A Generalized Canonical Bundle Formula

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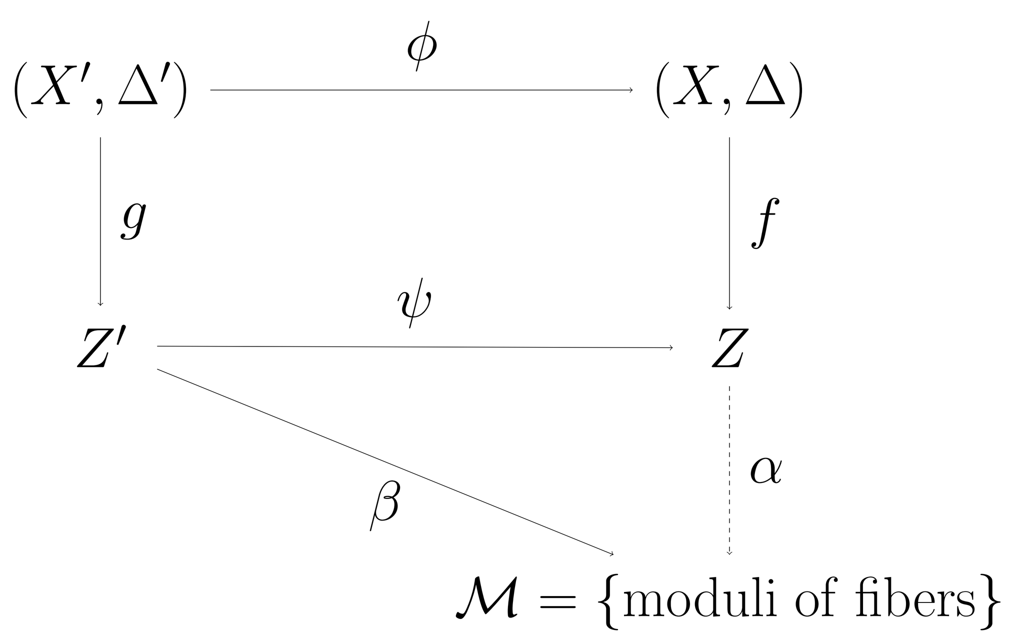
Background: The canonical bundle formula

Given a **relatively minimal elliptic surface** $f : S \to C$, one can write $K_S \sim_{\mathbb{Q}} f^*(K_C + B_C + M_C)$, where:

- B_C **measures the singularities** of the fibers, and is supported on the image of the singular fibers;
- $M_C = \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1)$, where $j : C \to \mathbb{P}^1$ **measures the variation** of the j-invariant of the smooth fibers.

Consider a log canonical pair (X, Δ) and a fibration $f : X \to Z$ with $f_* \mathcal{O}_X = \mathcal{O}_Z$ and $K_X + \Delta \sim_{\mathbb{Q},f} 0$. **Conjecturally**, we have



(1) $K_{X'} + \Delta' = \phi^*(K_X + \Delta)$, and g is birationally equivalent to f;
(2) $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$, $K_{X'} + \Delta' \sim_{\mathbb{Q}} g^*(K_{Z'} + B_{Z'} + M_{Z'})$, where $B_Z = \psi_* B_{Z'}$ and $B_{Z'}$ measure the singularities of the fibers;
(3) β resolves α, $M_Z = \psi_* M_{Z'}$, and $M_{Z'} = \beta^*(\text{ample } \mathbb{Q}\text{−divisor})$.

Properties **(1) and (2) are known**. In general, we do not know how to construct the moduli space $\mathcal{M}$. Using Hodge theory, **we get an approximation of (3)**:

(3') if Z' is "nice", $M_{Z'}$ is nef, and $M_Z = \psi_* M_{Z'}$.

Prokhorov and Shokurov formulated the following conjecture, which is known just if $\dim X - \dim Z = 1$ [PS09].

Conjecture 1. Fix the above setup and notation. Then:

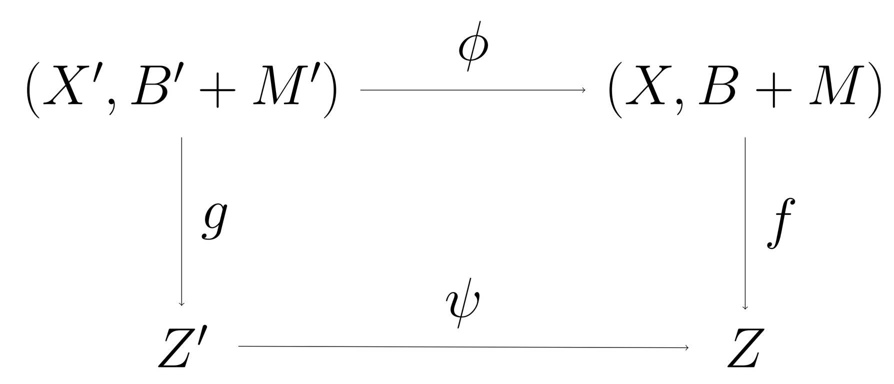
- $M_{Z'}$ is semi-ample;
- there exists an integer C_1 depending only on $\dim X$ and $\mathrm{coeff}(\Delta^h)$ such that $C_1 M_{Z'}$ is basepoint-free; and
- there exist an integer C_2 depending only on $\dim X_\eta$ and $\mathrm{coeff}(\Delta^h)$ such that $C_2(K_{X_\eta} + \Delta_\eta) \sim 0$.

Main Result

The essence of the outcome of the **canonical bundle formula** is captured by the notion of **generalized pairs** [BZ16]. These are **key tools in recent advances** in birational geometry.

Definition. A *generalized pair* is the datum of a normal variety Z, a birational morphism $Z' \xrightarrow{\psi} Z$, where Z' is normal, a $\mathbb{Q}$-boundary B_Z, and a $\mathbb{Q}$-Cartier divisor $M_{Z'}$ on Z' which is nef and such that $K_Z + B_Z + M_Z$ is $\mathbb{Q}$-Cartier, where $M_Z := \psi_* M_{Z'}$. We call B_Z the *boundary part* and M_Z the *nef part*.

Consider a generalized log canonical pair $(X, B + M)$ and a fibration with connected fibers $f : X \to Z$. Assume that we have $K_X + B + M \sim_{\mathbb{Q},f} 0$. For $\psi : Z' \to Z$ birational, we consider



(1) we can assume that X' is a model where the nef divisor $M_{X'}$ descends, and f and g are birationally equivalent;
(2) $K_X + B + M \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$, where B_Z measures the singularities of the fibers of f;
(3) $K_{X'} + B' + M' \sim_{\mathbb{Q}} g^*(K_{Z'} + B_{Z'} + M_{Z'})$, $B_{Z'}$ measures the singularities of the fibers of g, $M_Z = \psi_* M_{Z'}$ and $B_Z = \psi_* B_{Z'}$.

Theorem 2 ([Fil18, Theorem 1.4]). In the above setup, if Z' is high enough, we have:

- $M_{Z'}$ is nef;
- if $\rho : Z'' \to Z'$ is birational, then $M_{Z''} = \rho^* M_{Z'}$.

In particular, $(Z, B_Z + M_Z)$ is a generalized pair with the same class of singularities of $(X, B + M)$.

The **main tools** in the proof are **weak semi-stable reduction** and the **MMP**.

Applications

We obtain **generalized adjunction and inversion of adjunction** applying the generalized canonical bundle formula to the extraction of a generalized log canonical place:

Theorem 3 ([Fil18, Theorem 1.5, Theorem 1.6]). Let $(X, B + M)$ be a generalized pair, and W be a generalized log canonical center. Then, the following holds:

- there is a naturally induced generalized pair $(W^\nu, B_{W^\nu} + M_{W^\nu})$ on the normalization W^ν of W;
- $(X, B + M)$ is generalized log canonical in a neighborhood of W if and only if $(W^\nu, B_{W^\nu} + M_{W^\nu})$ is generalized log canonical.

We make progress towards Conjecture 1.

Corollary 4 ([Fil18, cf. Theorem 7.6]). Parts 1 and 2 of Conjecture 1 can be reduced to the following cases:

- $f : X \to Z$ is a K_X-Mori fiber space; or
- $K_{X_\eta} \sim_{\mathbb{Q}} 0$ and $\Delta^h = 0$.

If $\dim X - \dim Z = 2$, combining Corollary 4 together with results of Fujino [Fuj03], we have the following:

Corollary 5 ([Fil18, cf. Theorem 1.7]). Part 1 of Conjecture 1 holds true if $\dim X - \dim Z = 2$ and $X_{\overline{\eta}}$ is not $\mathbb{P}^2$. Furthermore, if $X_{\overline{\eta}}$ is smooth, minimal (in the sense of smooth surfaces) and not rational, all of Conjecture 1 holds true.

References

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