SHEET 16: LINEAR OPERATORS

We call a linear map from a vector space to itself a *linear operator*.

16.1 Eigenvalues and eigenvectors

The study of eigenvalues and eigenvectors arises from the desire to understand the vectors on which a linear operator acts just by scaling.

Definition 16.1. Let A be a linear operator and $\lambda \in F$. The set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ is called the λ-*eigenspace* of A. If any such vector is nonzero, we call it a λ-*eigenvector* of A and call λ an *eigenvalue* of A.

In other words, the λ-eigenspace of A is the null space of $A - \lambda I$. In particular, the 0-eigenspace of a matrix is just its null space.

Exercise 16.2. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

Some of the importance of eigenvalues lies in the following theorem.

Theorem 16.3. Suppose $A : V \to V$ is a linear operator and V has a basis $\mathbf{b}_1, \dots, \mathbf{b}_n$ consisting of eigenvectors of A (an *eigenbasis*). Let $B : F^n \to V$ be the isomorphism corresponding to that basis, as in Theorem 14.33, given by

$$B(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n.$$

Then the matrix of the linear map $B^{-1}AB : F^n \to F^n$ is a *diagonal matrix* (its off-diagonal entries are zero), and each diagonal entry is the eigenvalue of the corresponding eigenvector.

In other words, if there is an eigenbasis for a linear operator, its action on a vector is best understood by expressing that vector with respect to the eigenbasis: each coordinate is multiplied by the corresponding eigenvalue.

Exercise 16.4. Show that if $A \in M_{n\times n}(F)$ is a diagonal matrix, then $(A^k)_{ii} = (A_{ii})^k$ for any $k \in \mathbb{N}$.

Exercise 16.5. Let $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$. Check that $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of A. Use this to compute A^{10} by hand.

Remark 16.6. If we know an eigenvalue λ of A, we can find the corresponding eigenvectors by row reducing $A - \lambda I$ to find its null space. But we still have no effective means of finding eigenvalues in the first place, which amounts to finding the values of λ for which $A - \lambda I$ is *singular* (i.e., not invertible).

16.2 Volume and determinants

Consider the following notion of n-dimensional volume: the volume of a n-dimensional cube of side length ℓ is ℓ^n, and the volume of a general shape S is determined by approximating S by smaller and smaller n-dimensional cubes and adding their volumes. (This is analogous to how area under a curve was defined by approximation by rectangles.) To see how volume is transformed under linear operators, it is enough to find the scale factor by which the volume of the unit cube is transformed. A linear operator $A : F^n \to F^n$ transforms the unit cube, whose edges are $\mathbf{e}_1, \ldots, \mathbf{e}_n$, into the *parallelepiped* with edges $A\mathbf{e}_1, \ldots, A\mathbf{e}_n$, which are the columns of the square matrix A.

One can check that (a) adding one column to another does not change the volume of this parallelepiped, and (b) scaling one column by a factor c multiplies the volume by $|c|$. (Draw a picture to convince yourself of these in 2 dimensions.) In fact, (a) and (b) uniquely characterize the notion of volume.

The absolute value sign here is inconvenient. Instead of volume, we will consider a concept of signed volume, which changes by a factor c when any one column is scaled by c. It will then follow that the absolute value of signed volume satisfies (a) and (b) and hence agrees with our notion of volume. This idea is analogous to how we attach a minus sign to integrals that proceed from right to left in order to preserve the additivity property of integrals.

We will now make the above discussion of signed volume precise and applicable to a general field F.

Definition 16.7. Fix F and n. A *signed volume* is a function $V : M_{n\times n}(F) \to F$ whose value is unchanged if one column is added to an adjacent column, and changes by a factor of c if one column is multiplied by c. Call these two column operations *good moves.*

Lemma 16.8. We can swap adjacent columns using good moves.

Elementary column operations are the same as elementary row operations, but on columns.

Lemma 16.9. We can do elementary column operations using good moves.

Proposition 16.10. Let V be a signed volume. $V(A)$ is zero if A is singular; else it is $V(I)$ times the product of the scale factors that appear in any sequence of elementary column operations that reduce A to column echelon form. (A swap $C_i \leftrightarrow C_j$ yields a factor of -1.)

This proposition says that any signed volume V is fully determined by the scalar $V(I)$.

Exercise 16.11. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and assume that A is invertible. Calculate $V(A)$ in terms of $V(I)$.

Note that the zero function satisfies the conditions of Definition 16.7. We do not yet know that there is a nonzero signed volume. We will now construct one.

Definition 16.12. A *permutation* σ on n elements is a bijective function from $\{1, 2, \ldots, n\}$ to itself; the set of all such is denoted S_n.

An *inversion* in a permutation σ is a pair (i, j) such that $i < j$ but $\sigma(i) > \sigma(j)$; we will denote the number of inversions by $N(\sigma)$. The *sign* of a permutation σ is $\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)}$.

Definition 16.13. The determinant is the function $\det : M_{n\times n}(F) \to F$ defined by

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, A_{\sigma(1)1}\, A_{\sigma(2)2} \ldots A_{\sigma(n)n}.$$

We present this formula purely so that we know such a function exists. For practical calculations, except perhaps in dimensions 2 and 3, one should always find a determinant using row or column operations (or another efficient method). Applying a row operation to a matrix yields the same scale factor for the determinant as the corresponding column operation; this will follow from Theorem 16.25, but may be useful in exercises before then.

Proposition 16.14. The identity matrix I has determinant 1.

Exercise 16.15. Calculate $\det(A)$ for an arbitrary 2×2 matrix A.

Exercise 16.16. Given $\lambda \in F$, find $\det(\lambda A)$ in terms of $\det(A)$.

Lemma 16.17. If two adjacent columns are equal, the determinant is zero.

Proposition 16.18. The determinant is a signed volume.

Exercise 16.19. Use row or column operations to find $\det \begin{bmatrix} 1 & -1 & 0 & 3 \\ 1 & 0 & 3 & 4 \\ -2 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 \end{bmatrix}$.

Theorem 16.20. A square matrix A is singular if and only if its determinant is zero.

Corollary 16.21. The eigenvalues of a square matrix A are exactly the zeroes λ of the *characteristic polynomial* $\det(A - \lambda I)$, considered as a function of λ.

Proposition 16.22. The determinant of an *upper (or lower) triangular matrix*—one with all entries zero below (above) the main diagonal—is the product of its diagonal entries.

Exercise 16.23. Find the eigenvalues and eigenvectors of (a) $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 2 \\ 2 & -2 & 1 \end{bmatrix}$.

Hint: Use row operations on $A - \lambda I$, and try to avoid dividing by expressions that involve λ. Then use Remark 16.6 to find the eigenvectors.

Lemma 16.24. Let A be a fixed matrix. The function $V : M_{n\times n}(F) \to F$ defined by $V(B) = \det(AB)$ is a signed volume.

Theorem 16.25. The determinant is multiplicative: $\det(AB) = \det(A)\,\det(B)$.

Exercise 16.26. Is the determinant additive, meaning that $\det(A+B) = \det(A)+\det(B)$?

Corollary 16.27. The determinant of a matrix that has an eigenbasis is the product of its eigenvalues, where each eigenvalue is counted as many times as the dimension of its eigenspace.