An estimate for Brownian excursion measure

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Abstract

We prove an estimate for Brownian excursion measure that follows from basic properties of Brownian motion.

We give a proof of Lemma 11 from [1]. The proof is more detailed and at a more elementary level than is appropriate for that paper.

To summarise the relevant definitions, let B be a complex Brownian motion. For a domain D, let

$$\tau_D = \inf\{t : B_t \notin D\}.$$

We write h_D for the harmonic measure, i.e., if $V \subset \partial D$,

$$h_D(z, V) = \mathbf{P}^z\{\tau_D \in V\}.$$

If D is a domain and $V, W \subset \partial D$ are analytic boundary arcs, then the *excursion measure* in D between V and W is defined as

$$\mathcal{E}_D(V, W) = \int_V \mathcal{E}_D(v, W)\, |dv|,$$

where, if n_v is the inward pointing normal at v,

$$\mathcal{E}_D(v, W) = \lim_{\epsilon \downarrow 0} \epsilon^{-1}\, h_D(v + \epsilon n_v, W).$$

It is well known that $\mathcal{E}_D(V, W)$ is conformally invariant (see, e.g., [2, Proposition 5.8]), and therefore it makes sense even when V, W are boundary arcs that are not analytic.

If V, W are not boundary arcs but merely the images of simple curves (that may pass through D) then we set $\mathcal{E}_D(V, W) = \mathcal{E}_{D \setminus (V \cup W)}(V, W)$.

Lemma 1. *Let* $\gamma : [0,1] \to \overline{\mathbb{H}}$ *be a simple curve with* $\gamma(0) = 0$ *and* $\max_t |\gamma(t)| = r \le \frac{1}{4}$. *Let* $\eta : [0,1] \to \overline{\mathbb{H}}$ *be a simple curve with* $\eta(0) \in \mathbb{R}$ *and* $\operatorname{dist}(0,\eta) = 1$. *Then*

$$\mathcal{E}_{\mathbb{H}}(\gamma,\eta) \asymp r\,\big(\operatorname{diam}(\eta) \wedge 1\big).$$

Proof. Let $F : \mathbb{C} \setminus (\gamma \cup \gamma^*) \to \mathbb{C} \setminus s\overline{\mathbb{D}}$, for some $s > 0$, be the unique conformal map with $F(z) = z + O(1)$ as $z \to \infty$. (Here γ^* denotes the complex conjugate.) By the Schwarz lemma and the Koebe 1/4-theorem applied to $1/F(1/z)$, we have $r/4 \le s \le r \le \frac{1}{4}$. Moreover, by symmetry F restricts to a conformal map $f : \mathbb{H} \setminus \gamma \to \mathbb{H} \setminus s\overline{\mathbb{D}}$.

Let $\xi = f(\eta)$. By the Koebe distortion theorem, $R := \operatorname{dist}(0,\xi) \in \big[\frac{9}{16}, \frac{25}{16}\big]$ and $m := \frac{1}{2}\big(\operatorname{diam}(\xi) \wedge R\big) \asymp \operatorname{diam}(\eta) \wedge 1$. Since excursion measure is conformally invariant, it suffices to prove that

$$\mathcal{E}_{\mathbb{H}}(sC,\xi) \asymp s\,m.$$

To prove the lower bound, let ζ be a point in $\xi \cap R\,C$; by reflection we may assume that $\operatorname{Re}\zeta \ge 0$. We may choose a curve $\hat{\xi} \subset \xi$ such that $\zeta \in \hat{\xi}$ and $\max_{z\in\hat{\xi}} |z - \zeta| = m \le R/2$. First observe that

$$\mathcal{E}_{\mathbb{H}}(sC,\xi) \ge \mathcal{E}_{\mathbb{H}\cap(R/2)\mathbb{D}}(sC,V) \inf_{v\in V} h_{\mathbb{H}\setminus(sC\cup\hat{\xi})}(v,\hat{\xi}),$$

where $V = \frac{R}{2}\exp\big(i\pi\big[\frac{1}{4},\frac{3}{4}\big]\big) \subset \frac{R}{2}C$. From the Poisson kernel in a rectangle it is not hard to see that $\mathcal{E}_{\mathbb{H}\cap(R/2)\mathbb{D}}(sC,V) \asymp s$. We must now prove that $h_{\mathbb{H}\setminus(sC\cup\hat{\xi})}(v,\hat{\xi}) > c\,m$ for all $v \in V$. Let

$$\omega = \operatorname{Re}\zeta + 2Ri, \qquad \rho = |\omega - (\zeta + im/2)| \asymp 1,$$
$$B = B_\rho(\omega), \qquad \text{and} \quad W = B_{m/100}(\zeta + im/2) \cap \partial B.$$

By the Harnack inequality,

$$\inf_{v\in V} h_{\mathbb{H}\setminus(sC\cup\hat{\xi})}(v,\hat{\xi}) \asymp h_{\mathbb{H}\setminus(sC\cup\hat{\xi})}(\omega,\hat{\xi}) \ge h_B(\omega,W) \inf_{w\in W} h_{\mathbb{H}\setminus(sC\cup\hat{\xi})}(w,\hat{\xi}).$$

Since $h_B(\omega,W) \asymp m$, it suffices to prove that $h_{\mathbb{H}\setminus(sC\cup\hat{\xi})}(w,\hat{\xi}) > c$ for $w \in W$. Indeed, let E be the event that a Brownian motion travels at all times within distance $m/50$ of the path

$$\psi(t) = \zeta + e^{i\theta(t)} m/2, \qquad \text{where} \quad \theta(t) = \begin{cases} \pi/2 + t\,\pi/3, & 0 \le t \le 1, \\ 5\pi/6 - (t-1)\pi, & 1 \le t \le 5. \end{cases}$$

By considering the cases $\operatorname{Im}\zeta < R/2$ and $\operatorname{Im}\zeta \geq R/2$ separately, we can see that on the event E, the Brownian motion must hit $\hat{\xi}$ before hitting either sC or $\mathbb{R}$. By scaling, there exists a universal constant $c > 0$ such that $\mathbf{P}^w(E) > c$ for all $w \in W$. This gives the lower bound.

To prove the upper bound, note the estimate

$$\mathcal{E}_{\mathbb{H}}(sC, \xi) \leq \mathcal{E}_{\mathbb{H}}\Big(sC, \frac{R}{2}C\Big) \sup_{|z|=R/2} h_{\mathbb{H}}(z, \xi).$$

Since $\mathcal{E}_{\mathbb{H}}(sC, \frac{R}{2}C) \asymp s$ (again using the Poisson kernel in a rectangle) and $h_{\mathbb{H}}(z, \xi) < c\,m$ for $|z| = R/2$, the upper bound follows. $\square$

References

[1] L. S. Field and G. F. Lawler, Escape probability and transience for SLE, *Electron. J. Probab.* **20** (2015), no. 10, 1–14. `arXiv:1407.3314`

[2] G. F. Lawler, *Conformally invariant processes in the plane*, Mathematical Surveys and Monographs, 114, Amer. Math. Soc., Providence, RI, 2005.