SHEET 14: LINEAR ALGEBRA

Throughout this sheet, let F be a field. In examples, you need only consider the field $F = \mathbb{R}$.

14.1 Vector spaces

Definition 14.1. A *vector space* over F is a set V with two operations,

$$\begin{aligned} & V \times V \to V: \quad (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} \qquad \text{(vector addition)} \\ \text{and} \quad & F \times V \to V: \quad (\lambda, \mathbf{x}) \mapsto \lambda\mathbf{x} \qquad \text{(scalar multiplication)}, \end{aligned}$$

that satisfy the following axioms.

1. Addition is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$.
2. Addition is associative: $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.
3. There is an additive identity $\mathbf{0} \in V$ satisfying $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$.
4. For each $\mathbf{x} \in V$, there is an additive inverse $-\mathbf{x} \in V$ satisfying $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
5. Scalar multiplication by 1 fixes vectors: $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.
6. Scalar multiplication is compatible with F: $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$ for all $\lambda, \mu \in F$ and $\mathbf{x} \in V$.
7. Scalar multiplication distributes over vector addition and over scalar addition: $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$ and $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$ for all $\lambda, \mu \in F$ and $\mathbf{x}, \mathbf{y} \in V$.

In this context, elements of F are called *scalars* and elements of V are called *vectors*.

Definition 14.2. Let n be a nonnegative integer. The *coordinate space* $F^n = F \times \cdots \times F$ is the set of all n-tuples of elements of F, conventionally regarded as column vectors. Addition and scalar multiplication are defined componentwise; that is,

$$\text{if} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{then} \quad \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad \lambda\mathbf{x} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}.$$

We will write $x_1, \ldots, x_n$ for the components of a vector $\mathbf{x} \in F^n$, and other letters similarly. F^0 is the vector space consisting only of the zero vector, so $F^0 = \{\mathbf{0}\}$.

Proposition 14.3. The coordinate space F^n is a vector space.

Exercise 14.4. Consider the field $F = \mathbb{R}$. Which of the following are vector spaces and which are not? You need not provide full proof.

1. The set $P[0,1]$ of all polynomial functions $f : [0,1] \to \mathbb{R}$ with addition and scalar multiplication defined pointwise, namely,

$$\begin{aligned} (f+g)(x) &= f(x) + g(x), \\ (\lambda f)(x) &= \lambda \cdot f(x), \qquad\qquad f, g \in P[0,1], \quad \lambda \in \mathbb{R}. \end{aligned}$$

2. The set $P_n[0,1]$ of all polynomials in $P[0,1]$ of degree at most n, with addition and scalar multiplication defined pointwise as for $P[0,1]$.

3. The set $C[0,1]$ of continuous functions $f : [0,1] \to \mathbb{R}$ with operations defined pointwise.

4. The set of all discontinuous functions $f : [0,1] \to \mathbb{R}$ with operations defined pointwise.

Exercise 14.5. Consider the field $F = \mathbb{R}$ and the vector space $\mathbb{R}^2$. The vector $\mathbf{x} \in \mathbb{R}^2$ is often drawn as an arrow in the plane from the origin to (x_1, x_2), or sometimes as a translation of this arrow. The sum of vectors can be seen as the result of joining them "tip to tail".

Define vectors $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$. Draw $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x}+\mathbf{y}$, $3\mathbf{y}$, and $-\mathbf{x}$ in the plane.

Definition 14.6. Suppose V is a vector space. A nonempty subset $W \subset V$ is a *subspace* of V if it is closed under addition and scalar multiplication, which means that if $\mathbf{x}, \mathbf{y} \in W$ and $\lambda \in F$, then $\mathbf{x} + \mathbf{y} \in W$ and $\lambda \mathbf{x} \in W$.

Exercise 14.7. Which of the following sets are subspaces of $\mathbb{R}^3$?

1. $\{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$
2. $\{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}$
3. $\{\mathbf{x} \in \mathbb{R}^3 \mid x_1 x_2 x_3 = 1\}$
4. $\{\mathbf{x} \in \mathbb{R}^3 \mid x_3 = 0\}$
5. $\{\mathbf{x} \in \mathbb{R}^3 \mid x_3 = 1\}$

Lemma 14.8. Let V be a vector space, $\mathbf{x} \in V$ and $\lambda \in F$. Then $0\mathbf{x} = \mathbf{0}$ and $(-1)\mathbf{x} = -\mathbf{x}$.

Proposition 14.9. Any subspace W of a vector space V is itself a vector space.

Definition 14.10. Let V be a vector space and $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$. A *linear combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is an expression of the form

$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_n \mathbf{x}_n, \qquad \text{where } \lambda_1, \ldots, \lambda_n \in F. \tag{1}$$

A linear combination is said to be *trivial* if the scalars $\lambda_1, \ldots, \lambda_n$ in (1) are all zero.

For completeness, this definition includes the case of an empty sequence of $n = 0$ vectors. In this case, there are no scalars λ_i to pick, the linear combination is trivial (as all zero of the λ_i's are equal to 0) and the value of the linear combination is $\mathbf{0}$. (An empty sum is zero.)

Exercise 14.11. Write the polynomial $h \in P[0,1]$ given by $h(x) = 3x + 4$ as a linear combination of the polynomials f and g given by $f(x) = 2x + 1$ and $g(x) = x + 1$.

Definition 14.12. Suppose $X \subset V$. The *span* of X, written $\langle X \rangle$, is the set of linear combinations of finitely many vectors in X. We say that X *spans* $W \subset V$ if $\langle X \rangle = W$.

As follows from Definition 14.10, the span of the empty set is $\{\mathbf{0}\}$.

Proposition 14.13. If $X \subset V$, then $\langle X \rangle$ is a subspace of V.

Definition 14.14. A finite sequence of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$ is said to be *linearly dependent* if $\mathbf{0}$ is a nontrivial linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, and *linearly independent* otherwise.

Definition 14.15. A finite sequence of vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in V$ is a *basis* (plural: bases, pron. bā′sēz) for V if it is linearly independent and spans V.

Definition 14.16. In F^n, the *standard basis vectors* are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Proposition 14.17. The *standard basis* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a basis for F^n. (In particular, the empty sequence is a basis for $F^0 = \{\mathbf{0}\}$.)

Exercise 14.18. Find a basis for $\mathbb{R}^2$ that contains none of the standard basis vectors, nor any scalar multiple of them. Can you do the same for $\mathbb{R}^3$?

Proposition 14.19. If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a sequence of vectors in V, the following are equivalent.

1. The sequence $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is linearly dependent.
2. One of the vectors is a linear combination of the other vectors in the sequence.
3. One of the vectors is a linear combination of the vectors preceding it in the sequence. (Note that this is true in the case where $\mathbf{x}_1$ is a linear combination of the preceding vectors, of which there are none; that is, $\mathbf{x}_1 = \mathbf{0}$.)

Definition 14.20. A vector space is said to be *finite-dimensional* if it is spanned by finitely many vectors.

Exercise 14.21. Prove that F^n is finite-dimensional, but $P[0,1]$ is not finite-dimensional.

Lemma 14.22. Suppose that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ span V and $\mathbf{x}_i$ is a linear combination of the vectors that precede it. Then the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n$ span V.

Theorem 14.23. If V is finite-dimensional, then V has a basis.

14.2 Linear maps

Definition 14.24. Suppose V and W are vector spaces over the same field F. A function $L : V \to W$ is called a *linear map* if it respects addition and scalar multiplication; that is,

$$L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}) \quad \text{and} \quad L(\lambda\mathbf{x}) = \lambda L(\mathbf{x}) \qquad \text{for all } \mathbf{x}, \mathbf{y} \in V \text{ and } \lambda \in F.$$

Exercise 14.25. Show that the function $L : P[0,1] \to P[0,1]$ defined by $L(f) = f'$ is a linear map.

Exercise 14.26. Describe all linear maps from $\mathbb{R}^1$ to $\mathbb{R}^n$.

Lemma 14.27. *If $L : V \to W$ is a linear map then $L(\mathbf{0}) = \mathbf{0}$ and $L(-\mathbf{x}) = -L(\mathbf{x})$ for all $\mathbf{x} \in V$.*

Sums $L + M$ and scalar multiples λL of linear maps are defined pointwise, just as are sums of functions. To add two linear maps, their domains and codomains must match.

The composition $L \circ M$ of two linear maps is usually written LM. It exists so long as the domain of L is the codomain of M.

Proposition 14.28. *Sums, scalar multiples and compositions of linear maps are linear.*

Definition 14.29. The *kernel* of a linear map $L : V \to W$, denoted $\ker(L)$, is the set of all vectors $\mathbf{x} \in V$ such that $L(\mathbf{x}) = \mathbf{0}$.

Lemma 14.30. *In any vector space, $\lambda\mathbf{0} = \mathbf{0}$ for all $\lambda \in F$.*

Exercise 14.31. Suppose $L : V \to W$ is a linear map. Prove that $\ker(L)$ is a subspace of V.

Exercise 14.32. Show that a linear map $L : V \to W$ is injective if and only if $\ker(L) = \{\mathbf{0}\}$.

Definition 14.33. A bijective linear map is called an *isomorphism* of vector spaces. If there is an isomorphism from V to W, we say that V and W are *isomorphic*, and we write $V \cong W$.

Proposition 14.34. *A function $L : V \to W$ is an isomorphism if and only if the inverse function $L^{-1} : W \to V$ exists and is an isomorphism.*

Propositions 14.28 and 14.34 show that isomorphism is an equivalence relation.

Theorem 14.35. *Suppose that the sequence $\mathbf{b}_1, \ldots, \mathbf{b}_n$ is a basis of V. Then the map*

$$B : F^n \to V \qquad \text{defined by} \qquad B(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n$$

is an isomorphism. In particular, every vector $\mathbf{y} \in V$ can be expressed as a linear combination of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ in a unique way. The unique vector $\mathbf{x} \in F^n$ such that $B(\mathbf{x}) = \mathbf{y}$ is called the coordinate vector *of $\mathbf{y}$ with respect to the basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$.*

Exercise 14.36. Suppose $k \in \mathbb{N}$. Find an isomorphism $B : \mathbb{R}^n \to P_k[0,1]$ for some natural number n. What value of n did you choose?