < x/y < c$.

2.1 Complex Brownian motion

We say that a subset V of $\mathbb{C}$ is *nonpolar* if it is hit by Brownian motion. More precisely, V is nonpolar if for every $z \in \mathbb{C}$, the probability that a Brownian motion starting at z hits V is positive. Since Brownian motion is recurrent we can replace "is positive" with "equals one". In a slight abuse of terminology, we will call a domain (connected open subset) D of $\mathbb{C}$ nonpolar if ∂D is nonpolar.

2.1.1 Harmonic measure and excursion measure

If B_t is a complex Brownian motion and D is a domain, let

$$\tau_D = \inf\{t : B_t \notin D\}.$$

A domain D is nonpolar if and only if $\mathbf{P}^z\{\tau_D < \infty\} = 1$ for every z. In this case we define harmonic measure of D at $z \in D$ by

$$h_D(z, V) = \mathbf{P}^z\{B_{\tau_D} \in V\}.$$

If V is smooth then we can write

$$h_D(z, V) = \int_V h_D(z, w)\, |dw|,$$

where $h_D(z,w)$ is the *Poisson kernel.* If $z \in \partial D \setminus V$ and ∂D is smooth near z, we define the *excursion measure* of V in D from z by

$$\mathcal{E}_D(z,V) = \mathcal{E}(z,V;D) = \partial_{\mathbf{n}} h_D(z,V),$$

where $\mathbf{n} = \mathbf{n}_{z,D}$ denotes the unit inward normal at z. If V is smooth, we can write

$$\mathcal{E}_D(z,V) = \int_V h_{\partial D}(z,w)\,|dw|,$$

where $h_{\partial D}(z,w) := \partial_{\mathbf{n}} h_D(z,w)$ is the *excursion* or *boundary Poisson kernel.* (Here the derivative $\partial_{\mathbf{n}}$ is applied to the first variable.) One can also obtain the excursion Poisson kernel as the normal derivative in both variables of the Green's function; this establishes symmetry, $h_{\partial D}(z,w) = h_{\partial D}(w,z)$. If $f : D \to f(D)$ is a conformal transformation, then (assuming smoothness of f at boundary points at which f' is taken)

$$\begin{aligned} h_D(z,V) &= h_{f(D)}(f(z), f(V)), \\ h_D(z,w) &= |f'(w)|\, h_{f(D)}(f(z), f(w)), \\ \mathcal{E}_D(z,V) &= |f'(z)|\, \mathcal{E}_{f(D)}(f(z), f(V)), \\ h_{\partial D}(z,w) &= |f'(z)|\,|f'(w)|\, h_{\partial f(D)}(f(z), f(w)). \end{aligned}$$

2.1.2 Brownian bubble measure

If D is a nonpolar domain and $z \in \partial D$ is an analytic boundary point (i.e., ∂D is analytic in a neighborhood of z), the Brownian bubble measure $m_D(z)$ in D at z is a sigma-finite measure on loops $\gamma : [0,t_\gamma] \to \mathbb{C}$ with $\gamma(0) = \gamma(t_\gamma) = z$ and $\gamma(0,t_\gamma) \subset D$. It can be defined as the limit as $\epsilon \downarrow 0$ of $\pi\, \epsilon^{-1}\, h_D(z+\epsilon\mathbf{n}, z)$ times the probability measure on paths obtained from starting a Brownian motion at $z+\epsilon\mathbf{n}$ and conditioning so that the path leaves D at z. Here $\mathbf{n} = \mathbf{n}_{z,D}$ is the inward unit normal. If $\tilde{D} \subset D$ agrees with D in a neighborhood of z, then the bubble measure in $\tilde{D}$ at z, $m_{\tilde{D}}(z)$, is obtained from $m_D(z)$ by restriction. This is also an infinite measure but the difference $m_D(z) - m_{\tilde{D}}(z)$ is a finite measure. We will denote its total mass by

$$m(z; D, \tilde{D}) = \|m_D(z) - m_{\tilde{D}}(z)\|.$$

The normalization of m is chosen so that

$$m(0; \mathbb{H}, \mathbb{H} \cap \mathbb{D}) = 1. \tag{2.1}$$

Remark 2.1. The factor of π in the bubble measure was put in so that (2.1) holds. However, the loop measure in the next section does not have this factor, so we will have to divide it out again. For this paper, it would have been easier to have defined the bubble measure without the π but we will keep it in order to match definitions elsewhere.

The bubble measure is conformally covariant [7]: if $f : D \to f(D)$ is conformal and $z \in \partial D$ and $f(z)$ are smooth boundary points, then

$$f \circ m_D(z) = |f'(z)|^2 \, m_{f(D)}(f(z)). \tag{2.2}$$

2.1.3 Brownian loop measure

Definition 2.2. A *rooted loop* in a domain $D \subset \mathbb{C}$ is a continuous map $\gamma : [0, t_\gamma] \to D$ with $t_\gamma > 0$ and $\gamma(0) = \gamma(t_\gamma)$. Its *root* is $\gamma(0)$.

An *unrooted loop* in D is an equivalence class of rooted loops in D under the equivalence $\gamma \sim \gamma_s$ for all s, where $\gamma_s(t) = \gamma(s+t)$ (considering γ as a t_γ-periodic function) and $t_{\gamma_s} = t_\gamma$.

The Brownian loop measure μ_D^{loop} is a sigma-finite measure on unrooted loops in a domain D. The measure $\mu_{\mathbb{C}}^{\text{loop}}$ can be defined as follows.

- Consider the measure on triples $(z, t_\gamma, \tilde{\gamma})$ given by

$$\text{area} \times \frac{dt}{2\pi t^2} \times (\text{length 1 Brownian bridge from 0 in } \mathbb{C}) \, .$$

- Let

$$\gamma(s) = z + \sqrt{t_\gamma} \, \tilde{\gamma}(s/t_\gamma).$$

- Project this measure onto unrooted loops by forgetting the root.

Then μ_D^{loop} is defined to be $\mu_{\mathbb{C}}^{\text{loop}}$ restricted to loops in D.

We have defined the measure so that it satisfies the restriction property: if $D' \subset D$, then $\mu_{D'}^{\text{loop}}$ is μ_D^{loop} restricted to loops in D'. The other important feature of the Brownian loop measure is its conformal invariance, which was proved in [7], Proposition 6.

Proposition 2.3. *If $f : D \to f(D)$ is a conformal map, then $f \circ \mu_D^{\text{loop}} = \mu_{f(D)}^{\text{loop}}$.*

For computational purposes it is useful to write the measure in terms of the bubble measure, which can be done in many ways. We will use the following expression, which assigns to each unrooted loop the root furthest from the origin:

$$\mu^{\rm loop}_{\mathbb{C}} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} m_{\mathbb{D}_r}(re^{i\theta})\, r\, dr\, d\theta. \tag{2.3}$$

(For a proof, see [7], Proposition 7, and apply (2.2).) To be precise, we are considering the right hand side as a measure on unrooted loops. We can also assign to each unrooted loop the root closest to the origin:

$$\mu^{\rm loop}_{\mathbb{C}} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} m_{\mathcal{O}_r}(re^{i\theta})\, r\, dr\, d\theta. \tag{2.4}$$

If $\overline{\mathbb{D}}_r \subset D$, then the Brownian loop measure in D restricted to loops that intersect $\overline{\mathbb{D}}_r$ can be written as

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^{r} m_{D_s}(se^{i\theta})\, s\, ds\, d\theta, \tag{2.5}$$

where $D_s = D \cap \mathcal{O}_s$. If $r_1 < r$, then the Brownian loop measure restricted to loops in $\mathcal{O}_{r_1}$ that intersect $\overline{\mathbb{D}}_r$ is given by

$$\frac{1}{\pi}\int_0^{2\pi}\int_{r_1}^{r} m_{D_s}(se^{i\theta})\, s\, ds\, d\theta.$$

Using this and appropriate properties of the bubble measure we can conclude the following.

Lemma 2.4. *For every $0 < s < r < \infty$ and $d > 0$, the loop measure of the set of loops in $\mathcal{O}_s$ of diameter at least d that intersect $\mathbb{D}_r$ is finite.*

Remark 2.5. This result is not true for $s = 0$. The Brownian loop measure of loops in $\mathbb{C}$ of diameter greater than d that intersect the unit disk is infinite. See, e.g., Lemma 4.17 below.

Conformal invariance implies that the Brownian loop measure of loops in $A_{r,2r}$ that separate the origin from infinity is the same for all r. It is easy to see that this measure is positive and the last lemma shows that it is finite.

It follows that the measure of the set of loops that surround the origin is infinite.

If $V_1, V_2, \ldots$ are closed subsets of the Riemann sphere and D is a nonpolar domain, then

$$\Lambda(V_1, V_2, \ldots, V_k; D)$$

is defined to be the loop measure of the set of loops in D that intersect all of the sets $V_1, \ldots, V_k$. Note that

$$\Lambda(V_1, V_2, \ldots, V_k; D) \\ = \Lambda(V_1, V_2, \ldots, V_{k+1}; D) + \Lambda(V_1, V_2, \ldots, V_k; D \setminus V_{k+1}). \quad (2.6)$$

If $V_1, V_2, \ldots, V_k$ are the traces of simple curves that pass through the origin, then the comment in the last paragraph shows that for all $r > 0$,

$$\Lambda(V_1, V_2, \ldots, V_k; \mathbb{D}_r) = \infty.$$

2.2 Conformal mapping

2.2.1 Univalent functions and capacity

A *univalent function* is a one-to-one holomorphic function. We will need the following version of the growth and distortion theorems for univalent functions. For a proof, see [2], Theorem 3.21 and Proposition 3.30.

Proposition 2.6. *If f is univalent on $\mathcal{O}_\rho$, $f(\infty) = \infty$ and $f'(\infty) = 1$, then if $r = \rho/|z| < 1$,*

$$f(z) = z + O(\rho),$$

$$\left(\frac{1-r}{1+r}\right)^3 \leq |f'(z)| \leq \left(\frac{1+r}{1-r}\right)^3.$$

We will use the fact that $|f(z)/z|$ and $|f'(z)|$ are both $1 + O(r)$ as $r \to 0$, uniformly in f. By the Koebe 1/4–theorem, this is as true for f^{-1} as for f.

Definition 2.7. A *hull* is a compact, connected set $K \subset \mathbb{C}$ larger than a single point. We denote by g_K the unique conformal map $g_K : \mathbb{C} \setminus K \to \mathcal{O}_t$, for some $t > 0$, with $g_K(\infty) = \infty$ and $g_K'(\infty) = 1$. The *capacity* of K is defined by $\operatorname{cap} K = \log t$.

Remark 2.8. If $0 \in K$ and $\operatorname{cap} K = \log t$, then the *radius* of K, $\operatorname{rad} K := \max\{|k| : k \in K\}$, lies in $[t, 4t]$ by the Schwarz lemma and the Koebe 1/4–theorem. In particular, $|g_K(z)/z|$ and $|g_K'(z)|$ are $1 + O(t/|z|)$ as $t/|z| \to 0$.

2.2.2 Conformal annuli

In this section we let $\delta_t = 1/\log(1/t)$, let $\mathcal{D}$ denote the set of simply connected domains D containing the origin with $\operatorname{dist}(0, \partial D) = 1$, and let $\mathcal{H}_t$ denote the set of hulls $K \subset \mathbb{D}$ of capacity $\log t$ containing the origin.

If $D \in \mathcal{D}$, let $\psi = \psi_D : D \to \mathbb{D}$ denote the unique conformal transformation with $\psi(0) = 0$ and $\psi'(0) > 0$. If $D \in \mathcal{D}$ and $K \in \mathcal{H}_t$, let ϕ denote a conformal transformation $\phi = \phi_{D,K} : D \setminus K \to A_{s,1}$. It is well known that this ϕ is defined uniquely up to a final rotation, and in particular, $s = s_{D,K}$ is a uniquely defined number reflecting the conformal type of the conformal annulus $D \setminus K$. We recall the classical fact that nested conformal annuli have nested values of s.

Lemma 2.9. *In this situation, $s \asymp t \asymp \operatorname{rad} K$. In particular, the expressions $O(s)$ and $O(t)$ are interchangeable, as are "$s \to 0$" and "$t \to 0$".*

Proof. Let $r = \operatorname{rad} K$. Remark 2.8 provides the bound $t \asymp r$.

Since $A_{r,1} \subset D \setminus K$, $s \le r$. Applying ψ and using the Koebe 1/4-theorem, it suffices to prove that $t = O(s)$ in the case $D = \mathbb{D}$. Now $g_K(\mathbb{D} \setminus K) \subset A_{t,1+O(t)} \subset A_{t,O(1)}$ by Proposition 2.6, and thus $t = O(s)$. □

Lemma 2.10. *Let $K \in \mathcal{H}_t$, $s = s_{\mathbb{D},K}$, $0 < r \le |z| < 1$, $|w| = 1$ and $t \to 0$. Then*

$$s = t\,[1 + O(t)],$$
$$h_{\mathbb{D}\setminus K}(z, K) = \delta_t \,\log|1/z|\,[1 + O_r(t)],$$
$$\mathcal{E}_{\mathbb{D}\setminus K}(w, K) = \delta_t\,[1 + O(t)].$$

Proof. Since

$$h_{A_{r,1}}(z, C_r) = \frac{\log|1/z|}{\log(1/r)},$$

it suffices to consider $|z| = r$. By conformal invariance of Brownian motion,

$$h_{\mathbb{D}\setminus K}(z, K) = h_{g_K(\mathbb{D}\setminus K)}(g_K(z), C_t).$$

By Proposition 2.6, $|g_K(y)| = 1+O(t)$ for $y \in C_1$, and $|g_K(z)| = r\,[1+O_r(t)]$. It follows that

$$A_{t,1-O(t)} \subset g_K(\mathbb{D} \setminus K) \subset A_{t,1+O(t)}.$$

We conclude that $s = t\,[1 + O(t)]$ and

$$h_{\mathbb{D}\setminus K}(z, K) = \frac{\log(1/r\,[1 + O_r(t)])}{\log(1/t\,[1 + O(t)])} = \delta_t\,\log(1/r)\,[1 + O_r(t)].$$

The remaining estimates follow. □

We define an *inner transformation* of a conformal annulus $A \subset \mathbb{C}$ to be a conformal map of A fixing and preserving the orientation of one of its boundary components, which we call the *outer boundary*.

Proposition 2.11. *Let A be a conformal annulus of conformal type s, and $s \to 0$. Then for any inner transformation χ of A, $|\chi'| = 1 + O(s)$ on A's outer boundary.*

Proof. By a conformal map, we may assume that $A \subset \mathbb{D}$ with outer boundary $\partial\mathbb{D}$. Moreover, it suffices to consider $\chi : A \to A_{s,1}$ and to estimate the derivative at 1. Let $K := \mathbb{D} \setminus A$ have capacity $\log t$, and let $L(z) = \log \chi(e^{iz}) = u(z)+iv(z)$, defined in a neighborhood of the origin using Schwarz reflection. Since u vanishes on the real axis, $\partial_x u(0) = 0$. Since

$$\frac{\log |\chi(z)|}{\log s} = h_{A_{s,1}}(\chi(z), C_s) = h_{\mathbb{D}\setminus K}(z, K),$$

we see that

$$-\partial_y u(0) = \log(1/s)\, \mathcal{E}_{\mathbb{D}\setminus K}(1, K) = \frac{\delta_t}{\delta_s}\,[1 + O(t)] = 1 + O(t).$$

Therefore, $L'(0) = i + O(t)$. By the chain rule, $|\chi'(1)| = 1 + O(t)$. □

The following estimate allows us to extend Lemma 2.10 to general domains $D \in \mathcal{D}$.

Lemma 2.12. *Let $K \in \mathcal{H}_t$. Let f be univalent on $\mathbb{D}$. Let $t \to 0$. Then*

$$\operatorname{cap} f(K) = \log(|f'(0)|\, t) + O(t).$$

Proof. By a similarity we may assume that $f(0) = 0$ and $f'(0) = 1$. By the growth theorem, $f(z) = z\,[1 + O(z)]$ as $z \to 0$, and hence $\operatorname{rad} f(K) = [1 + O(t)] \operatorname{rad} K$. By Remark 2.8, $\operatorname{cap} f(K) = \operatorname{cap} K + O(1)$.

Let B_s be a complex Brownian motion. When we ask to start B_s at infinity, we mean to let the starting point be distributed uniformly on a circle of sufficiently large radius.

Let k be a constant to be determined. Let

$$\begin{aligned} \sigma &= \inf\{s : B_s \in K\}, & \tilde{\sigma} &= \inf\{s : B_s \in f(K)\}, \\ X &= \log|kt/f(B_\sigma)|, & \tilde{X} &= \log|kt/B_{\tilde{\sigma}}|, \\ X_0 &= \mathbf{E}^\infty[X], & \tilde{X}_0 &= \mathbf{E}^\infty[\tilde{X}]. \end{aligned}$$

We will use the fact that $\operatorname{cap} K = \mathbf{E}^\infty[\log|B_\sigma|]$ and $\operatorname{cap} f(K) = \mathbf{E}^\infty[\log|B_{\tilde{\sigma}}|]$ (for a proof, see [2], Corollary 3.34). Since $\log|B_\sigma| = \log|f(B_\sigma)| + O(t)$, it suffices to prove that $X_0 = \tilde{X}_0 + O(t)$. We already know that $X_0 = \tilde{X}_0 + O(1)$.

Let

$$\begin{aligned} \tau &= \inf\{s : B_s \in C_{kt}\}, & \tilde{\tau} &= \inf\{s : B_s \in f(C_{kt})\}, \\ \rho &= \inf\{s > \tau : B_s \in C_{1/2}\}, & \tilde{\rho} &= \inf\{s > \tilde{\tau} : B_s \in f(C_{1/2})\}, \\ V &= \{\rho < \sigma\}, & \tilde{V} &= \{\tilde{\rho} < \tilde{\sigma}\}. \end{aligned}$$

By Proposition 2.6 applied to g_K, we may choose the universal constant $k > 8$ so that $\mathbf{P}^z(V) \asymp \delta_t$ for $z \in C_{kt}$, and we may also insist that $f(K) \subseteq C_{kt/2}$. This implies that $\tilde{X}_0 \asymp 1$ and hence

$$\mathbf{E}^z[\tilde{X}] \asymp 1, \qquad 3kt/4 < |z| < 5kt/4, \tag{2.7}$$

by Harnack's inequality. Let

$$u(z) = \mathbf{E}^z[X] - \tilde{X}_0, \qquad u^* = \max_{z \in C_{1/2}} |u(z)|,$$

which exists since u is harmonic off K.

Let $z \in C_{1/2}$. Denote by μ the distribution of $f(B_\tau)$ if B_s is started at z. By the strong Markov property and conformal invariance of Brownian motion, $\mathbf{P}^z(V) = \mathbf{P}^\mu(\tilde{V})$ and also

$$\mathbf{E}^z[X] = \mathbf{E}^\mu[\tilde{X}] + \mathbf{P}^z(V)\big(\mathbf{E}^z[X \mid V] - \mathbf{E}^\mu[\tilde{X} \mid \tilde{V}]\big). \tag{2.8}$$

We claim

$$\begin{aligned} \mathbf{E}^\mu[\tilde{X}] &= \tilde{X}_0 + O(t), && (2.9) \\ \mathbf{E}^\mu[\tilde{X} \mid \tilde{V}] &= \tilde{X}_0 + O(t), && (2.10) \\ \big|\mathbf{E}^z[X \mid V] - \tilde{X}_0\big| &\le u^*. && (2.11) \end{aligned}$$

The strong Markov property applied to ρ yields (2.11). The strong Markov property applied to $\tilde{\rho}$, the estimate

$$h_{\mathcal{O}_{kt}}(w, kte^{i\theta}) = [1 + O(t)]/2\pi kt, \qquad |w| \geq 1/8,$$

and (2.7) together yield (2.10). If $w = O(t)$, we know $f(w) = w + O(t^2)$. Moreover, if $h(w) = \mathbf{E}^w[\tilde{X}]$ then $|\nabla h(w)| = O(1/t)$ for $7kt/8 < |w| < 9kt/8$ by (2.7) and standard derivative estimates for harmonic functions. We conclude that $\mathbf{E}^{f(w)}[\tilde{X}] = \mathbf{E}^w[\tilde{X}] + O(t)$ for $w \in C_{kt}$. It follows that $\mathbf{E}^\mu[\tilde{X}] = \mathbf{E}^\nu[\tilde{X}] + O(t)$ where ν is the hitting distribution on C_{kt} of Brownian motion started at z. But ν is uniform up to a relative error of $O(t)$, which implies that $\mathbf{E}^\nu[\tilde{X}] = \tilde{X}_0 + O(t)$. Combining these observations yields (2.9).

If we apply (2.9–2.11) to (2.8), we conclude that

$$\left|\mathbf{E}^z[X] - \tilde{X}_0\right| \leq O(t) + \mathbf{P}^z(V)[u^* + O(t)],$$

so

$$u^* \leq O(t) + O(\delta_t)\, u^*,$$

so $u^* = O(t)$. Therefore, $X_0 = \tilde{X}_0 + O(t)$. □

Lemma 2.13. *Let $D \in \mathcal{D}$, $K \in \mathcal{H}_t$, $\psi = \psi_D$ and $s = s_{D,K}$. Let $0 < r < 1$ and $z \in D$ with $r \leq |z|$. Let $t \to 0$. Then*

$$s = \psi'(0)\, t\, [1 + O(t)],$$
$$h_{D\smallsetminus K}(z, K) = \delta_{\psi'(0)\, t}\, \log|1/\psi(z)|\, [1 + O_r(t)].$$

Moreover, if $w \in \partial D$ is a smooth boundary point,

$$\mathcal{E}_{D\smallsetminus K}(w, K) = \delta_{\psi'(0)\, t}\, |\psi'(w)|\, [1 + O(t)].$$

In particular, if also $\tilde{K} \in \mathcal{H}_t$ then $\mathcal{E}_{D\smallsetminus \tilde{K}}(w, \tilde{K}) = \mathcal{E}_{D\smallsetminus K}(w, K)\, [1 + O(t)]$.

Proof. Let $\log \tilde{t} = \operatorname{cap} \psi(K)$. By Lemma 2.12, $\operatorname{cap} \psi(K) = \log(\psi'(0)\, t) + O(t)$, so $\tilde{t} = \psi'(0)\, t\, [1 + O(t)]$ and $\delta_{\tilde{t}} = \delta_{\psi'(0)\, t}\, [1 + O(t)]$. By Lemma 2.10, we conclude that $s = \tilde{t}\, [1 + O(\tilde{t})]$ and

$$h_{D\smallsetminus K}(z, K) = h_{\mathbb{D}\smallsetminus \psi(K)}(\psi(z), \psi(K)) = \delta_{\tilde{t}}\, \log|1/\psi(z)|\, [1 + O_r(\tilde{t})],$$

using the fact that $|\psi(z)| \geq r/16$. The remaining estimates follow. □

2.3 Schramm–Loewner evolution

2.3.1 Chordal, radial and whole-plane SLE

Fix $\kappa > 0$. The *Loewner equation* in the upper half-plane $\mathbb{H}$ is the ODE

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \quad z \in \mathbb{H},$$

where $a = 2/\kappa$ and the *driving function* U_t is continuous and real-valued. Chordal SLE_κ in $\mathbb{H}$ from 0 to ∞, which we denote by $\mu_{\mathbb{H}}(0, \infty)$, is the random family of conformal maps g_t induced by this ODE when U_t is a standard Brownian motion.

We recall several facts about SLE that were first proved in the seminal paper of Rohde and Schramm [10]. Chordal SLE_κ is generated by a continuous curve $\gamma : [0, \infty) \to \mathbb{H}$ in the sense that for each t, the set H_t of points $z \in \mathbb{H}$ for which the solution exists beyond time t is the unbounded component of $\mathbb{H} \smallsetminus \gamma_t$. (The most delicate case of $\kappa = 8$ was unresolved in [10] but was proved in [9].) Moreover, $\gamma(t) \to \infty$ as $t \to \infty$. For $\kappa \leq 4$ the curve γ is simple, for $4 < \kappa < 8$ it touches itself, and for $\kappa \geq 8$ it fills the half-plane.

The g_t are easily seen to satisfy the scaling rule $r^{-1}\, g_{r^2 t}(rz) \stackrel{\mathrm{d}}{=} g_t(z)$ for $r > 0$, and hence $\gamma \stackrel{\mathrm{d}}{=} r\gamma$, up to a reparametrization. As a consequence, the chordal SLE_κ curve γ from z to w in any simply connected domain D, where $z, w \in \partial D$, may be defined as the conformal image of SLE_κ in $\mathbb{H}$ from 0 to ∞, modulo reparametrization. Chordal SLE_κ from z to w in D is then conformally invariant and has the domain Markov property: if τ is a stopping time, then conditional on the initial segment γ_τ, the remainder of the curve has the law of SLE_κ in $D \smallsetminus \gamma_\tau$ from $\gamma(\tau)$ to w. (In the case $4 < \kappa < 8$, this means the component of $D \smallsetminus \gamma_\tau$ adjoining w.)

Radial SLE_κ is a probability measure on curves joining a boundary point to an interior point of a simply connected domain D. It can be obtained from a similar ODE, the radial Loewner equation in the disk, or can be obtained from chordal SLE_κ by weighting by an appropriate local martingale—see Proposition 2.14. In particular, the paths of chordal and radial SLE from the same point are locally mutually absolutely continuous. Radial SLE_κ is also conformally invariant and satisfies the domain Markov property.

Whole-plane SLE_κ is a variant on radial SLE_κ in which the curve γ_t connects two marked points on the Riemann sphere $\hat{\mathbb{C}}$. The domain Markov property then takes the following form: conditional on an initial segment γ_τ, the remainder of the curve has the law of radial SLE_κ in $\hat{\mathbb{C}} \setminus \gamma_\tau$ from $\gamma(\tau)$ to the target point. This property determines the measure on curves, up to reparametrization. We denote whole-plane SLE_κ from 0 to ∞ by $\mu_{\mathbb{C}}(0, \infty)$. In this paper we adopt the convention that the whole-plane SLE curve γ is always parametrized so that for each $t > 0$, $\operatorname{cap} \gamma_t = \log t$. We will use the notation $g_t = g_{\gamma_t}$. As mentioned before, we have $\operatorname{rad} \gamma_t \in [t, 4t]$ and

$$g_t'(z) = 1 + O(t), \quad t \leq 1/8, \quad |z| \geq 1.$$

2.3.2 The SLE partition function

Let D be a simply connected domain, z a smooth boundary point and w either an interior point or a smooth boundary point. We will consider SLE from z to w in D as a finite measure

$$\mu_D(z, w) = \Psi_D(z, w)\, \mu_D^{\#}(z, w),$$

where $\Psi_D(z, w)$ is the partition function and $\mu_D^{\#}(z, w)$ is the corresponding probability measure. For chordal and radial SLE_κ the partition function is determined by the conformal covariance rule (1.1), together with the normalizations $\Psi_{\mathbb{H}}(0, \infty) = \Psi_{\mathbb{D}}(1, 0) = 1$.

We will frequently use ratios of partition functions $\Psi_{D'}(z, w')/\Psi_D(z, w)$ even in the case where $\partial D'$ and ∂D are not smooth at z. This ratio is well-defined so long as D' and D agree in a neighborhood of z. Indeed, we can write

$$\frac{\Psi_{D'}(z, w')}{\Psi_D(z, w)} = \frac{|g'(w')|^{b_{w'}}\, \Psi_{g(D')}\big(g(z), g(w')\big)}{|g'(w)|^{b_w}\, \Psi_{g(D)}\big(g(z), g(w)\big)}$$

for any conformal map g defined on $D \cup D'$ with $g(z)$ a smooth boundary point, where $b_w, b_{w'}$ are the appropriate scaling exponents for w, w' in D, D' respectively. Moreover, in the case of non-smooth boundary points we will use the notation

$$\frac{d\mu_{D'}(z, w')}{d\mu_D(z, w)} := \frac{\Psi_{D'}(z, w')}{\Psi_D(z, w)}\, \frac{d\mu_{D'}^{\#}(z, w')}{d\mu_D^{\#}(z, w)},$$

even though there are no measures $\mu_D(z, w), \mu_{D'}(z, w')$ as such.

When we say "weighting paths locally" by the quantity Q_t, we mean the following. Take the unique continuous local martingale $M_t = C_t\, Q_t$ with respect to the implied filtration $\mathcal{F}_t$, where $C_0 = 1$ and C_t is a process of bounded variation called the *compensator* of Q_t. Then consider the paths up to an appropriate stopping time τ in the new measure $\mathbf{Q}_\tau$ defined by $\mathbf{Q}_\tau(A) = \mathbf{E}(AM_{t\wedge\tau})$ for $A \in \mathcal{F}_t$.

Girsanov's theorem says that this change of measure can be interpreted as adding a drift to the underlying Brownian motion. Using this argument, the following proposition can be proved. See [4, 3, 6] for more detail on the partition function viewpoint of SLE.

Proposition 2.14. *Let D and D' be simply connected domains agreeing in a neighborhood U of the smooth boundary point z, and let w, w' be interior or smooth boundary points of D, D' respectively. Let τ be a stopping time for the SLE$_\kappa$ process $\mu_D(z,w)$ such that γ_τ does not hit the boundary of D or D' outside U. Then $\mu_{D'}(z,w')$, up to time τ, can be obtained by weighting the paths of $\mu_D(z,w)$ locally by the ratio of partition functions*

$$\frac{\Psi_{D'\smallsetminus\gamma_t}(\gamma(t), w')}{\Psi_{D\smallsetminus\gamma_t}(\gamma(t), w)}. \tag{2.12}$$

If also $D = D'$, then the compensator is trivial, so that (2.12) *is the Radon-Nikodym derivative of the measures on the initial segment γ_t.*

Remark 2.15. If $4 < \kappa < 8$, we ignore any extra connected components that appear in $D \smallsetminus \gamma_t$ for the purpose of the partition function.

2.3.3 Annuli and multiply connected domains

From now on let $\kappa \leq 4$. Computing the compensator to

$$\frac{\Psi_{D_1\smallsetminus\gamma_t}(\gamma(t), w)}{\Psi_{D\smallsetminus\gamma_t}(\gamma(t), w)},$$

one can deduce the boundary perturbation rule (1.4). This rule permits the definition of SLE$_\kappa$ in general domains $D \subset \mathbb{C}$ as in [6]. There is a unique extension of the simply-connected domain definition so that (1.4) and the conformal covariance rule hold across all domains D and endpoints z, w (smooth points, if on the boundary). To be specific, there are two cases defined: boundary/boundary SLE, generalizing chordal SLE, and boundary/bulk SLE, generalizing radial SLE.

The case of conformal annuli D and smooth boundary points z, w on different boundary components is called *crossing annulus SLE*. In this case, the partition function $\Psi_D(z,w)$ is known to be finite—see [6]—which permits us to discuss the crossing annulus SLE probability measure $\mu_D^{\#}(z,w)$. This probability measure is the same as that used by Zhan in his proof of reversibility of whole-plane SLE [14].

Proposition 2.16. *Let D be a simply connected domain and $A \subset D$ a conformal annulus such that ∂D is one component of ∂A. Let $z \in \partial D$, $w' \in \partial A \setminus \partial D$ and $w \in D$, and suppose w' is a smooth boundary point. Let τ be a stopping time for radial SLE $\mu_D(z,w)$ such that γ_τ does not leave A. On the initial segment γ_τ, the Radon–Nikodym derivative of crossing annulus SLE $\mu_A(z,w')$ with respect to radial SLE $\mu_D(z,w)$ is*

$$\frac{d\mu_A(z,w')}{d\mu_D(z,w)}(\gamma_\tau) = \exp\Big\{\frac{\mathbf{c}}{2}\Lambda(\gamma_\tau, D\setminus A; D)\Big\}\frac{\Psi_{A\setminus\gamma_\tau}(\gamma(\tau),w')}{\Psi_{D\setminus\gamma_\tau}(\gamma(\tau),w)}.$$

Proof. Fix a partial curve γ_τ starting from z in A. Find a simply connected domain $U \subset A$ which contains γ_τ and agrees with D near z and w'. Let $\tilde{w} \in U$ be arbitrary. On the partial curves γ_τ, we can write down the Radon-Nikodym derivatives

$$\frac{d\mu_A(z,w')}{d\mu_U(z,w')}, \quad \frac{d\mu_U(z,w')}{d\mu_U(z,\tilde{w})}, \quad \frac{d\mu_U(z,\tilde{w})}{d\mu_D(z,\tilde{w})} \quad \text{and} \quad \frac{d\mu_D(z,\tilde{w})}{d\mu_D(z,w)}$$

using Proposition 2.14 and the boundary perturbation rule (1.4). Taking their product and using the decomposition rule (2.6) yields the proposition. □

We will also use Theorem 4.6 from [6], which states that crossing annulus SLE converges to radial SLE:

Theorem 2.17. *There exist $c < \infty$, $q > 0$ such that the following holds. Let $t > 0$ and let γ_t denote an initial segment of a path in $\mathbb{D}$ starting at 1 such that if $g : \mathbb{D}\setminus\gamma_t \to \mathbb{D}$ is a conformal transformation with $g(0) = 0, g'(0) > 0$, then $g'(0) = e^t$. Suppose that $\log(1/r) \geq t + 2$, $0 \leq \theta < 2\pi$, and let $\mu_1 = \mu_{\mathbb{D}}(1,0)$, $\mu_2 = \mu^{\#}_{A_{r,1}}(1, re^{i\theta})$, both considered as probability measures on initial segments γ_t. Let $Y = d\mu_2/d\mu_1$. Then*

$$|Y(\gamma_t) - 1| \leq c\,(re^t)^q. \tag{2.13}$$

Moreover, there exists $c_0 \in (0,\infty)$ such that

$$\Psi_{A_{r,1}}(1, re^{i\theta}) = c_0\, r^{\tilde{b}-b}\,[\log(1/r)]^{\mathbf{c}/2}\,[1 + O(r^q)]. \tag{2.14}$$

Corollary 2.18. *Radial SLE* $\mu_{\mathbb{D}}(1,0)$ *is the weak limit of*

$$c_0^{-1}\, r^{b-\tilde{b}}\, [\log(1/r)]^{-\mathbf{c}/2}\, \mu_{A_{r,1}}(1, re^{i\theta})$$

as $r \downarrow 0$*, uniformly in* θ*.*

This follows from Theorem 2.17 together with the continuity of radial SLE at its terminal point, which was proved in [5]. By Lemma 2.9, we conclude:

Corollary 2.19. *Let* γ *be a curve starting at* 0 *parametrized so that* $\operatorname{cap}\gamma_t = \log t$*,* D *a simply connected domain containing* 0*, and* $w \in \partial D$*. The radial SLE probability measure* $\mu_D^{\#}(w,0)$ *is the weak limit of the annulus SLE probability measures* $\mu_{D\smallsetminus\gamma_t}^{\#}(w,\gamma(t))$ *as* $t \downarrow 0$*, uniformly in* γ*.*

3 Radial SLE from the interior

In this section we prove Theorem 1.2, leaving the proof of Theorem 1.1 until Sect. 4. Throughout this section we assume that $\kappa \le 4$. Our approach is to *define* a measure $\mu_D(0,w)$ for simply connected D that satisfies (1.7) and then to prove that it is the reversal of $\mu_D(w,0)$.

3.1 Definition

Let D be a simply connected domain containing the origin. Recall the normalized loop measure

$$\Lambda^*(V_1,V_2) = \lim_{r\downarrow 0}[\Lambda(V_1,V_2;\mathcal{O}_r) - \log\log(1/r)],$$

where $\Lambda(V_1,V_2;\mathcal{O}_r)$ is the Brownian loop measure of the loops that hit V_1 and V_2 but do not come within distance r of the origin.

Definition 3.1. Let T be a positive stopping time for whole-plane SLE $\mu_{\mathbb{C}}(0,\infty)$ such that with probability 1, $\gamma_T \subset D$. *Radial SLE from the interior of* D *to* $w \in \partial D$ is a measure $\mu_D^T(0,w)$ on curves γ_T up to time T. It is defined by its density with respect to whole-plane SLE, which is

$$\frac{d\mu_D^T(0,w)}{d\mu_{\mathbb{C}}(0,\infty)}(\gamma_T) = c_1 \exp\Big\{\frac{\mathbf{c}}{2}\Lambda^*(\gamma_T, D^c)\Big\}\frac{\Psi_{D\smallsetminus\gamma_T}(\gamma(T),w)}{\Psi_{\mathbb{C}\smallsetminus\gamma_T}(\gamma(T),\infty)}. \tag{3.1}$$

In this formula, the last factor is the ratio of partition functions for annulus and whole-plane SLE, and $1/c_1 = c_0$ is the constant from Theorem 2.17.

Proposition 3.2. *Let $T \geq \tau$ be stopping times for whole-plane SLE such that with probability 1, $\gamma_T \subset D$. The measure $\mu_D^T(0,w)$ on γ_T, considered as a measure on the initial segment γ_τ, is the same as the measure $\mu_D^\tau(0,w)$.*

Proof. Using the loop measure decomposition (4.23), we can factor the density as $\left(d\mu_D^T(0,w)/d\mu_{\mathbb{C}}(0,\infty)\right)(\gamma_T) = XY$, where

$$X = c_1 \exp\Big\{\frac{\mathbf{c}}{2}\Lambda^*(\gamma_\tau, D^c)\Big\}, \quad Y = \exp\Big\{\frac{\mathbf{c}}{2}\Lambda(\gamma_T, D^c; \gamma_\tau^c)\Big\}\frac{\Psi_{D\smallsetminus\gamma_T}(\gamma(T),w)}{\Psi_{\mathbb{C}\smallsetminus\gamma_T}(\gamma(T),\infty)}.$$

Note that X is $\mathcal{F}_\tau$–measurable. By Proposition 2.16, conditional on $\mathcal{F}_\tau$,

$$Y = \frac{d\mu_{D\smallsetminus\gamma_\tau}(\gamma(\tau),w)}{d\mu_{\mathbb{C}\smallsetminus\gamma_\tau}(\gamma(\tau),\infty)}(\gamma_T), \quad \text{so} \quad \mathbf{E}(Y \mid \mathcal{F}_\tau) = \frac{\Psi_{D\smallsetminus\gamma_\tau}(\gamma(\tau),w)}{\Psi_{\mathbb{C}\smallsetminus\gamma_\tau}(\gamma(\tau),\infty)}$$

by the domain Markov property for whole-plane SLE. It follows that

$$\mathbf{E}\left[\frac{d\mu_D^T(0,w)}{d\mu_{\mathbb{C}}(0,\infty)}(\gamma_T) \,\middle|\, \mathcal{F}_\tau\right] = X\,\mathbf{E}(Y \mid \mathcal{F}_\tau) = \frac{d\mu_D^\tau(0,w)}{d\mu_{\mathbb{C}}(0,\infty)}(\gamma_\tau). \qquad \square$$

This proposition tells us that radial SLE from the interior is defined consistently across stopping times T. In particular, the total mass of the measure is independent of T. This means we may consider radial SLE from the interior as a well-defined measure on curves from 0 in D stopped before hitting ∂D. We will denote this measure by $\mu_D(0,w)$ and its partition function by $\Psi_D(0,w) = \|\mu_D(0,w)\|$.

3.2 Density estimate

The key to our analysis of radial SLE from the interior is the following estimate on the normalized loop measure of the loops that hit both boundary components of a conformal annulus. We prove a substantially stronger estimate than is needed for this paper, because we think that the result is interesting in its own right.

Proposition 3.3. *There exists $c < \infty$ such that if $D \in \mathcal{D}$, $t \leq 1/8$ and K is a hull of capacity $\log t$ containing the origin, then*

$$|\Lambda^*(K, \partial D) + \log\log(1/\psi'(0)\,t)| \leq c\,t,$$

where $\psi : D \to \mathbb{D}$ is the unique conformal transformation with $\psi(0) = 0$ and $\psi'(0) > 0$.

The proof is independent of any SLE notions, and we defer it to Sect. 4.4.

We can now estimate the density of radial SLE from the interior with respect to whole-plane SLE. Recall the notation $\delta_t = 1/\log(1/t)$.

Proposition 3.4. *Let $D \in \mathcal{D}$ and let w be a smooth boundary point of D. Let ψ be a conformal transformation from D onto $\mathbb{D}$ fixing 0. Let τ be a stopping time for whole-plane SLE such that with probability 1, $0 < \tau \le t$. Then, as $t \to 0$,*

$$\frac{d\mu_D(0,w)}{d\mu_{\mathbb{C}}(0,\infty)}(\gamma_\tau) = |\psi'(0)|^{\tilde{b}}\,|\psi'(w)|^{b}\,\left[1 + O(\delta_t)\right]. \tag{3.2}$$

Proof. Let $t < 1/4$, so that whole-plane SLE from 0 to ∞ does not leave D by time t. By definition

$$\frac{d\mu_D(0,w)}{d\mu_{\mathbb{C}}(0,\infty)}(\gamma_t) = c_1\,\exp\left\{\frac{\mathbf{c}}{2}\Lambda^*(\gamma_t, D^c)\right\}\frac{\Psi_{D\setminus\gamma_t}(\gamma(t),w)}{\Psi_{\mathbb{C}\setminus\gamma_t}(\gamma(t),\infty)}.$$

It follows from Proposition 3.3 that

$$\exp\left\{\frac{\mathbf{c}}{2}\Lambda^*(\gamma_t, D^c)\right\} = [\log(1/t)]^{-\mathbf{c}/2}\,\left[1 + O(\delta_t)\right].$$

Recall the conformal map $g_t : \mathbb{C}\setminus\gamma_t \to \mathcal{O}_t$, which satisfies $g_t(\infty) = \infty$ and $g_t'(\infty) = 1$. If $U_t := g_t(\gamma(t))$, we have

$$\frac{\Psi_{D\setminus\gamma_t}(\gamma(t),w)}{\Psi_{\mathbb{C}\setminus\gamma_t}(\gamma(t),\infty)} = \frac{|g_t'(w)|^b\,\Psi_{g_t(D)}(U_t, g_t(w))}{\Psi_{\mathcal{O}_t}(U_t,\infty)}.$$

Since $|g_t'(w)| = 1 + O(t)$ by Proposition 2.6 and $\Psi_{\mathcal{O}_t}(U_t,\infty) = t^{\tilde{b}-b}$, we get

$$\frac{d\mu_D(0,w)}{d\mu_{\mathbb{C}}(0,\infty)}(\gamma_t) = c_1\,[\log(1/t)]^{-\mathbf{c}/2}\,t^{b-\tilde{b}}\,\Psi_{g_t(D)}(U_t, g_t(w))\,[1 + O(\delta_t)].$$

Let $\rho : g_t(D) \to A_{s,1}$ be a conformal transformation taking $g_t(\partial D)$ to C_1. Then $\rho\circ g_t$ is a conformal transformation of $D\setminus\gamma_t$ onto $A_{s,1}$. By Lemma 2.13 we know that

$$s = |\psi'(0)|\,t\,[1 + O(t)].$$

By Proposition 2.11 applied to $\rho\circ g_t\circ\psi^{-1}$ and $z \mapsto (t/s)\,\rho(z)$ respectively,

$$|\rho'(g_t(w))| = |\psi'(w)|\,[1 + O(t)],$$
$$|\rho'(U_t)| = |\psi'(0)|\,[1 + O(t)].$$

Hence,

$$\Psi_{g_t(D)}(U_t, g_t(w)) = [1 + O(t)]\,|\psi'(0)|^b\,|\psi'(w)|^b\,\Psi_{A_{s,1}}(\rho(U_t), \rho \circ g_t(w)).$$

Using (2.14) we see that

$$\Psi_{A_{s,1}}(\rho(U_t), \rho \circ g_t(w)) = [1 + O(\delta_t)]\,c_0\,|\psi'(0)|^{\tilde{b}-b}\,t^{\tilde{b}-b}\,[\log(1/t)]^{\mathbf{c}/2}.$$

Combining all of these estimates, we have the proposition. □

Taking expectations in (3.2) and then comparing with the conformal covariance rule (1.1), we get the following corollary.

Corollary 3.5. *The partition function for radial SLE from the interior is finite and equals*

$$\Psi_D(0, w) = |\psi'(0)|^{\tilde{b}}\,|\psi'(w)|^b.$$

Hence, it agrees with the partition function $\Psi_D(w, 0)$ *of ordinary radial SLE.*

3.3 Agreement with reversed radial SLE

Proposition 3.6 (Domain Markov property)**.** *Let* γ *be radial SLE from the interior, following the law* $\mu^\#_D(0, w)$*. Let* τ *be a stopping time as in Definition 3.1. Conditional on the starting segment* γ_τ*, the remainder of* γ *has the law of crossing annulus SLE* $\mu^\#_{D\smallsetminus\gamma_\tau}(\gamma(\tau), \infty)$ *in the slit domain.*

Proof. Let $T \geq \tau$ be a stopping time as in Definition 3.1. By the domain Markov property for a whole-plane SLE curve γ following the law $\mu_{\mathbb{C}}(0, \infty)$, conditional on $\mathcal{F}_\tau$, the remainder of γ has the law of radial SLE $\mu^\#_{\mathbb{C}\smallsetminus\gamma_\tau}(\gamma(\tau), \infty)$ in the slit domain.

Conditional on $\mathcal{F}_\tau$, the density of $\mu_D(0, w)$ on γ_T with respect to whole-plane SLE $\mu_{\mathbb{C}}(0, \infty)$ (i.e., with respect to the measure $\mu^\#_{\mathbb{C}\smallsetminus\gamma_\tau}(\gamma(\tau), \infty)$) is

$$\begin{aligned}&\frac{d\mu_D(0, w)}{d\mu_{\mathbb{C}}(0, \infty)}(\gamma_T) \Big/ \frac{d\mu_D(0, w)}{d\mu_{\mathbb{C}}(0, \infty)}(\gamma_\tau)\\ &\qquad = \exp\Big\{\frac{\mathbf{c}}{2}\Lambda(\gamma_T, \partial D; \mathbb{C}\smallsetminus\gamma_\tau)\Big\}\frac{\Psi_{D\smallsetminus\gamma_T}(\gamma(T), w)}{\Psi_{\mathbb{C}\smallsetminus\gamma_T}(\gamma(T), \infty)} \Big/ \frac{\Psi_{D\smallsetminus\gamma_\tau}(\gamma(\tau), w)}{\Psi_{\mathbb{C}\smallsetminus\gamma_\tau}(\gamma(\tau), \infty)}.\end{aligned}$$

By Proposition 2.16, we know that this quantity is the density of crossing annulus SLE $\mu^\#_{D\smallsetminus\gamma_\tau}(\gamma(\tau), w)$ with respect to radial SLE $\mu^\#_{\mathbb{C}\smallsetminus\gamma_\tau}(\gamma(\tau), \infty)$, up to time T. □

Proposition 3.7. *The definition of radial SLE from the interior gives a random curve $\gamma : (0, T_D) \to D$ with $\gamma(0+) = 0$ and $\gamma(T_D-) = w$, where T_D is the random time at which γ leaves D.*

Proof. Let t be small enough that $\gamma_t \subset D$ deterministically. By Proposition 3.6, conditional on γ_t, the remainder of γ is crossing annulus SLE in D to w. Because crossing annulus SLE is defined to be absolutely continuous with respect to chordal SLE, it is continuous up to its terminal point, and therefore $\gamma(T_D-) = w$. □

Proposition 3.8. *As a probability measure, radial SLE from the interior $\mu_D^\#(0, w)$ is the reversal of radial SLE $\mu_D^\#(w, 0)$.*

Proof. Let γ be a radial SLE curve from the interior, following the law $\mu_D^\#(0, w)$. Conditional on the initial segment γ_t, the remainder of the curve has the law of annulus SLE in the slit domain, $\mu^\#_{D \smallsetminus \gamma_t}(\gamma(t), w)$. Since annulus SLE is reversible, this is the reversal of $\mu^\#_{D \smallsetminus \gamma_t}(w, \gamma(t))$. By Corollary 2.19, the weak limit of these measures as $t \to 0$, uniformly in γ, is the reversal of $\mu_D^\#(w, 0)$. But the weak limit of the measures on post-t segments of γ is $\mu_D^\#(0, w)$ itself, whence the result follows. □

The non-probability measure for radial SLE from the interior, $\mu_D(0, w)$, is therefore the reversal of radial SLE $\mu_D(w, 0)$, because the corresponding partition functions and probability measures agree. This concludes the proof of Theorem 1.2.

3.4 Multiply connected domains

If D is not simply connected, we can still define $\mu_D(0, w)$ by (3.1), which may be regarded as bulk/boundary SLE in the terminology of [6]. For each curve γ from 0 to w in D, we may find a simply connected $D_1 \subset D$ agreeing with D near w. Now we use (2.6) and (4.23) to observe that

$$\frac{d\mu_D(0, w; D_1)}{d\mu_{D_1}(0, w)}(\gamma) = \exp\Big\{-\frac{\mathbf{c}}{2}\Lambda(\gamma, D_1^c; D)\Big\}.$$

where $\mu_D(0, w; D_1)$ denotes $\mu_D(0, w)$ restricted to curves that lie in D_1. But boundary/bulk SLE is defined in [6] to satisfy this restriction rule. Moreover, we know that $\mu_{D_1}(0, w)$ is the reversal of $\mu_{D_1}(w, 0)$. We conclude that Corollary 1.3 holds and bulk/boundary SLE is conformally covariant.

4 Normalizing the Brownian loop measure

The aim of this section is to prove Theorem 1.1 and Proposition 3.3.

Invariance of Λ^* under Möbius transformations implies that the definition (1.6) does not change if we shrink down at a point on the Riemann sphere other than the origin. In other words,

$$\Lambda^*(V_1, V_2) = \lim_{r \downarrow 0} [\Lambda(V_1, V_2; \mathcal{O}_r(z)) - \log\log(1/r)], \tag{4.1}$$

$$\Lambda^*(V_1, V_2) = \lim_{R \to \infty} [\Lambda(V_1, V_2; \mathbb{D}_R) - \log\log R]. \tag{4.2}$$

Theorem 4.21 establishes the existence of the limit in (1.6). Theorem 4.26 proves the alternate forms (4.1) and (4.2). If f is a Möbius transformation, then conformal invariance of the loop measure implies

$$\Lambda(V_1, V_2; \mathcal{O}_r) = \Lambda(f(V_1), f(V_2); f(\mathcal{O}_r)).$$

Invariance of Λ^* under dilations, translations, and inversions can be deduced from this and (1.6), (4.1), and (4.2), respectively.

The proof of Theorem 1.1 really only uses standard arguments about planar Brownian motion but we need to control the error terms. In order to make the proof easier to understand, we have split it into three subsections. The first subsection considers estimates for planar Brownian motion. Readers who are well acquainted with planar Brownian motion may wish to skip this subsection and refer back as necessary. This subsection assumes knowledge of planar Brownian motion as in [2, Chapter 2]. The next subsection discusses the Brownian (boundary) bubble measure and gives estimates for it. The Brownian loop measure is a measure on unrooted loops, but for computational purposes it is often easier to associate to each unrooted loop a particular rooted loop yielding an expression in terms of Brownian bubbles. The third subsection proves the main theorem by giving estimates for the loop measure. The last subsection proves Proposition 3.3.

4.1 Lemmas about Brownian motion

The exact form of the Poisson kernel in the unit disk shows that there is a c such that for all $|z| \leq 1/2$ and $|w| = 1$,

$$|2\pi\, h_{\mathbb{D}}(z, w) - 1| \leq c\,|z|.$$

By taking an inversion, we get that if $|z| \geq 2$,

$$|2\pi\, h_{\mathcal{O}}(z,w) - 1| \leq \frac{c}{|z|}. \tag{4.3}$$

It is standard that

$$h_{A_R}(z, C_R) = \frac{\log|z|}{\log R}, \quad 1 < |z| < R. \tag{4.4}$$

In particular,

$$\mathcal{E}_{A_R}(1, C_R) = \frac{1}{\log R}, \quad \mathcal{E}_{A_R}(R, C_1) = \frac{1}{R\,\log R}, \tag{4.5}$$

If $V \subset \partial D$ is smooth, let $\overline{h}_D(z,w;V) = h_D(z,w)/h_D(z,V)$ for $w \in V$. In other words, $\overline{h}_D(z,w;V)$ is the density of the exit distribution of a Brownian motion *conditioned* so that it exits at V. We similarly define $\overline{h}_{\partial D}(z,w;V)$.

Lemma 4.1. *There exists $c < \infty$ such that the following holds. Suppose $R > 0$ and D is a domain with $A_R \subset D \subset \mathcal{O}$. Then*

$$|2\pi\,\overline{h}_D(z,w;C_1) - 1| \leq c\,\frac{\log R}{R}, \quad |w| = 1,\, z \in D \cap \overline{\mathcal{O}}_{R/2}. \tag{4.6}$$

Remark 4.2. The conclusion of this lemma is very reasonable. If a Brownian motion starting at a point z far from the origin exits D at C_1, then the hitting distribution is almost uniform. This uses the fact that $D \cap \mathbb{D}_R$ is the same as A_R. The important result is the estimate of the error term.

Proof. Assume $|w| = 1$. Let $\tau = \tau_D$ and let $\partial^* = \partial D \cap \mathcal{O}$. It suffices to prove the estimate for $|z| = R/2$. For every $|\zeta| \geq R/2$, (4.3) gives

$$|2\pi h_{\mathcal{O}}(\zeta, w) - 1| \leq \frac{c}{R}. \tag{4.7}$$

Note that

$$h_{\mathcal{O}}(z,w) = h_D(z,w) + \mathbf{E}^z[h_{\mathcal{O}}(B_\tau, w); B_\tau \in \partial^*].$$

Using (4.7), we get

$$2\pi\mathbf{E}^z[h_{\mathcal{O}}(B_\tau, w); B_\tau \in \partial^*] = h_D(z, \partial^*)\,[1 + O(R^{-1})].$$

Therefore,

$$2\pi\, h_D(z,w) = h_D(z, C_1) + O(R^{-1}). \tag{4.8}$$

Since $h_D(z, C_1)$ is bounded below by the probability of reaching C_1 before C_R, (4.4) implies

$$h_D(z, C_1) \geq \frac{\log 2}{\log R},$$

and hence (4.8) implies

$$2\pi \, h_D(z, w) = h_D(z, C_1) \left[1 + O\left(\frac{\log R}{R} \right) \right]. \qquad \square$$

Corollary 4.3. *There exists $c < \infty$ such that if $R \geq 2$, $|z| = 1$, $|w| = R$, then*

$$\left| h_{\partial A_R}(z, w) - \frac{1}{2\pi R \, \log R} \right| \leq \frac{c}{R^2}. \tag{4.9}$$

Proof. Recall that $h_{\partial A_r}(z, w) = h_{\partial A_r}(w, z)$. We know from (4.5) that

$$\int_{C_1} h_{\partial A_R}(w, \zeta) \, |d\zeta| = \frac{1}{R \log R}.$$

Also, by definition,

$$h_{\partial A_R}(w, z) = \frac{\overline{h}_{\partial A_R}(w, z)}{R \, \log R}.$$

Note that $\overline{h}_{\partial A_R}(w, z)$ is bounded by the minimum and maximum values of $\overline{h}_{A_R}(\hat{w}, z)$ over $|\hat{w}| = R/2$, which by (4.6) satisfy

$$\overline{h}_{A_R}(\hat{w}, z) = \frac{1}{2\pi} + O\left(\frac{\log R}{R} \right). \qquad \square$$

Lemma 4.4. *Suppose D is a nonpolar domain containing $\mathbb{D}$. If $0 < s < 1$, let $D_s = D \cap \mathcal{O}_s$. Then if $s < r \leq 1/2$ and $|z| = r$,*

$$\frac{\log r}{\log s} \leq h_{D_s}(z, C_s) \leq \frac{\log r}{\log s} \left[1 - \frac{p \, \log 2}{(1-p) \log(1/r)} \right]^{-1},$$

where

$$p = p_D = \sup_{|\tilde{w}|=1} h_{D_{1/2}}(\tilde{w}, C_{1/2}) < 1.$$

Proof. The inequality $p_D < 1$ follows immediately from the fact that D is nonpolar and contains $\mathbb{D}$.

Let $T = T_s = \inf\{t : B_t \in C_s \cup C_1\}$ and $\sigma = \sigma_{s,r} = \inf\{t \geq T : B_t \in C_r\}$. Then if $|z| \leq 1/2$,

$$\mathbf{P}^z\{B_{\tau_{D_s}} \in C_s\} = \mathbf{P}^z\{B_T \in C_s\} + \mathbf{P}^z\{B_T \in C_1, B_{\tau_{D_s}} \in C_s\}.$$

By (4.4),

$$\mathbf{P}^z\{B_T \in C_s\} = \frac{\log r}{\log s},$$

which gives the lower bound. Let

$$q = q(r,s,D) = \sup_{|\tilde{z}|=r} \mathbf{P}^{\tilde{z}}\left\{B_{\tau_{D_s}} \in C_s\right\},$$

$$u = u(r,D) = \sup_{|\tilde{w}|=1} \mathbf{P}^{\tilde{w}}\left\{B_{\tau_{D_r}} \in C_r\right\}.$$

Then,

$$\mathbf{P}^z\{B_T \in C_1, B_{\tau_{D_s}} \in C_s\} \\ \leq \mathbf{P}^z\{\sigma < \tau_{D_s} \mid B_T \in C_1\}\, \mathbf{P}^z\{B_{\tau_{D_s}} \in C_s \mid B_T \in C_1, \sigma < \tau_{D_s}\} \leq u\, q.$$

Applying this to the maximizing $\tilde{z}$, gives

$$q \leq \frac{\log r}{\log s} + u\, q, \qquad q \leq \frac{\log r}{(1-u)\,\log s}.$$

By (4.4), the probability that a Brownian motion starting on $C_{1/2}$ reaches C_r before reaching C_1 is $\log 2/\log(1/r)$. Using a similar argument as in the previous paragraph, we see that

$$u \leq p\,\frac{\log 2}{\log(1/r)} + p\,u, \qquad u \leq \frac{p}{1-p}\,\frac{\log 2}{\log(1/r)}. \tag{4.9 bis}$$

□

Proposition 4.5. *Suppose D is a nonpolar domain containing the origin. Then there exists $c = c_D < \infty$ such that if $0 < r \leq 1/2$, $D_r = D \cap \mathcal{O}_r$, and $z \in D, |z| \geq 1$,*

$$h_{D_r}(z, C_r) \leq \frac{c}{\log(1/r)}. \tag{4.10}$$

Also, if $|w| = r$,

$$h_{D_r}(z, w) \leq \frac{c}{r\,\log(1/r)}. \tag{4.11}$$

Proof. Find $0 < \beta < 1/2$ such that $\partial D \cap \mathcal{O}_{2\beta}$ is nonpolar. It suffices to prove (4.10) for $r < \beta$. Since $\partial D \cap \mathcal{O}_{2\beta}$ is nonpolar, there exists $q = q_{D,\beta} > 0$ such that for every $|z| \geq 2\beta$, the probability that a Brownian motion starting at z leaves D before reaching C_β is at least q. If $r < \beta$, the probability that a Brownian motion starting at C_β reaches C_r before reaching $C_{2\beta}$ is

$$p(r) = \log 2 / \log(2\beta/r) \leq \frac{c_1}{\log(1/r)}.$$

Let $Q(r) = \sup_{|z|\geq 2\beta} h_{D_r}(z, C_r)$. Then arguing similarly to the previous proof, we have

$$Q(r) \leq (1-q)\ [p(r) + [1-p(r)]\, Q(r)] \leq p(r) + (1-q)\, Q(r),$$

which yields $Q(r) \leq p(r)/q$. This gives (4.10) and (4.11) follows from

$$h_{D_r}(z, w) \leq h_{D_{2r}}(z, C_{2r}) \sup_{|\zeta|=2r} h_{\mathcal{O}_r}(\zeta, w). \qquad \square$$

Proposition 4.6. *There exists $c < \infty$ such that the following holds. Suppose $|z| = 1/2$ and $0 < s < r < 1/8$. Let $D_{s,r} = \mathcal{O}_s \cap \mathcal{O}_r(z)$. Then for $|w| \geq 1$,*

$$\left| \frac{\log(rs)}{\log r} h_{D_{s,r}}(w, C_s) - 1 \right| \leq \frac{c}{\log(1/r)}. \tag{4.12}$$

Proof. Without loss of generality, we assume $z = 1/2$. Let L denote the line $\{x + iy : x = 1/4\}$. Let $\tau = \tau_{D_{s,r}}$, T the first time a Brownian motion reaches $C_r \cup C_r(z)$, and σ the first time after T that the Brownian motion returns to L. By symmetry, for every $w \in L$,

$$\mathbf{P}^w\{B_T \in C_r\} = \frac{1}{2}.$$

Using Lemma 4.4, we get

$$\mathbf{P}^w\{\tau < \sigma \mid B_T \in C_r\} = \frac{\log r}{\log s} \left[1 + O\left(\frac{1}{\log(1/r)} \right) \right].$$

Therefore, for every $w \in L$,

$$\mathbf{P}^w\{\tau < \sigma; B_\tau \in C_r(z)\} = \frac{1}{2},$$

$$\mathbf{P}^w\{\tau < \sigma; B_\tau \in C_s\} = \frac{\log r}{2\log s}\left[1 + O\left(\frac{1}{\log(1/r)}\right)\right].$$

This establishes (4.12) for $w \in L$. If $|w| \geq 1$, then the probability of reaching $\mathcal{O}_r(z)$ before reaching L is $O(1/\log(1/r))$ and the probability of reaching $\mathcal{O}_s$ before reaching L is $O(1/\log(1/s))$. Using this we get (4.12) for $|w| \geq 1$. □

Remark 4.7. The end of the proof uses a well known fact. Suppose one performs independent trials with three possible outcomes with probabilities $p, q, 1-p-q$, respectively. Then the probability that an outcome of the first type occurs before one of the second type is $p/(p+q)$.

4.2 Brownian bubble measure

Let D be a nonpolar domain, $z \in \partial D$ an analytic boundary point and $\tilde{D} \subset D$ a domain that agrees with D near z. From the definition of the Brownian bubble measure, we see that if $\partial\tilde{D} \cap D$ is smooth

$$m(z; D, \tilde{D}) = \pi \int_{\partial\tilde{D}\cap D} h_{\partial\tilde{D}}(z, w)\, h_D(w, z)\, |dw|.$$

This is also equal to $\pi\,\partial_{\mathbf{n}} f(z)$ for the function $f(\zeta) = h_D(\zeta, z) - h_{\tilde{D}}(\zeta, z)$. Let $h_{D,-}(V, z), h_{D,+}(V, z)$ denote the infimum and supremum, respectively, of $h_D(w, z)$ over $w \in V$. Then a simple estimate is

$$h_{D,-}(\partial\tilde{D} \cap D, z) \leq \frac{m(z; D, \tilde{D})}{\pi\,\mathcal{E}_{\tilde{D}}(z, \partial\tilde{D} \cap D)} \leq h_{D,+}(\partial\tilde{D} \cap D, z). \tag{4.13}$$

Lemma 4.8. *If $R > 1$, let*

$$\rho(R) = m(1; \mathcal{O}, A_R).$$

There exists $c < \infty$ such that for all $R \geq 2$,

$$\left|\rho(R) - \frac{1}{2\log R}\right| \leq \frac{c}{R\,\log R}.$$

Remark 4.9. Rotational invariance implies that $m(z; \mathcal{O}, A_R) = \rho(R)$ for all $|z| = 1$.

Proof. By (4.5),

$$\mathcal{E}(1, C_R; A_R) = \frac{1}{\log R}.$$

and by (4.3),

$$2\pi\, h_{\mathcal{O}}(w, 1) = 1 + O(R^{-1})$$

for $w \in C_R$. We now use (4.13). □

The next lemma generalizes this to domains D with $A_R \subset D \subset \mathcal{O}$. The result is similar but the error term is a little larger. Note that the q in the next lemma equals 1 if $D = \mathcal{O}$.

Lemma 4.10. *Suppose $R \geq 2$ and D is a domain satisfying $A_R \subset D \subset \mathcal{O}$. Let q be the probability that a Brownian motion started uniformly on C_R exits D at C_1, i.e.,*

$$q = q(R, D) = \frac{1}{2\pi R} \int_{C_R} h_D(z, C_1)\, |dz|.$$

Then if $|w| = 1$,

$$m(w; D, A_R) = \frac{q}{2\,\log R} \left[1 + O\left(\frac{\log R}{R}\right)\right], \tag{4.14}$$

$$m(w; \mathcal{O}, D) = \frac{1-q}{2\,\log R} + O\left(\frac{q \log R + 1}{R \log R}\right). \tag{4.15}$$

Proof. By definition,

$$m(w; D, A_R) = \pi \int_{C_R} h_{\partial A_R}(w, z)\, h_D(z, w)\, |dz|.$$

For $z \in C_R$, by (4.9) we know that

$$h_{\partial A_R}(w, z) = \frac{1}{2\pi\, R\, \log R} \left[1 + O\left(\frac{\log R}{R}\right)\right],$$

and by (4.6) we know that

$$h_D(z, w) = \frac{1}{2\pi}\, h_D(z, C_1) \left[1 + O\left(\frac{\log R}{R}\right)\right].$$

Combining these gives (4.14), and (4.15) follows from Lemma 4.8 and

$$m(w; \mathcal{O}, A_R) = m(w; D, A_R) + m(w; \mathcal{O}, D).$$ □

Corollary 4.11. *There exists $c < \infty$ such that the following is true. Suppose $R \geq 2$ and D is a domain with $A_R \subset D \subset \mathcal{O}$. Suppose $\partial D \cap \mathcal{O}_R$ is nonpolar and hence*

$$p = p_{R,D} := \sup_{|z|=R} h_{D \cap \mathcal{O}_{R/2}}(z, C_{R/2}) < 1.$$

Then, if $|w| = 1$,

$$\left| m(w; \mathcal{O}, D) - \frac{1}{2 \log R} \right| \leq \frac{c}{(1-p) \log^2 R}.$$

Proof. Let q be as in the previous lemma. By (4.9 bis) we see that

$$q \leq \frac{p \log 2}{(1-p) \log R}.$$

and hence the result follows from (4.15). □

Remark 4.12. We will used scaled versions of this corollary. For example, if D is a nonpolar domain containing $\mathbb{D}$, $r < 1/2$, $D_r = D \cap \mathcal{O}_r$, and $|w| = r$,

$$\left| r^2 \, m(w; \mathcal{O}_r, D_r) - \frac{1}{2 \log(1/r)} \right| \leq \frac{c}{(1-p) \log^2(1/r)},$$

where

$$p = \sup_{|z|=1} h_{D_{1/2}}(z, C_{1/2}).$$

Proposition 4.13. *Suppose V is a nonpolar closed set, $z \neq 0$, and $z, 0 \notin V$. For $0 < r, s < \infty$, let*

$$D_{s,r} = \mathcal{O}_s \cap \mathcal{O}_r(z).$$

Then as $s, r \downarrow 0$, if $|w| = s$,

$$\frac{1}{\pi} \, m(w; D_{s,r}, D_{s,r} \setminus V) = \frac{1}{2\pi s^2 \, \log(1/s)} \, \frac{\log r}{\log(rs)} \, \left[1 + O\left(\delta_{r,s}\right)\right],$$

where $\delta_{r,s} = (\log(1/r))^{-1} + (\log(1/s))^{-1}$.

Remark 4.14. The implicit constants in the $O(\cdot)$ term depend on V, z but not on w.

Proof. We will use (4.13) and write $\delta = \delta_{r,s}$. By scaling we may assume $z = 2$ and let $d = \min\{2, \mathrm{dist}(0, V), \mathrm{dist}(2, V)\}$. We will only consider $r, s \le d/2$. By (4.5),

$$\mathcal{E}(w, C_d; A_{s,d}) = s^{-1}\, \mathcal{E}(w/s, C_{d/s}; A_{d/s}) = \frac{1}{s\, \log(1/s)}\, [1 + O(\delta)]\,.$$

Using this and (4.4) we can see that

$$\mathcal{E}(w, V; D_{s,r} \smallsetminus V) = \frac{1}{s\, \log(1/s)}\, [1 + O(\delta)]\,.$$

For $\zeta \in V$, (4.12) gives

$$h_{D_{s,r}}(\zeta, C_s) = \frac{\log r}{\log(rs)}\, [1 + O(\delta)]\,. \tag{4.16}$$

Therefore, by (4.6),

$$h_{D_{s,r}}(\zeta, w) = \frac{\log r}{2\pi s \log(rs)}\, [1 + O(\delta)]\,.$$

□

The next proposition is the analogue of Proposition 4.13 with $z = \infty$.

Proposition 4.15. *Suppose V is a nonpolar compact set, with $0 \notin V$. For $0 < s, r < \infty$, let*

$$D_{s,r} = \mathcal{O}_s \cap \mathbb{D}_{1/r}.$$

Then, as $s, r \downarrow 0$, if $|w| = s$,

$$\frac{1}{\pi}\, m(w; D_{s,r}, D_{s,r} \smallsetminus V) = \frac{1}{2\pi s^2\, \log(1/s)}\, \frac{\log r}{\log(rs)}\, [1 + O\left(\delta_{r,s}\right)]\,,$$

where $\delta_{r,s} = (\log(1/r))^{-1} + (\log(1/s))^{-1}$.

Proof. The proof is the same as for the previous proposition. In fact, it is slightly easier because (4.16) is justified by (4.4). □

4.3 Brownian loop measure

Lemma 4.16. *Suppose V_1, V_2 are closed sets and D is a domain. Let*

$$V^j = \overline{A_{e^{j-1},e^j}}, \quad \mathcal{O}^j = \mathcal{O}_{e^j}, \quad D^j = D \cap \mathcal{O}^j.$$

Then

$$\Lambda(V_1, V_2; D) = \sum_{j=-\infty}^{\infty} \Lambda(V_1, V_2, V^{j+1}; D^j). \tag{4.17}$$

Proof. For each unrooted loop, consider the point on the loop closest to the origin. The measure of the set of loops for which the distance to the origin is exactly e^j for some integer j is 0. For each loop, there is a unique j such that the loop is in $\mathcal{O}^j$ but not in $\mathcal{O}^{j+1}$. Except for a set of loops of measure zero, such a loop intersects V^{j+1} but does not intersect V^k for $k < j+1$, and hence each loop is counted exactly once on the right-hand side of (4.17). □

Lemma 4.17. *There exists $c < \infty$ such that if $0 < s < 1, R \geq 2$,*

$$\left| \Lambda(C_1, C_R; \mathcal{O}_s) - \log \left[\frac{\log(R/s)}{\log R} \right] \right| \leq \frac{c}{R \log R}.$$

In particular, there exists $c < \infty$ such that if $R \geq 2/s > 4$,

$$\left| \Lambda(C_1, C_R; \mathcal{O}_s) - \frac{\log(1/s)}{\log R} \right| \leq \frac{c \log^2(1/s)}{\log^2 R}.$$

Proof. By (2.5), rotational invariance, and the scaling rule, we get

$$\Lambda(C_1, C_R; \mathcal{O}_s) = 2 \int_s^1 r\, m(r; \mathcal{O}_r, A_{r,R})\, dr = 2 \int_s^1 r^{-1}\, \rho(R/r)\, dr,$$

where ρ is as in Lemma 4.8. From that lemma, we know that

$$\rho(R/r) = \frac{1}{2 \log(R/r)} + O\left(\frac{r}{R \log(R/r)} \right),$$

and hence

$$\Lambda(C_1, C_R; \mathcal{O}_s) = O\left(\frac{1}{R \log R} \right) + \int_s^1 \frac{1}{r\,(\log R - \log r)}\, dr.$$

The first assertion follows by integrating and the second from the expansion

$$\log \left[\frac{\log(R/s)}{\log R} \right] = \frac{\log(1/s)}{\log R} + O\!\left(\frac{\log^2(1/s)}{\log^2 R} \right).$$

□

Lemma 4.18. *Suppose V is a closed, nonpolar set with $0 \notin V$ and $\alpha > 1$. There exists $c = c_{V,\alpha} < \infty$ such that for sufficiently small r,*

$$\left|\Lambda(V, \mathcal{O}_r \setminus \mathcal{O}_{\alpha r}; \mathcal{O}_r) - \frac{\log \alpha}{\log(1/r)}\right| \leq \frac{c}{\log^2(1/r)}.$$

Proof. By scaling, we may assume that $\operatorname{dist}(0, V) = 1$. It suffices to prove the result for r sufficiently small. By (2.5), we have

$$\Lambda(V, \mathcal{O}_r \setminus \mathcal{O}_{\alpha r}; \mathcal{O}_r) = \frac{1}{\pi} \int_0^{2\pi} \int_r^{\alpha r} m(se^{i\theta}; \mathcal{O}_s, D_s)\, s\, ds\, d\theta,$$

where $D_s = \mathcal{O}_s \setminus V$. By Corollary 4.11, for $r \leq s \leq \alpha r$,

$$m(se^{i\theta}; \mathcal{O}_s, D_s) = \frac{1}{2s^2\, \log(1/s)} \left[1 + O\left(\frac{1}{\log(1/r)}\right)\right].$$

Therefore,

$$\Lambda(V, \mathcal{O}_r \setminus \mathcal{O}_{\alpha r}; \mathcal{O}_r) = \left[1 + O\left(\frac{1}{\log(1/r)}\right)\right] \int_r^{\alpha r} \frac{ds}{s\, \log(1/s)}.$$

Also,

$$\begin{aligned} \int_r^{\alpha r} \frac{ds}{s\, \log(1/s)} &= \log\log\left(\frac{1}{r}\right) - \log\log\left(\frac{1}{\alpha r}\right) \\ &= \frac{\log \alpha}{\log(1/r)} + O\left(\frac{1}{\log^2(1/r)}\right). \end{aligned}$$

□

Lemma 4.19. *Suppose V_1, V_2 are nonpolar closed subsets of the Riemann sphere with $0 \notin V_1$. Then there exists $c = c_{V_1,V_2} < \infty$ such that for all $r \leq \operatorname{dist}(0, V_1)/2$,*

$$\Lambda(V_1, C_r; \mathbb{C} \setminus V_2) \leq \frac{c}{\log(1/r)}. \tag{4.18}$$

Proof. Constants in this proof depend on V_1, V_2. Without loss of generality assume $0 \notin V_2$ and let $D_r = \mathcal{O}_r \setminus V_2$. We will first prove the result for $r \leq r_0 = [\operatorname{dist}(0, V_1) \wedge \operatorname{dist}(0, V_2)]/2$. By (2.5), we have

$$\Lambda(V_1, C_r; \mathbb{C} \setminus V_2) = \frac{1}{\pi} \int_{|z| \leq r} m(z; D_{|z|}, D_{|z|} \setminus V_1)\, dA(z). \tag{4.19}$$

By (4.11),

$$h_{D_r}(w,z) \leq \frac{c}{r\,\log(1/r)}, \qquad w \in V_1, \quad |z| = r.$$

By comparison with an annulus, we get

$$\mathcal{E}_{D_r \setminus V_1}(z, V_1) \leq \frac{c}{r\,\log(1/r)}, \quad |z| = r.$$

Using (4.13), we then have

$$\frac{1}{\pi}\, m(z; D_{|z|}, D_{|z|} \setminus V_1) \leq \frac{c}{|z|^2\, \log^2(1/|z|)}.$$

By integrating, we get (4.18) for $r \leq r_0$.

Let $r_1 = \operatorname{dist}(0, V_1)/2$ and note that

$$\Lambda(V_1, C_{r_1}; \mathbb{C} \setminus V_2) = \Lambda(V_1, C_{r_0}; \mathbb{C} \setminus V_2) + \Lambda(V_1, C_{r_1}; \mathcal{O}_{r_0} \setminus V_2).$$

Using Lemma 2.4 we can see that $\Lambda(V_1, C_{r_1}; \mathcal{O}_{r_0} \setminus V_2) < \infty$. Therefore,

$$\Lambda(V_1, C_{r_1}; \mathbb{C} \setminus V_2) < \infty,$$

and we can conclude (4.18) for $r_0 \leq r \leq r_1$ with a different constant. □

Corollary 4.20. *Suppose V_1, V_2 are disjoint closed subsets of the Riemann sphere and D is a nonpolar domain. Then*

$$\Lambda(V_1, V_2; D) < \infty.$$

Proof. Assume $0 \notin V_1$. Lemma 4.19 shows that $\Lambda(V_1, \overline{\mathbb{D}}_s; D) < \infty$ for some $s > 0$. Note that

$$\Lambda(V_1, V_2; D) \leq \Lambda(V_1, \overline{\mathbb{D}}_s; D) + \Lambda(V_1, V_2; \mathcal{O}_s).$$

Since at least one of V_1, V_2 is compact, Lemma 2.4 implies that

$$\Lambda(V_1, V_2; \mathcal{O}_s) < \infty.$$ □

Theorem 4.21. *Suppose V_1, V_2 are disjoint, nonpolar closed subsets of the Riemann sphere. Then the limit*

$$\Lambda^*(V_1, V_2) = \lim_{r \downarrow 0} \left[\Lambda(V_1, V_2; \mathcal{O}_r) - \log\log(1/r)\right] \tag{4.20}$$

exists.

Proof. Without loss of generality, assume that $\mathrm{dist}(0, V_1) \geq 2$ and let $\mathcal{O}^k = \mathcal{O}_{e^{-k}}$. Let $\hat{V}_2 \subset V_2$ be a nonpolar closed subset with $0 \notin \hat{V}_2$. Constants in the proof depend on V_1, V_2. Since $\Lambda(V_1, V_2; \mathcal{O}_r)$ increases as r decreases to 0, it suffices to establish the limit

$$\lim_{k\to\infty} \left[\Lambda(V_1, V_2; \mathcal{O}^k) - \log k\right].$$

Repeated application of (2.6) shows that if $k \geq 1$,

$$\Lambda(V_1, V_2; \mathcal{O}^k) = \Lambda(V_1, V_2; \mathcal{O}^0) + \sum_{j=1}^{k} \Lambda(V_1, V_2, \mathcal{O}^{j-1} \setminus \mathcal{O}^j; \mathcal{O}^j).$$

Similarly, for fixed k, (2.6) implies

$$\begin{aligned}\Lambda(V_1, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k) - \Lambda(V_1, V_2, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k)&\\ = \Lambda(V_1, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k \setminus V_2)&\\ \leq \Lambda(V_1, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k \setminus \hat{V}_2).&\end{aligned}$$

From Lemma 4.18, we can see that

$$\Lambda(V_1, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k) = \frac{1}{k} + O_{V_1}\left(\frac{1}{k^2}\right),$$

and hence the limit

$$\lim_{k\to\infty} \left[-\log k + \sum_{j=1}^{k} \Lambda(V_1, \mathcal{O}^{k-1} \setminus \mathcal{O}^k; \mathcal{O}^k)\right]$$

exists and is finite. By Lemma 4.19, we see that

$$\sum_{j=k}^{\infty} \Lambda(V_1, \mathcal{O}^{j-1} \setminus \mathcal{O}^j; \mathcal{O}^j \setminus \hat{V}_2) = \Lambda(V_1, \overline{\mathbb{D}}^k; \mathbb{C} \setminus \hat{V}_2) \leq \frac{c}{k},$$

and hence

$$\sum_{j=k}^{\infty} \left[\Lambda(V_1, \mathcal{O}^{j-1} \setminus \mathcal{O}^j; \mathcal{O}^j) - \Lambda(V_1, V_2, \mathcal{O}^{j-1} \setminus \mathcal{O}^j; \mathcal{O}^j)\right] \leq \frac{c}{k},$$

where the constant c depends on V_1 and $\hat{V}_2$ but not otherwise on V_2. □

Remark 4.22. It follows from the proof that

$$\Lambda^*(V_1, V_2) = \Lambda(V_1, V_2; \mathcal{O}^k) - \log k + O\left(\frac{1}{k}\right),$$

where the $O(\cdot)$ term depends on V_1 and $\hat{V}_2$ but not otherwise on V_2. As a consequence we can see that if $0 \notin V_1$ and $V_{2,r} = V_2 \cap \{|z| \geq r\}$, then

$$\lim_{r\downarrow 0} \Lambda^*(V_1, V_{2,r}) = \Lambda^*(V_1, V_2). \tag{4.21}$$

The definition of Λ^* in (4.20) seems to make the origin a special point. Theorem 4.26 shows that this is not the case.

Lemma 4.23. *Suppose V is a nonpolar closed set, $z \neq 0$ and $0 \notin V$. Let $\alpha > 0$. There exist c, r_0 (depending on z, V, α) such that if $0 < r < r_0$,*

$$|\Lambda(V, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) - \log 2| \leq \frac{c}{\log(1/r)}.$$

Proof. We will first assume $z \notin V$. For $s \leq r$, let $D_s = \mathcal{O}_s \cap \mathcal{O}_{\alpha r}(z)$. As in (2.5),

$$\Lambda(V, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) = \frac{1}{\pi} \int_{|w|\leq r} m(w; D_{|w|}, D_{|w|} \setminus V)\, dA(w).$$

By Proposition 4.13, if $|w| = s \leq r$,

$$\frac{1}{\pi}\, m(w; D_s, D_s \setminus V) = \frac{1}{2\pi s^2 \log(1/s)} \frac{\log r}{\log(rs)} \left[1 + O\left(\frac{1}{\log(1/r)}\right)\right],$$

and therefore,

$$\Lambda(V, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) = \log r \int_0^r \frac{ds}{s \log(1/s) \log(rs)} \left[1 + O\left(\frac{1}{\log(1/r)}\right)\right].$$

A straightforward computation gives

$$\log r \int_0^r \frac{ds}{s \log(1/s) \log(rs)} = \log 2.$$

This finishes the proof for $z \notin V$.

If $z \in V$, let $V_1 \subset V$ be a closed nonpolar set with $z \notin V_1$. Then (2.6) implies

$$\Lambda(V, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) \\ = \Lambda(V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) + \Lambda(V \setminus V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z) \setminus V_1).$$

Since the previous paragraph applies to V_1 it suffices to show that

$$\Lambda(V \setminus V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z) \setminus V_1) = O\left(\frac{1}{\log(1/r)}\right).$$

We can write

$$\Lambda(V \setminus V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z) \setminus V_1) = \frac{1}{\pi} \int_{|w| \leq r} m(w; D_s \setminus V_1, D_s \setminus V) \, dA(w).$$

By using (4.14) and (4.10) we can see that

$$m(w; D_s \setminus V_1, D_s \setminus V) \leq \frac{c}{s^2 \log^2(1/s)},$$

and hence

$$\Lambda(V \setminus V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z) \setminus V_1) \leq c \int_0^r \frac{ds}{s \log^2(1/s)} \leq \frac{c}{\log(1/r)}. \qquad \square$$

The following is the equivalent lemma for $z = \infty$. It can be proved similarly or by conformal transformation.

Lemma 4.24. *Suppose V is a nonpolar closed set, and $0 \notin V$. Let $\alpha > 0$. There exists c, r_0 (depending on V, α) such that if $0 < r < r_0$,*

$$\left| \Lambda(V, \mathbb{C} \setminus \mathcal{O}_r; \mathbb{D}_{\alpha/r}) - \log 2 \right| \leq \frac{c}{\log(1/r)}.$$

We extend this to k closed sets.

Lemma 4.25. *Suppose $V_1, \ldots, V_k$ are closed nonpolar subsets of $\mathbb{C}$ that do not contain 0. Let $z \neq 0$ and $\alpha > 0$. There exist c, r_0 (depending on $z, \alpha, V_1, \ldots, V_k$) such that if $0 < r < r_0$,*

$$|\Lambda(V_1, \ldots, V_k, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) - \log 2| \leq \frac{c}{\log(1/r)},$$
$$\left|\Lambda(V_1, \ldots, V_k, \mathbb{C} \setminus \mathcal{O}_r; \mathbb{D}_{\alpha/r}) - \log 2\right| \leq \frac{c}{\log(1/r)}.$$

Proof. If $k = 2$, inclusion-exclusion implies

$$\Lambda(V_1 \cup V_2, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) + \Lambda(V_1, V_2, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z))$$
$$= \Lambda(V_1, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)) + \Lambda(V_2, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_{\alpha r}(z)).$$

Since Lemma 4.23 applies to $V_1 \cup V_2, V_1, V_2$, we get the result. The cases $k > 2$ and $z = \infty$ are done similarly. □

Theorem 4.26. *Suppose V_1, V_2 are disjoint, nonpolar closed subsets of the Riemann sphere and $z \in \mathbb{C}$. Then*

$$\Lambda^*(V_1, V_2) = \lim_{r \downarrow 0} \left[\Lambda(V_1, V_2; \mathcal{O}_r(z)) - \log\log(1/r)\right].$$

Moreover,

$$\Lambda^*(V_1, V_2) = \lim_{R \to \infty} \left[\Lambda(V_1, V_2; \mathbb{D}_R) - \log\log R\right].$$

Proof. We will assume $0 \notin V_1$. Using (4.20), we see that it suffices to prove that

$$\lim_{r \downarrow 0} \left[\Lambda(V_1, V_2; \mathcal{O}_r(z)) - \Lambda(V_1, V_2; \mathcal{O}_r)\right] = 0.$$

Note that

$$\Lambda(V_1, V_2; \mathcal{O}_r(z)) - \Lambda(V_1, V_2; \mathcal{O}_r)$$
$$= \Lambda(V_1, V_2, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_r(z)) - \Lambda(V_1, V_2, \mathbb{C} \setminus \mathcal{O}_r(z); \mathcal{O}_r).$$

Lemma 4.25 implies

$$\Lambda(V_1, V_2, \mathbb{C} \setminus \mathcal{O}_r; \mathcal{O}_r(z)) = \log 2 + O\left(\frac{1}{\log(1/r)}\right), \tag{4.22}$$

where the constants in the error term depend on z, V_1, V_2. Similarly, using translation invariance of the loop measure, we can see that

$$\Lambda(V_1, V_2, \mathbb{C} \setminus \mathcal{O}_r(z); \mathcal{O}_r) = \log 2 + O\left(\frac{1}{\log(1/r)}\right).$$

The case $z = \infty$ is done similarly. □

If $V_1, V_2, \ldots, V_k$ are pairwise disjoint nonpolar closed subsets of the Riemann sphere, we define similarly

$$\Lambda^*(V_1, \ldots, V_k) = \lim_{r \downarrow 0} \left[\Lambda(V_1, V_2, \ldots, V_k; \mathcal{O}_r) - \log\log(1/r)\right].$$

One can prove the existence of the limit in the same way or we can use the relation

$$\Lambda^*(V_1, \ldots, V_k) = \Lambda^*(V_1, \ldots, V_{k+1}) + \Lambda(V_1, \ldots, V_k; \mathbb{C} \setminus V_{k+1}). \tag{4.23}$$

4.4 Estimate on the loops that cross an annulus

We conclude the discussion of the normalized loop measure by supplying the proof of Proposition 3.3, which was stated on page 21.

Proof. Since $\partial D \cap \mathbb{D} = \emptyset$, if $R > 1$,

$$\Lambda(K, \partial D; \mathbb{D}_R) = \Lambda(K, C_1; \mathbb{D}_R) - \Lambda(K, C_1; D \cap \mathbb{D}_R).$$

Taking limits as $R \to \infty$,

$$\Lambda^*(K, \partial D) = \lim_{R\to\infty} \left[\Lambda(K, C_1; \mathbb{D}_R) - \log\log R\right] - \Lambda(K, C_1; D).$$

Hence it suffices to show that

$$\lim_{R\to\infty} \left[\Lambda(K, C_1; \mathbb{D}_R) - \log\log R\right] = -\log\log(1/t) + O(t) \tag{4.24}$$

and

$$\Lambda(K, C_1; D) = \log\left(1 + \frac{\log\psi'(0)}{\log t}\right) + O(t). \tag{4.25}$$

Suppose $t < 1/8$ and $K \in \mathcal{H}_t$. Then, by (2.3), if $R > 1$,

$$\Lambda(K, C_1; \mathbb{D}_R) = \frac{1}{\pi}\int_0^{2\pi}\int_1^R m(re^{i\theta}; \mathbb{D}_r, \mathbb{D}_r \setminus K)\, r\, dr\, d\theta.$$

If $z \in K$, $r \geq 1$, $\theta \in [0, 2\pi]$, then

$$h_{\mathbb{D}_r}(z, re^{i\theta}) = \frac{1}{2\pi r}\left[1 + O(t/r)\right],$$

and hence

$$\frac{1}{\pi} m(re^{i\theta}; \mathbb{D}_r, \mathbb{D}_r \setminus K) = \mathcal{E}_{\mathbb{D}_r\setminus K}(re^{i\theta}, K)\, \frac{1}{2\pi r}\left[1 + O(t/r)\right].$$

Lemma 2.10 and conformal covariance gives

$$\mathcal{E}_{\mathbb{D}_r\setminus K}(re^{i\theta}, K) = r^{-1}\, \mathcal{E}_{\mathbb{D}\setminus(r^{-1}K)}(e^{i\theta}, r^{-1}K) = \frac{1}{r\log(r/t)} + O(t/r^2).$$

Therefore,

$$\frac{r}{\pi}\int_0^{2\pi} m(re^{i\theta}; \mathbb{D}_r, \mathbb{D}_r \setminus K)\, d\theta = \frac{1}{r\log(r/t)} + O(t/r^2)$$

and

$$\Lambda(K, C_1; \mathbb{D}_R) = \int_1^R \left[\frac{1}{r \log(r/t)} + O(t/r^2) \right] dr = \log\left[\frac{\log(R/t)}{\log(1/t)} \right] + O(t).$$

This gives (4.24).

Let D_r denote the connected component of $D \cap \mathbb{D}_r$ containing the origin. Then using (2.3) again, we get

$$\Lambda(K, C_1; D) = \frac{1}{\pi} \int_0^{2\pi} \int_1^\infty m(re^{i\theta}; D_r, D_r \setminus K)\, r\, dr\, d\theta.$$

Our first claim is

$$\Lambda(K, C_1; D) = \Lambda(C_t, C_1; D)\, [1 + O(t)].$$

In fact, for every $r \geq 1$,

$$m(re^{i\theta}; D_r, D_r \setminus K) = m(re^{i\theta}; D_r, D_r \setminus \overline{\mathbb{D}}_t)\, [1 + O(t)].$$

Indeed, this estimate follows from the two estimates

$$h_{D_r}(z, re^{i\theta}) = h_{D_r}(0, re^{i\theta})\, [1 + O(t)], \quad |z| \leq 4t,$$

$$\mathcal{E}_{D_r \setminus K}(re^{i\theta}, K) = \mathcal{E}_{D_r \setminus \overline{\mathbb{D}}_t}(re^{i\theta}, \overline{\mathbb{D}}_t)\, [1 + O(t)].$$

The first follows from the fact that $h_{D_r}(\cdot, re^{i\theta})$ is a positive harmonic function on $\mathbb{D}$. The second may be found in Lemma 2.13.

To compute $\Lambda(C_t, C_1; D)$ we use (2.4) to write

$$\Lambda(C_t, C_1; D) = \frac{1}{\pi} \int_0^{2\pi} \int_0^t m_{\mathcal{O}_r}(re^{i\theta}; D, \mathbb{D})\, r\, dr\, d\theta.$$

Also, for $0 < r < t$,

$$\begin{aligned} \frac{1}{\pi}\, m_{\mathcal{O}_r}(re^{i\theta}; D, \mathbb{D}) &= \int_{C_1} h_{\partial A_{r,1}}(re^{i\theta}, w)\, h_{D \cap \mathcal{O}_r}(w, re^{i\theta})\, |dw| \\ &= \frac{1}{r} \left[\frac{\delta_r}{2\pi} + O(r) \right] \int_0^{2\pi} h_{D \cap \mathcal{O}_r}(e^{iy}, re^{i\theta})\, dy \end{aligned}$$

by (4.9). Let $q_D(w, r)$ denote the probability that a Brownian motion starting at w hits C_r before leaving D. Then (4.6) implies that for $|w| = 1$,

$$h_{D \cap \mathcal{O}_r}(w, re^{i\theta}) = \frac{q_D(w, r)}{2\pi r}\, [1 + O(r/\delta_r)].$$

Therefore,

$$r \int_0^{2\pi} h_{D\cap\mathcal{O}_r}(e^{iy}, re^{i\theta})\, dy = \mathbf{E}[q_D(B_\tau, r)]\,[1 + O(r/\delta_r)],$$

$$\frac{r}{\pi}\, m_{\mathcal{O}_r}(re^{i\theta}; D, \mathbb{D}) = \frac{\delta_r}{2\pi r}\,\mathbf{E}[q_D(B_\tau, r)]\,[1 + O(r/\delta_r)],$$

where B_t is a Brownian motion started at 0 and τ is the first time t with $|B_t| = 1$. Let $\psi : D \to \mathbb{D}$ be the unique conformal transformation with $\psi(0) = 0$, $\psi'(0) > 0$. We have

$$\mathbb{D}_{\psi'(0)\, r - O(r^2)} \subset \psi(\mathbb{D}_r) \subset \mathbb{D}_{\psi'(0)\, r + O(r^2)}.$$

If $D \in \mathcal{D}$, then $|\psi(w)| \geq 1/16$ for $|w| = 1$, and hence

$$q_D(w, r) = \frac{\log|\psi(w)|}{\log[\psi'(0)\, r + O(r^2)]} = -\delta_{\psi'(0)\, r}\,[\log|\psi(w)|]\;[1 + O(r\delta_r)]\,.$$

Hence

$$\mathbf{E}[q_D(B_\tau, r)] = -\delta_{\psi'(0)\, r}\,\mathbf{E}[\log|\psi(B_\tau)|]\;[1 + O(r\delta_r)]\,.$$

By considering the harmonic function $H(z) = \log|\psi(z)/z|$, we see that

$$\mathbf{E}[\log|\psi(B_\tau)|] = \log \psi'(0).$$

Hence,

$$\frac{r}{\pi} \int_0^{2\pi} m_{\mathcal{O}_r}(re^{i\theta}, D, \mathbb{D})\, d\theta = -\frac{\delta_r\, \delta_{\psi'(0)\, r}}{r}\,[\log \psi'(0)]\,[1 + O(r/\delta_r)].$$

Also,

$$\int_0^t \frac{\delta_r\, \delta_{\psi'(0)\, r}}{r}\, dr = -\frac{1}{\log \psi'(0)} \log\left(1 + \frac{\log \psi'(0)}{\log t}\right),$$

$$0 \leq \int_0^t \delta_{\psi'(0)\, r}\, dr \leq t\, \delta_{\psi'(0)\, t}.$$

This gives (4.25). □

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