Boundedness of Log Canonical Surface Generalized Pairs

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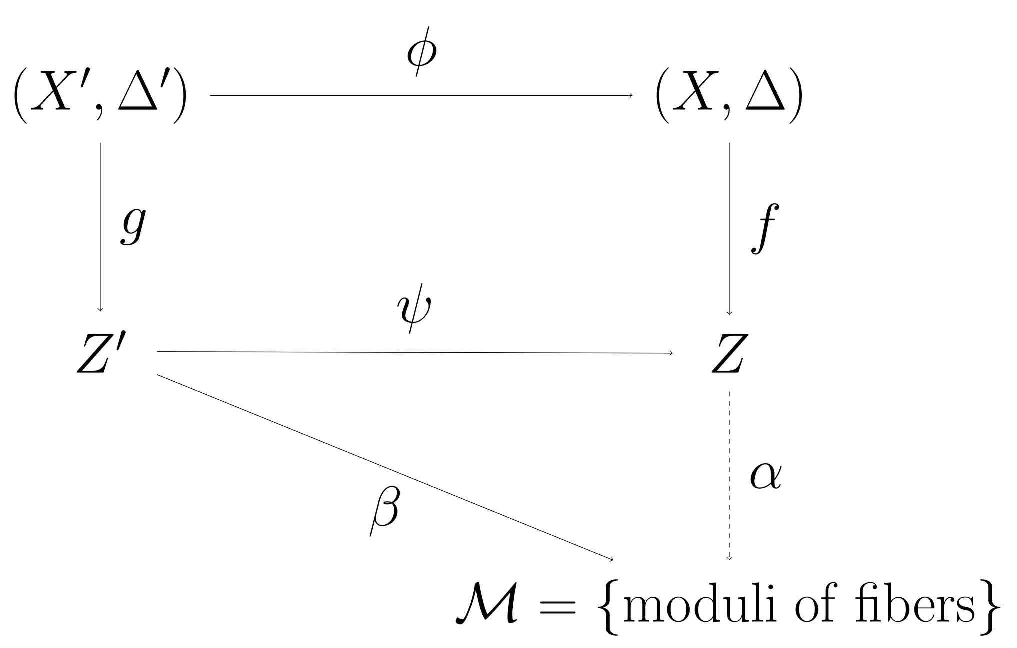
Background: The canonical bundle formula

Given a **relatively minimal elliptic surface** $f : S \to C$, one can write $K_S \sim_{\mathbb{Q}} f^*(K_C + B_C + M_C)$, where:

- B_C **measures the singularities** of the fibers, and is supported on the image of the singular fibers;
- $M_C = \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}^1$, where $j : C \to \mathbb{P}^1$ **measures the variation** of the j-invariant of the smooth fibers.

Consider a log canonical pair (X, Δ) and a fibration with connected fibers $f : X \to Z$. Assume that $K_X + \Delta \sim_{\mathbb{Q},f} 0$. **Conjecturally**, we have



(1) $K_{X'} + \Delta' = \phi^*(K_X + \Delta)$, and g is birationally equivalent to f;

(2) $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$, $K_{X'} + \Delta' \sim_{\mathbb{Q}} g^*(K_{Z'} + B_{Z'} + M_{Z'})$, where $B_Z = \psi_* B_{Z'}$ and $B_{Z'}$ measure the singularities of the fibers;

(3) β resolves α, $M_Z = \psi_* M_{Z'}$, and $M_{Z'} = \beta^*(\text{ample } \mathbb{Q}\text{-divisor})$.

Properties **(1) and (2) are known**. In general, we do not know how to construct the moduli space $\mathcal{M}$. Using Hodge theory, **we get an approximation of (3)**:

(3') if Z' is "nice", $M_{Z'}$ is nef, and $M_Z = \psi_* M_{Z'}$.

We capture the essence of the outcome of the **canonical bundle formula** with the following definition [BZ16].

Definition. *A generalized (sub)-pair is the datum of a normal variety Z, equipped with a birational morphism $Z' \xrightarrow{\psi} Z$, where Z' is normal, an $\mathbb{R}$-(sub)-boundary B_Z, and an $\mathbb{R}$-Cartier divisor $M_{Z'}$ on Z' which is nef and such that $K_Z + B_Z + M_Z$ is $\mathbb{R}$-Cartier, where $M_Z \coloneqq \psi_* M_{Z'}$. We call B_Z the boundary part and M_Z the nef part.*

Main Results

A first step towards the construction of moduli spaces of varieties of general type deals with their **boundedness**.

Definition. *A set of pairs $\mathcal{D}$ is log bounded if there is a projective morphism of quasi-projective varieties $\pi : \mathcal{X} \to T$ and a divisor $\mathcal{B}$ on $\mathcal{X}$ such that for every $(X, \Delta) \in \mathcal{D}$ there is $t \in T$ such that $(X, \mathrm{Supp}(\Delta)) \cong (\mathcal{X}_t, \mathcal{B}_t)$.*

In [HMX16], Hacon, McKernan and Xu prove **log boundedness of pairs of log general type** under the assumptions:

- (X, Δ) is semi-log canonical;
- $K_X + \Delta$ is ample, of fixed volume v;
- $\mathrm{coeff}(\Delta) \in \Lambda$, where Λ is a fixed DCC set (e.g. $\{1 - \frac{1}{n}\} \cup \{1\}$).

A **key step** is to prove that the set **volumes** $\mathrm{vol}(X, K_X + \Delta)$ with $\mathrm{coeff}(\Delta) \in \Lambda$ is a **DCC set**.

Since **generalized pairs have played a major role in recent developments** in birational geometry, we ask the following.

Question: Does boundedness extend to generalized pairs?

We focus on the case of surfaces, and prove the following.

Theorem 1 ([Fil17, Theorem 1.3])**.** *Generalized pairs of dimension 2 are log bounded if we assume the following:*

- *$(Z, B_Z + M_Z)$ is generalized semi-log canonical;*
- *$K_Z + B_Z + M_Z$ is ample, of fixed volume v;*
- *$\mathrm{coeff}(B_Z) \in \Lambda$, where $\Lambda \subset [0, 1]$ is a fixed DCC set;*
- *rM_Z is integral Cartier, where $r \in \mathbb{N}$ is fixed.*

A key step is the study of the distribution of volumes of generalized pairs of dimension 2.

Theorem 2 ([Fil17, Theorem 1.2])**.** *Let $\Lambda \subset [0, 1]$ be a DCC set, and let r be a positive integer. Then the set*

$$\mathfrak{V} \coloneqq \{ \mathrm{vol}(Z, K_Z + B_Z + M_Z) | (Z, B_Z) \text{ is lc surface}, \mathrm{coeff}(B_Z) \in \Lambda,\ rM_Z \text{ is nef and Cartier} \}$$

satisfies the DCC property.

Main Ingredients and Applications

The **key steps** in the proof are:

- Deformation invariance of volumes for log smooth generalized log canonical generalized pairs with generalized log canonical singularities;
- Let $(Z, B_Z + M_Z)$ be a surface generalized pair with Z smooth and rM_Z Cartier. Then, $Z' \xrightarrow{\psi} Z$ is the blow-up of at most $r^2(M_Z^2 - M_{Z'}^2)$ points of Z, possibly infinitely close.

We have to overcome technical issues, as **not all the good properties of pairs extend to generalized pairs**.

We construct a log smooth family $(\mathcal{X}, \mathcal{B} + \mathcal{M}) \to T$ such that:

- $K_{\mathcal{X}} + \mathcal{B} + \mathcal{M}$ is relatively nef and big over T;
- the generalized pluricanonical ring $R(\mathcal{X}_t, K_{\mathcal{X}_t} + \mathcal{B}_{\mathcal{X}_t} + \mathcal{M}_{\mathcal{X}_t})$ is not deformation invariant;
- for a general fiber $\mathcal{X}_t$, $R(\mathcal{X}_t, K_{\mathcal{X}_t} + \mathcal{B}_{\mathcal{X}_t} + \mathcal{M}_{\mathcal{X}_t})$ is not finitely generated.

Boundedness of generalized pairs leads to **boundedness of the bases of the Iitaka fibration**. We exploit it to prove the following.

Theorem 3. *Fix a positive number v. Let $\mathfrak{D}(v)$ be the set of minimal terminal threefolds X such that:*

- *the Kodaira dimension is 2, and the 2-volume of K_X is v;*
- *the Iitaka fibration $f : X \to S$ admits a section.*

Then $\mathfrak{D}(v)$ is bounded modulo flops.

References

[BZ16] C. Birkar and D-Q. Zhang. "Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs". In: *Publications mathématiques de l'IHÉS* 123.2 (2016), pp. 283–331.

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[HMX16] C. D. Hacon, J. McKernan, and C. Xu. "Boundedness of varieties of log general type". In: (2016). arXiv: `1606.07715v1 [math.AG]`.