The Cone Theorem

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These notes are supposed to be a handout for the student seminar in algebraic geometry at the University of Utah. In this seminar, we will give an overview of the cone theorem, together with motivations and consequences of such result. These notes are not meant to be comprehensive either exhaustive. A detailed discussion could be found in Kollar and Mori's "Birational Geometry of Algebraic Varieties".

Introduction

One of the main goals of algebraic geometry is to classify varieties in families. One way to approach this deep and vast problem is to study varieties up to birational equivalence. This leads to the following questions: how to determine a suitable representative for an equivalence class? In what sense can a variety be nicer or simpler than a birational one?

This circle of ideas moved from the study of surfaces by the Italian school, trying to generalize the concept of minimal surface to higher dimension. Starting with Mori in the 80's, the unpredicted problems in achieving this goal stimulated the development of completely new ideas and techniques.

In the proceeding, unless otherwise mentioned, varieties will be assumed to be defined on the field of complex numbers $\mathbb{C}$.

The case of surfaces and -1-curves

Given a proper smooth surface S, we can naturally define intersection products between curves $C_1, C_2 \subset S$: in case they meet transversally, the intersection product $C_1 \cdot C_2$ is given by the sum of their intersection points. More generally, in case they do not share connected components, the geometric interpretation is still valid, but needs to be wighted by local intesection multiplicities. It can also be showed that this product is well defined and can be extended to curves sharing a connected component. One way to see that is to write a linear equivalent representative of C_1 as difference of cycles that meet C_2 transversally (this can be achieved with ample cycles, which move in families). A second interpretation comes from the equivalent definition of $C_1 \cdot C_2$ as $\deg \mathcal{O}_{C_1}(C_2)$.

One of the main tools of study and classification for smooth surfaces comes, as we will see later, from an analysis of the above product. In particular, this product defines an equivalence relation on cycles, called numerical equivalence: two cycles are in the same coset if they differ by a cycle C such that $C \cdot D = 0$ for any cycle D (C is called numerically trivial). The quotient group is called Neron-Severi group of S, denoted by $NS(S)$, and has rank $\rho(S)$, called Picard number[1].

In our discussion, the first important result about surfaces is due to the Italian school of early twentieth century.

Theorem (Castelnuovo)**.** If X is a smooth proper surface, then there is a nontrivial birational morphism $f : X \to Y$ to a smooth surface Y iff X contains a smooth rational f-exceptional curve with self-intersection -1.

Observation. Castelnuovo's contractibility criterion can be consider a very satisfactory first step. Indeed, it is telling us that anything redundant in our surface locally analitically looks like a (sequence of) blow up of $\mathbb{C}^2$ at a point.

Numerical equivalence and cones

The above mentioned intersection product and numerical equivalence can be generalized to proper varieties of higher dimension. The major thing to notice is that now curves are no more divisors, and therefore they will not correspond to line bundles anymore. In this more general setting, the pairing will be between the group of Cartier divisors $\mathrm{Div}(X)$ and the group of one-cycles $Z_1(X)$. Since Cartier divisors are represented by locally principal Weil divisors, the geometric interpretation of the pairing still holds for a divisor D and a curve C such that $C \not\subset D$. In general, the product $D \cdot C$ is again given by $\deg \mathcal{O}_C(D)$.

In general, it is useful to have the flexibility to allow rational or even real coefficients in the groups above. In such case, we will make our choice evident, such as in $\mathrm{Div}(X)_{\mathbb{Q}}$. This need for more coefficients has various reasons. First of all, our variety X might not be smooth. In such case, there might be Weil divisors that are not Cartier, but a multiple of whose is Cartier. For such a divisor D, we want to define the product $D \cdot C$ for a curve C. Assuming that $mD = D'$ is Cartier, we just set $D \cdot C = \frac{D' \cdot C}{m}$.

Example. Call C the cone over a conic in $\mathbb{P}^3$. Then a line L through the vertex V is a so called $\mathbb{Q}$-Cartier divisor: L itself is a Weil but not Cartier divisor, while $2L$ is a Cartier divisor. In particular, $2L$ is given by a hyperplane cut (a hyperplane tangent to the cone), thus it is linearly equivalent to a conic section H. Here curves are divisors, since we are on a surface, so it makes sense to compute L^2. One sees geometrically that $H \cdot L = 1$, and therefore we have $L^2 = \frac{1}{2}$.

Once we allow rational coefficients, it is sometimes useful to allow real ones simply for completeness and limit convergence reasons.

The natural pairing $\mathrm{Div(X)}_{\mathbb{R}} \times \mathrm{Z_1(X)}_{\mathbb{R}} \to \mathbb{R}$ induces a numerical equivalence relation on both factors. This induces a perfect pairing between $N^1(X)_{\mathbb{R}}$, the

[1]There are different definitions of the Neron-Severi group. These do not in general match; on the other hand they match up to torsion, wich is what matters for our discussion.

Neron-Severi group extended with real coefficients, and $N_1(X)_{\mathbb{R}}$, the group of one-cycles with real coefficients modulo numerical equivalence. In each of these vector spaces there are natural cones to consider. Our main focus will be on the cone of effective curves in $N_1(X)_{\mathbb{R}}$: we will denote by $NE(X) = \{\sum a_i\,[C_i]\,|C_i \subset X$ effective curve, $a_i \geq 0\}$ the cone of curves, and by $\overline{NE}(X)$ the closed cone of curves, i.e. the closure of $NE(X)$ in the Euclidean topology. Given a divisor class $D \in N^1(X)_{\mathbb{R}}$, we denote by $\overline{NE}(X)_{D\geq 0} = \overline{NE}(X) \cap \{D \geq 0\}$, where $\{D \geq 0\}$ denotes the cone in $N_1(X)_{\mathbb{R}}$ where D is non-negative as functional.

Intersection with K_X and rational curves

One main character has not come into play yet: the canonical bundle ω_X. We will assume our varieties to be normal, so that ω_X is a reflexive sheaf given by a Weil divisor. Although no preferrable choice of divisor D satisfying $\mathcal{O}_X(D) \cong \omega_X$ exists in general, we will talk about the canonical divisor K_X. Ineed, we will be mostly interested in the numerical properties of K_X, which do not depend on the choice of a representative over another.

Castelnuovo's theorem suggests that the presence of rational curves influences the geometry of a variety. The formalism of cones will allow us to interpret such a result from another point of view, and to generalize it.

The first results of Mori in the direction of a generalization of what was known about surfaces rely on the bend and break lemmas. His idea was that families of curves with certain geometric properties would degenerate producing rational curves. A discussion of this beautiful topic goes beyond the purpose of these notes; therefore we refer to Kollar and Mori's nice exposition in the book "Birational Geometry of Algebraic Varieties". Via these ideas, relying on the presence of rational curves under certain assumptions, Mori proved the following.

Theorem (Mori)**.** Let X be a smooth projective variety and H an ample divisor on X. Assume that there is an irreducible curve $C' \subset X$ such that $-(K_X \cdot C') > 0$. Then there is a rational curve $E \subset X$ such that

$$\dim X + 1 \geq -(K_X \cdot E) > 0.$$

Now, we recall that a divisor class D (or a line bundle) is called nef (i.e. numerically effective) if for any irreducible curve $C \subset X$ we have $D \cdot C \geq 0$. Thus, if we consider a smooth projective surface S whose canonical divisor K_S is not nef, we can produce interesting rational curves. Since $\dim X = 2$, we have that the rational curve produced has $K_S \cdot E \in \{-1, -2, -3\}$. Since we are in the smooth case, we can use adjunction, i.e. $(K_S + E) \cdot E = \deg \omega_E = -2$. Thus, the three cases for $K_S \cdot E$ correspond to $E^2 = -1$, $E^2 = 0$, and $E^2 = 1$.

We immediately recognize that the first case corresponds to Castelnuovo's contractibility criterion. Indeed, we can go even backwards: if we know that we have a -1-curve C, using adjunction we recover that $K_S \cdot C = -1$. Therefore Castelnuovo's criterion falls into this analysis of the failure of K_S to be nef. This one is also supposed to be the general case: $E^2 = 0$ and $E^2 = 1$ correspond to two particular families of surfaces. In the first case, S is birational to a ruled surface: we can think of $E^=0$ as the fibers of the ruling having intersection 0. The second case, $E^2 = 1$, will correspond to a surface birational to $\mathbb{P}^2$: we can think of $\mathbb{P}^2$ itself, where given a line L, we have $L^2 = 1$, and $K_{\mathbb{P}^2} \cong -2L$.

The cone theorem (smooth case)

From what we discussed above, we now have a better idea why having a nef canonical bundle is desirable. On the one hand, our goal is to understand what makes K_X not nef. On the other hand, a generalization of Castelnuovo's contractibility criterion is needed, in order to get rid of the curves with negative intersection with K_X.

Here is where the machinery about cones comes into play. We recall that a face of a cone C in a vector space V is called extremal if for any $a, b \in V$ with $a + b \in C$, then $a, b \in C$. The first main result is the following.

Theorem (Cone Theorem (smooth case))**.** Let X be a non-singular projective variety.

(1) There are countably many rational curves $C_i \subset X$ such that $0 < -(K_X \cdot C_i) \leq \dim X + 1$, and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0} [C_i] .$$

(2) For any $\varepsilon > 0$ and ample divisor H,

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X + \varepsilon H) \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0} [C_i] .$$

Observation. The $\mathbb{R}_{\geq 0} [C_i]$'s are extremal rays (i.e. one dimensional faces) of the closed cone $\overline{NE}(X)$. Furthermore, we point out that (1) above is a formal consequence of (2) via geometric properties of cones. This fact is important for the generalizations of the theorem, since we have the freedom to prove a statement perturbing K_X by a small ample class.

This important theorem shows how the failure of K_X to be nef is given by rational curves, and the families they span. What is left, is to understand how to get rid of them. What we are looking for are contraction morphisms.

Definition. Let X be a projective variety and $F \subset \overline{NE}(X)$ an extremal face. A morphism $\text{cont}_F : X \to Z$ is called contraction of F if the following hold:

(1) $\text{cont}_F(C)$ is a point for an irreducible curve $C \subset X$ iff $[C] \in F$,

(2) $(\text{cont}_F)_* \mathcal{O}_X = \mathcal{O}_Z$.

Observation. Property (1) in the definition above prescribes what the map should be set theoretically. Property (2) is saying that cont_F should be its own Stein factorization.

Contraction morphisms do not in general exist. Given this new piece of vocabulary, we can rephrase our previous question as follows: are there contraction morphisms for the $\mathbb{R}_{\geq 0}[C_i]$'s? As we will see, the answer is just partly positive. Namely, we will be able to contract those rays, but we will in general fall off the smooth case. Because of this, it will be better to state the result just for surfaces and threefolds.

Theorem. Let S be a smooth projective surface and $R \subset \overline{NE}(S)$ an extremal ray such that $K_S \cdot R < 0$. Then the contraction morphism $\mathrm{cont}_R : S \to Z$ exists and is one of the following types:

(1) Z is a smooth surface and S is obtained from Z blowing up a closed point; $\rho(Z) = \rho(S) - 1$.

(2) Z is a smooth curve and S is a minimal ruled surface over Z; $\rho(S) = 2$.

(3) Z is a point, and $S \cong \mathbb{P}^2$.

Proof. [Sketch] Call C a rational curve generating R, i.e. $R = \mathbb{R}_{\geq 0}[C]$. The three cases correspond to the different possible signs of C^2. If $C^2 < 0$, adjunction, together with $K_S \cdot C < 0$, tells us that $C^2 = -1$. Therefore (1) is equivalent to Castelnuovo's criterion.

If $C^2 = 0$, we argue as follows. Since C is an effective divisor, $H^2(S, \mathcal{O}_S(mC)) = 0$ for $m >> 0$. Indeed, fixed an ample class H, we know $C \cdot H > 0$. On the other hand, $H^2(S, \mathcal{O}_S(mC)) \cong H^0(S, \mathcal{O}_S(K_S - mC))\check{}$. If the latter has sections, then it is equivalent to an effective divisor, and therefore $(K_S - mC) \cdot H > 0$. On the other hand, for $m >> 0$ this quantity is negative, and the claim follows. Thus, by Riemann-Roch we know $h^0(S, \mathcal{O}_S(mC)) \geq \chi(S, \mathcal{O}_S(mC)) = -\frac{K_S \cdot C}{2} + \chi(\mathcal{O}_S) \geq 2$ for $m >> 0$. Since $C^2 = 0$, it means that the general element of $|mC|$ can be moved away from C. Therefore $|mC|$ is base point free, and defines a morphism f. The contraction map is then the Stein factorization of f.

Finally, if $C^2 > 0$, a result about the cone of curves tells us it is in the interior of $\overline{NE}(S)$. On the other hand, we are assuming it is extremal. Therefore $\rho(S) = 1$, and C is an ample divisor, since S is projective and any divisor class is a multiple of the one of C. Therefore, a map contracting an ample divisor contracts the variety to a point. Then it can be shown that all of this together implies that S is $\mathbb{P}^2$. □

This shows that in the case of surfaces our study up to birational equivalence reduces essentially to two cases: minimal ruled surfaces, and smooth surfaces with nef canonical bundle. On the other hand, the case of surfaces gives problems.

Theorem. Let X be a non-singular projective threefold and $\mathrm{cont}_R : X \to Y$ the contraction of a K_X-negative extremal ray $R \subset \overline{NE}(X)$. Then the following is a list of the possibilities for cont_R:

(E) : (Exceptional) $\dim Y = 3$, cont_R is birational and there are five types of local behaviour near the contracted surface:

(E1) : cont_R is the (inverse of) blow-up of a smooth curve in the smooth threefold Y.

(E2) : cont_R is the (inverse of) blow-up of a smooth point of the smooth threefold Y.

(E3) : cont_R is the (inverse of) blow-up of an ordinary double point of Y. (Locally analytically, an ordinary double point is given by the equation $x^2 + y^2 + z^2 + w^2 = 0$.)

(E4) : cont_R is the (inverse of) blow-up of a point of Y which is locally analytically given by the equation $x^2 + y^2 + z^2 + w^3 = 0$.

(E5) : cont_R contracts a smooth $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$ to a point of multiplicity 4 on Y which is locally analytically the quotient of $\mathbb{C}^3$ by the involution $(x, y, z) \mapsto (-x, -y, -z)$.

(C) : (Conic bundle) $\dim Y = 2$ and cont_R is a fibration whose fibers are plane conics. (General fibers are smooth.)

(D) : (Del Pezzo fibration) $\dim Y = 1$ and general fibers of cont_R are Del Pezzo surfaces.

(F) : (Fano variety) $\dim Y = 0$, $-K_X$ is ample and hence is a Fano variety.

Observation. On the one hand, we have a generalization of what happened for surfaces: (C) and (D) correspond to ruled surfaces, (F) to the $\mathbb{P}^2$-case, (E1) and (E2) to Castelnuovo's criterion. On the other hand, we have introduced new patterns: (E3), (E4) and (E5) introduce singular varieties, which did not occur in the case of surfaces.

The case of threefolds illustrates how increasing the dimension introduces new patterns. Nevertheless, the case of smooth threefolds still does not show how bad things could be. Indeed, all the cases described in (E) involve contraction of divisors. A phenomenon that in the smooth setting first appears in fourfolds (in threefolds in the singular case) is the one of small contractions. A birational morphism $f : X \to Y$ is called small contraction if the exceptional locus $Ex(f)$ has codimension at least two. In such a case, we introduce very bad singularities. Indeed, assume X is normal and D is an effective Cartier divisor that meets transversally the exceptional locus. In particular, there is a contracted curve C such that $D \cdot C > 0$. On the other hand, the Weil divisor D' given by the image of D can not be $\mathbb{Q}$-Cartier. Indeed, assume mD' is Cartier. Then mD and $f^*(mD')$ are two line bundles that are isomorphic out of a set of codimension at least two. Since X is normal, the isomorphism extends globally. On the one hand, $mD \cdot C > 0$, while, on the other hand, $f^*(mD') \cdot C = 0$. This leads to a contradiction, and thus D' can not be Cartier.

Example. The easiest example of small contraction in the smooth case is the following. Consider $\mathbb{P}^2$ and the vector bundle $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ over it. Assume X is a compactification of such variety. Then any line in $\mathbb{P}^2$ generates a negative extremal ray. If we contract it, we contract the whole $\mathbb{P}^2$ (indeed it is covered by lines equivalent to the one we picked), while outside this section the map is an isomorphism. More precisely, consider the compactification given by $X = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)) = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})$. Then, call $\pi : X \to \mathbb{P}^2$ and $E = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$. Using the formula for the canonical bundle of a projective bundle, we have $K_X = -3\xi + \pi^*(K_{\mathbb{P}^2} + \det E)$, where ξ is the class of $\mathcal{O}_X(1)$. Identify $\mathbb{P}^2$ with the section given by the surjection $E \to \mathcal{O}_{\mathbb{P}^2}$. Then $\mathcal{O}_X(1)|_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$. Since $\det E = \mathcal{O}_{\mathbb{P}^2}$, we have $K_X \cdot L = -4$ for a line $L \subset \mathbb{P}^2$. The contraction map is then given by the linear series $|\mathcal{O}_X(1)|$. Indeed $H^0(X, \mathcal{O}_X(1)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \oplus H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \oplus H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})$, thus the sections coming from the $\mathcal{O}_{\mathbb{P}^2}(1)$ tell points apart except from our fixed section $\mathbb{P}^2$: there the only nonvanishing section comes from $\mathcal{O}_{\mathbb{P}^2}$, and therefore $\mathbb{P}^2$ gets contracted via this map.

Pairs

Since the contraction of extremal rays might lead out of the smooth setting, it is necessary to determine a class of varieties for which Mori's program is stable. More precisely, we would like to have a class of varieties for which a generalization of the cone theorem holds, and for which contraction morphisms exist and do not lead out of the fixed class. First of all, we will require our varieties to be normal. This is very desirable: although some Weil divisors might not correspond to line bundles, all of them correspond to a reflexive sheaf.

One main problem we have to fix is that the canonical divisor[2] might be not Cartier, or even $\mathbb{Q}$-Cartier. This is one of the reasons why we introduce the notion of pair: a pair (X, Δ) is the datum of a normal variety X and a boundary (Weil) $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (i.e. it is a Weil divisor such that $m(K_X + \Delta)$ is Cartier for some $m \in \mathbb{N}_{>0}$). Another reason why the notion of pair is useful is the flexibility it allows: in many settings we might want to perturb Δ to reduce to a known case, and this would not be possible if we had to stay stick to just K_X.

Now that a notion of pair is introduced, we have to give a notion of for pairs to be more or less singular. First, we start with an example.

Example. Consider $f : X \to Y$ a birational morphism of smooth varieties. Then we know that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and that $f^*(K_Y) = K_X + E$, where E is an exceptional divisor. Indeed, we can think of the pullback of local holomorphic forms: since the map is birational, canonical sheaves will agree along the isomorphic open subsets. On the other hand, where the rank of the jacobian of f drops, a non-vanishing form is pulled back to a vanishing one. Therefore differential forms in $f^*(K_Y)$ are differential forms vanishing along the exceptional locus. Thus, $f^*(K_Y) = K_X - E$ for E exceptional. If we write $E = \sum a_i E_i$, with the E_i's irreducible, we have then $a_i \in \mathbb{N}_{>0}$. Projection formulas then gives the equality $f_*(K_X - E) = f_*(f^*(K_Y)) = K_Y$, since $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.

The example illustrates how it is interesting to consider the coefficients a_i appearing in $f^*(K_Y) = K_X + \sum a_i E_i$. Then, we can investigate the same notion for a pair. Consider a pair (X, Δ), and a birational morphism of normal varieties

[2] It is well defined in the normal setting. Take a big open set contained in the smooth locus of X, and consider the canonical bundle there. By normality of X, this extends uniquely to a reflexive sheaf, which is then given by a Weil divisor, which we will denote K_X.

$f : X \to Y$. Then, we can write $K_Y + f_*^{-1}\Delta = f^*(K_X + \Delta) + \sum a_i(E_i, X, \Delta)E_i$, where $f_*^{-1}\Delta$ denotes the strict transform of Δ and the E_i's are f-exceptional. In the previous example, where X and Y were smooth and no boundary component was introduced, we expected the a_i's to be positive. It is not always the case.

Example. Consider a cone over a rational curve of degree n. If we blow up the vertex, we get a resolution of singularities. Call the map $\pi : X \to Y$. The above situation can be constructed from $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, blowing down a copy of $\mathbb{P}^1$ via the linear series $|\mathcal{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))}(1)|$. From this description, we see that the exceptional divisor E has self intersection $E^2 = -n$. We know that $K_X = f^*(K_Y) + aE$. On the other hand, on X we can use adjunction to compute $K_E = K_{\mathbb{P}^1}$. Thus, we have $-2 = (K_X + E) \cdot E = (f^*(K_Y) + (a+1)E) \cdot E = (a+1)E^2 = -n(a+1)$. Hence, we get $a = -\frac{n-2}{n} > -1$.

By a similar argument, considering a cone over a curve of higher genus, we get $a = -1$ for an elliptic curve, and $a < -1$ for a curve of higher genus. This example also shows that to a simpler singularity, i.e. in the case the base of the cone is $\mathbb{P}^1$, corresponds a bigger coefficient a, and the coefficient gets more negative the worse the singularity is.

The example illustrates how we should think of the coefficients a_i's: the more negative they are, the worse singularities are present. Therefore, we can define what the discrepancy of a pair (X, Δ).

Definition. The discrepancy of (X, Δ) is

$$\operatorname{discrep}(X, \Delta) = \inf_E \{a(E, X, \Delta) | E \text{ is an exceptional divisor over } X\},$$

where by exceptional divisor over X we denote any exceptional divisor coming from a birational morphism $f : Y \to X$ from a normal variety Y.

The idea behind this definition is that $\operatorname{discrep}(X, \Delta)$ detects how bad the pair (X, Δ) is. Although the definition is quite technical, there are ways to compute directly a discrepancy. More precisely, in most cases it is enough to find a resolution of singularities $f : Y \to X$ where $f_*^{-1}\Delta$ and all the exceptional divisors are in simple normal crossing position[3], and then run an algorithm involving their coefficients. To conclude the overview about pairs, here are a few technical facts and definitions.

Proposition. Either $\operatorname{discrep}(X, \Delta) = -\infty$, or $-1 \leq \operatorname{discrep}(X, \Delta) \leq 1$.

Definition. Given a pair (X, Δ) with X normal variety, and $\Delta = \sum a_i D_i$ sum of distinct prime divisors, we say it is *klt* (Kawamata log terminal) if $\operatorname{discrep}(X, \Delta) > -1$ and $\lfloor \Delta \rfloor \leq 0$, where the round down of a divisor is defined as $\lfloor \Delta \rfloor = \sum \lfloor a_i \rfloor D_i$.

Definition. Given a pair (X, Δ) with X normal variety, and $\Delta = \sum a_i D_i$ sum of distinct prime divisors with $0 \leq a_i \leq 1$, we say it is *dlt* (divisorial log terminal) if there is a closed subset $Z \subset X$ such that:

[3]We recall that a divisor D is said to have simple normal support if all of its irreducible components are smooth, and they meet the most transversally possible: in local analytic coordinates, the singular locus of D is given by $z_1 \cdot \ldots \cdot z_d$. Then, D is a simple normal crossing divisor if all the coefficients of its components are 1.

(1) $Z \setminus Z$ is smooth and $\Delta|_{X\setminus Z}$ has simple normal crossing support.

(2) If $f : Y \to X$ is birational and $E \subset Y$ is an irreducible divisor such that $f(E) \subset Z$ then $a(E, X, \Delta) > -1$

Cone Theorem (general case)

Now we have sketched all the ingredients needed to have a more general version of the cone theorem. Before the statement, we recall that a variety is said $\mathbb{Q}$-factorial if any Weil divisor is $\mathbb{Q}$-Cartier.

Theorem (Cone Theorem (general case))**.** Let (X, Δ) be a projective *klt* pair with Δ effective. Then:

(1) There are countably many rational curves $C_i \subset X$ such that $0 < -(K_X + \Delta) \cdot C_i \leq 2 \dim X$, and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta \geq 0} + \sum \mathbb{R}_{\geq 0} [C_i] .$$

(2) For any $\varepsilon > 0$ and ample divisor H,

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta+\varepsilon H)\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0} [C_i] .$$

(3) Let $F \subset \overline{NE}(X)$ be a $(K_X+\Delta)$-negative extremal face. Then there is a unique morphism $\mathrm{cont}_F : X \to Z$ to a projective variety such that $(\mathrm{cont}_*)\mathcal{O}_X = \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point by cont_F iff $[C] \in F$. The morphism cont_F is called contraction of F.

(4) Let F and $\mathrm{cont}_F : X \to Z$ be as in (3). Let L be a line bundle on X such that $l \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Z on Z such that $L \cong \mathrm{cont}_F^* L_Z$.

Then we have a corollary that shows how in the Fano contraction (i.e. the dimension of Z drops) and divisorial contraction case we do not get worse singularities.

Corollary. Let (X, Δ) be a projective *klt* pair and $R \subset \overline{NE}(X)$ a $(K_X+\Delta)$-negative extremal ray with contraction morphism $g_R : X \to Z$. Assume that X is $\mathbb{Q}$-factorial and g_F is either a divisorial or a Fano contraction. Then Z is also $\mathbb{Q}$-factorial.

Example. In case the dimension in $g_R :\to Z$ drops, we talk about Fano contraction for the following reason.By generic smoothness, we have that the general fiber F of g_R is a smooth variety. For simplicity assume X is smooth and $\Delta = 0$. Then, we can compute K_F using repeatedly adjunction. Say $F = g_R^{-1}(p)$, and that p is locally given by p hyperplane equations in Z. Then these equations pull back to X and locally in a neighborhood of F they cut out F itself. Then we can think of F as being given by $H_1, \ldots, H_p$ hypersurfaces meeting transversally (around F). Thus, we can use adjunction in X to compute K_{H_1}, then adjunction in H_1 to compute $K_{H_1 \cap H_2}$, etc.. Therefore, we get $K_F = (K_X + H_1 + \ldots + H_p)|_F = (K_X)|_F$. The last equality follows from the fact that the H_i's vanish along F by construction. Now, the cone theorem tells us that any curve in $C \subset F$ is so that $0 > C \cdot K_X = C \cdot K_F$. This, by Kleiman's criterion, is then enough to conclude that $-K_F$ is ample.

Flips and the MMP

Now that we have a more general version of the cone theorem, we can try to sketch how to use it in order to produce a suitable representative in a birational class. As mentioned before, small contractions introduce very bad singularities. One last ingredient we need is to describe how to get around this problem.

In our setting, small contractions rise in the following way. We have a pair (X, Δ) to which we can apply the cone theorem, and we consider an extremal ray R. Then, the contraction map $g_R : X \to Z$ is our small contraction. By construction, $-(K_X + \Delta)$ is g_R-ample: this means that $-(K_X + \Delta) \cdot C > 0$ for any contracted curve C. The idea is to try to replace a our (X, Δ) with a suitable modification (X^+, Δ^+) such that $K_{X^+} + \Delta^+$ is h-ample, where $h : X^+ \to Z$. This is made more precise by the following.

Definition. Let X be a normal shceme and D a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. A $(K_X + D)$-flipping contraction is a proper birational morphism $f : X \to Y$ to a normal scheme such that the excetpional locus of f has codimension at least two, and $-(K_X + D)$ is f-ample.

A normal scheme X^+ together with a proper birational morphism $f^+ : X^+ \to Y$ is called a $(K_X + D)$-flip of f if:

(1) $K_{X^+} + D^+$ is $\mathbb{Q}$-Cartier, where D^+ is the birational transform of D on X^+.

(2) $K_{X^+} + D^+$ is f^+-ample.

(3) The exceptional locus of f^+ has codimension at least two.

It can be showed that if a $(K_X + D)$-flip exists, then it is unique. Thus, back to our setting, we are interested in finding the $(K_X + \Delta)$-flip of g_R. Furthermore, we want to stay consistent with the kind of singularities and pairs we start with. It turns out that *dlt* pairs are stable under the following operations.

Program (MMP for $\mathbb{Q}$-factorial *dlt* pairs)**.** Consider a *dlt* pair $(X, \Delta) = (X_0, \Delta_0)$. In order to achieve a simpler model for it, we try to run the following.

Step 0 *(Initial datum)* Assume that we already constructed a *dlt* pair (X_i, Δ_i) with X_i $\mathbb{Q}$-factorial.

Step 1 *(Preparation)* If $K_{X_i} + \Delta_i$ is nef, go to step 3, case (2). If not, we establish the following results.

(1) (Cone Theorem) $\overline{NE}(X_i) = \overline{NE}(X_i)_{K_{X_i}+\Delta_i \geq 0} + \sum \mathbb{R}_{\geq 0} C_i$.

(2) (Contraction Theorem) Any $K_{X_i} + \Delta_i$-negative extremal ray can be contracted.

Step 2 *(Birational transformations)* If $\mathrm{cont}_{R_i} : X_i \to Y_i$ is birational, then we produce a new pair as follows.

(1) (Divisorial contraction) If cont_{R_i} is a divisorial contraction, then set $X_{i+1} = Y_i$ adn $\Delta_{i+1} = (\mathrm{cont}_{R_i})_* \Delta_i$.

(2) (Flipping contraction) If cont_{R_i} is a flipping contraction, then set $(X_{i+1}, \Delta_{i+1}) = (X_i^+, \Delta_i^+)$, the flip of cont_{R_i}.

In both cases, we produce a *dlt*-pair (X_{i+1}, Δ_{i+1}) with X_{i+1} $\mathbb{Q}$-factorial. Thus, go back to Step 0.

Step 3 *(Final outcome)* We expect that eventually the procedure stops, and we get one of the following two possibilities.

(1) (Fano contraction) If cont_{R_i} is a Fano contraction, then set $X^*, \Delta*) = (X_i, \Delta_i)$.

(2) (Minimal model) If $K_{X_i} + \Delta_i$ is nef then set $X^*, \Delta^*) = (X_i, \Delta_i)$.