SHEET 15: MATRICES

15.1 The matrix of a linear map between coordinate spaces

Definition 15.1. An $m \times n$ *matrix* A (plural: matrices, pron. mā′trisēz) is a rectangular array of elements of F with m rows and n columns. Its i, j *entry* A_{ij} is the value in the ith row of the jth column, counting from left to right and top to bottom. The set of all $m \times n$ matrices is denoted $M_{m\times n}(F)$. A column vector $\mathbf{x} \in F^n$ is considered to be an $n \times 1$ matrix.

Note that $M_{m\times n}(F)$ is a vector space over F with addition and scalar multiplication performed entry by entry, i.e., $(A + B)_{ij} = A_{ij} + B_{ij}$ and $(\lambda A)_{ij} = \lambda\, A_{ij}$. For example,

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & -1 & 4 \end{bmatrix} + 2 \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}.$$

Proposition 15.2. Let $\mathcal{L}(F^n, F^m)$ be the vector space of all linear maps $L : F^n \to F^m$, with addition and scalar multiplication defined pointwise. There is an isomorphism $\Psi : \mathcal{L}(F^n, F^m) \to M_{m\times n}(F)$ taking each linear map $L : F^n \to F^m$ to the $m \times n$ matrix $\Psi(L)$ whose ith column is $L(\mathbf{e}_i)$. We say that the matrix $\Psi(L)$ *represents* the linear map L.

The very natural isomorphism Ψ means that linear maps $L : F^n \to F^m$ correspond exactly to $m \times n$ matrices $\Psi(L)$ (and vice versa) in a way that respects addition and scalar multiplication. In a slight abuse of notation, we will identify L with $\Psi(L)$, so that a linear map $L : F^n \to F^m$ is simultaneously an $m \times n$ matrix L and L_{ij} its i, j entry. For example, the linear map $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y + z \\ 3x + y \end{pmatrix}$ is identified with the matrix $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$.

Proposition 15.3. Let $A : F^n \to F^m$ be a linear map and $\mathbf{x} \in F^n$. Then $A(\mathbf{x})$ is the vector whose ith entry is the scalar

$$\sum_{j=1}^{n} A_{ij} x_j. \tag{1}$$

Exercise 15.4. Express the identity map $I : F^n \to F^n$, defined by $I(\mathbf{x}) = \mathbf{x}$, as a matrix.

Exercise 15.5.

1. Give a matrix which represents a linear map $R : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each vector in $\mathbb{R}^2$ by a right angle counterclockwise.
2. Give a matrix representing a linear map $P : \mathbb{R}^3 \to \mathbb{R}^3$ that is a *projection*, which means that $P^2 = P$, and whose range is $\{\mathbf{x} \in \mathbb{R}^3 : x_3 = 0\}$. Is there more than one such map?

Definition 15.6 (Matrix multiplication)**.** The product AB of two matrices A and B is exactly the matrix that represents the composition of linear maps $AB = A \circ B$.

In order to compose two linear maps AB, the codomain of B must be the domain of A. Therefore, in order to form the product AB, B must have as many rows as A has columns.

Proposition 15.7. Matrix multiplication is associative. *Hint: first show that composition of functions is associative, and apply this to the linear maps that the matrices represent.*

Proposition 15.8. If $A : F^n \to F^m$ and $B : F^p \to F^n$ are linear maps and $C = AB$, show that the i, k entry of the matrix representing C is given by

$$C_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}. \tag{2}$$

Comparing (2) with (1), we see that applying a linear map $A : F^n \to F^m$ to a vector $\mathbf{x} \in F^n$ gives the matrix product $A\mathbf{x}$.

Exercise 15.9. Is multiplication of $n \times n$ matrices commutative? Distributive over addition?

Definition 15.10. The *null space* of a matrix A, denoted $N(A)$, is the kernel of the associated linear map A. If A is a matrix, the span of the columns of A is called the *column space* of A and denoted $C(A)$.

Exercise 15.11. Find the null spaces of the two matrices defined in Exercise 15.5.

Proposition 15.12. If A is a linear map between coordinate spaces, the column space $C(A)$ is the range of the linear map A, and is a vector space.

Definition 15.13. The *transpose* of a matrix A is the matrix A^T whose i, j entry is the j, i entry of A. The *row space* of A, denoted $R(A)$, is the span of the rows of A, so $R(A) = C(A^T)$.

Definition 15.14. The *dot product* of two vectors $\mathbf{x}, \mathbf{y} \in F^n$ is

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Vectors are said to be *perpendicular*, or *orthogonal*, if their dot product is 0.

Exercise 15.15. Find the set of vectors in $\mathbb{R}^3$ perpendicular to $\mathbf{e}_1$.

Exercise 15.16. Fix a vector $\mathbf{x} \in F^n$. Prove that the set of vectors perpendicular to $\mathbf{x}$ is a subspace of F^n.

Theorem 15.17. The null space of a matrix is the set of all vectors that are perpendicular to each vector in the row space.

15.2 Gaussian elimination

Definition 15.18. The *elementary row operations* on a matrix are the operations (a) $R_i \to R_i + sR_j$, (b) $R_i \to sR_i$ and (c) $R_i \leftrightarrow R_j$, for any $i \neq j$ and $s \in F$, $s \neq 0$. Here R_i is the ith row of the matrix and $\to$ means "is replaced by".

Definition 15.19. An *elementary matrix* E is a matrix that is equal to the identity matrix except for exactly one of the following changes:

(a) there are numbers $i \neq j$ such that $E_{ij} = s$ for some $s \neq 0$, or

(b) there is a number i such that $E_{ii} = s$ for some $s \neq 0$, or

(c) there are numbers $i \neq j$ such that $E_{ij} = E_{ji} = 1$ and $E_{ii} = E_{jj} = 0$.

Proposition 15.20. For each elementary row operation, there is an elementary matrix E such that the row operation transforms a matrix M into EM.

Definition 15.21. A matrix that represents an isomorphism is called *invertible.*

Lemma 15.22. If A and B are matrices such that $AB = I$ and $BA = I$, then A and B are invertible. (As an aside, for now A may be an $m \times n$ matrix and B an $n \times m$ matrix. It will follow from Corollary 15.29, however, that $m = n$, so invertible matrices must be square.)

Lemma 15.23. The elementary matrices are invertible.

Theorem 15.24. Suppose a matrix A is transformed by elementary row operations into B.

1. A and B have the same row space.
2. There is an isomorphism $L : C(A) \to C(B)$ that sends each column of A to the corresponding column of B.

Definition 15.25. A *leading entry* of a matrix is one that is the first nonzero entry in its row. A matrix is in *row echelon form* if all leading entries are 1, leading entries (also called *pivots*) proceed from left to right down the matrix, and any zero rows are at the bottom. It is *reduced* if, in addition, all entries above a pivot in the same column as the pivot are zero.

Exercise 15.26. Use elementary row operations to put the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix}$$

in reduced row echelon form.

Proposition 15.27. Any matrix can be transformed into reduced row echelon form by elementary row operations. This is called *Gaussian elimination* or *row reduction.*

Proposition 15.28. Let A be an $m \times n$ matrix and B a reduced row echelon form of A. Let J be the set of all $j \in \{1, 2, \ldots, n\}$ such that B has no pivot in column j (the *non-pivot indices*). Then for each choice of scalars y_j over all $j \in J$ there is a unique vector $\mathbf{x} \in N(A)$ such that $x_j = y_j$ for all $j \in J$. *Hint: use Theorem 15.17.*

In other words, this proposition says that a vector in the null space of A is uniquely determined by its coordinates that have non-pivot indices, which may be chosen arbitrarily.

Corollary 15.29. If a linear map $A : F^n \to F^m$ is injective, then $m \geq n$.

Theorem 15.30. Let V be a finite-dimensional vector space. Then all bases of V have the same number of vectors. This number is called the *dimension* of V and written $\dim V$.

Corollary 15.31. If V and W are finite-dimensional vector spaces and $V \cong W$, then $\dim V = \dim W$.

Lemma 15.32. Let A be an $m \times n$ matrix and B a reduced row echelon form of A. The columns of A where B has a leading 1 form a basis for $C(A)$.

Theorem 15.33. The row and column spaces of a matrix have equal dimension, namely the number of leading 1s in its reduced row echelon form. This number is called the *rank* of the matrix.

Theorem 15.34 (Rank–Nullity Theorem)**.** If A is an $m \times n$ matrix, then $\dim C(A) + \dim N(A) = n$. (The *nullity* of A is $\dim N(A)$.)

Definition 15.35. Suppose A is an $m \times n$ matrix and B is an $m \times p$ matrix. The matrix of A *augmented by* B is the matrix $[A \mid B]$ formed by adjoining A and B. That is, $[A \mid B]_{ij} = A_{ij}$ if $1 \leq j \leq n$ and $[A \mid B]_{ij} = B_{i(j-n)}$ if $n + 1 \leq j \leq n + p$.

Exercise 15.36. Suppose $A \in M_{n \times m}(F)$ and $\mathbf{b} \in F^n$. Show that there is a vector $\mathbf{x} \in F^m$ such that $A\mathbf{x} = \mathbf{b}$ if and only if the row echelon form of the augmented matrix $[A \mid \mathbf{b}]$ has no leading 1 in the right column. If so, describe how to find a solution $\mathbf{x}_0$, and show that the general solution is given by $\mathbf{x} = \mathbf{x}_0 + \mathbf{y}$ for any $\mathbf{y} \in N(A)$.

Exercise 15.37. Show that a square matrix A is invertible if and only if its reduced row echelon form is the identity matrix I, and if so, A^{-1} can be computed as the right half of the reduced row echelon form of the matrix $[A \mid I]$.

Exercise 15.38. Compute the inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$.

Exercise 15.39. Compute the inverse of an arbitrary 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What algebraic condition on the entries of A is necessary and sufficient for the inverse A^{-1} to exist?