Generic Vanishing Fails for Surfaces in Positive Characteristic

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Generic Vanishing and Motivations

Generic Vanishing is a very powerful tool. Discovered first by Green and Lazarsfeld [GL87], it admits the following formulation.

Theorem (Generic Vanishing, [Hac04] and [PP11])**.** Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A defined over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. The following are equivalent:

- $\operatorname{codim}_{\widehat{A}} V^i(\mathcal{F}) \geq i$ for all $i \geq 0$, where the cohomology support loci are defined as $V^i(\mathcal{F}) := \{y \in \widehat{A} | h^i(A, \mathcal{F} \otimes \mathcal{L}_y) > 0\}$;
- for any sufficiently ample line bundle L on $\widehat{A}$, we have the vanishing $H^i(A, \mathcal{F} \otimes \widehat{L}) = 0$ for all $i > 0$;
- $Rp_{\widehat{A},*}(p_A^* D_A(\mathcal{F}) \otimes \mathcal{L}) \simeq R^0 p_{\widehat{A},*}(p_A^* D_A(\mathcal{F}) \otimes \mathcal{L})$.

Here $D_A(-)$ denotes the dualizing functor on $D^b(A)$, $\mathcal{L}$ is the Poincaré line bundle on $A \times \widehat{A}$, while p_A and $p_{\widehat{A}}$ denote the two projections.

Hacon and Kovács showed that **Generic Vanishing fails in positive characteristic** [HK13]. Their counterexample builds on the failure of Grauert-Riemenschneider Vanishing. Therefore, **their strategy does not hold in dimension 2**. Furthermore, **the failure of Generic Vanishing is supported on the exceptional locus of the Albanese morphism**.

Questions: How about the case of surfaces? What if the Albanese morphism is finite?

Idea of Proof

The proof relies on the **failure of Kodaira Vanishing**. There exists a smooth surface $\tilde{X}$ with an ample effective divisor $\tilde{D}$ such that

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\tilde{D})) \neq 0.$$

We construct a **Cartesian diagram** of finite morphisms

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{\;h\;} & S \\ \downarrow f & & \downarrow a \\ \mathbb{P}^2 & \xleftarrow{\;g\;} & A \end{array}$$

to obtain a counterexample S with a finite morphism to a principally polarized abelian variety. The ample class $H := h^*\tilde{D}$ violates Kodaira Vanishing. Up to taking **cyclic covers** of S, we can assume that $H - a^*\Theta$ is ample and effective. Then, we consider the Cartesian square

$$\begin{array}{ccc} S & \xleftarrow{\;\varphi\;} & S_n \\ \downarrow a & & \downarrow a_n \\ A & \xleftarrow{\;\mathbf{n}\;} & A \end{array}$$

We can show

$$H^1(S_n, \omega_{S_n} \otimes \mathcal{O}_{S_n}(a_n^*(n^2\Theta))) \to H^1(S_n, \omega_{S_n} \otimes \mathcal{O}_{S_n}(\varphi^* H)) \to 0.$$

If $a_*\omega_S$ is a GV-sheaf, then one can show $H^1(S_n, \omega_{S_n} \otimes \mathcal{O}_{S_n}(a_n^*(n^2\Theta))) = 0$. On the other hand, $H^1(S_n, \omega_{S_n} \otimes \mathcal{O}_{S_n}(\varphi^* H)) \neq 0$. Thus, S violates Generic Vanishing.

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Main Results

In [Fil17], we prove the following.

Theorem 1. For any prime $p \geq 3$ there exists a smooth surface S and a principally polarized abelian surface (A, Θ) such that

- there is a finite map $a : S \to A$ of degree coprime with p;
- there is an ample and effective divisor H on S such that we have $H^1(S, \mathcal{O}_S(-H)) \neq 0$;
- $a_*\omega_S$ is not a GV-sheaf.

By the quasi-isomorphism $Rq_{\widehat{Y},*}(q_A^* D_A(a_*\omega_S) \otimes \mathcal{L}) \simeq Rp_{\widehat{A},*}(\mathcal{P})[n]$, we recover another manifestation of the failure of Generic Vanishing.

Corollary 2. Let S and A be as in Theorem 1. Consider the morphism

$$\lambda := a \times id : S \times \widehat{A} \to A \times \widehat{A}.$$

Let $\mathcal{L}$ denote the Poincaré line bundle on $A \times \widehat{A}$, and $p_{\widehat{A}}$ the projection

$$p_{\widehat{A}} : S \times \widehat{A} \to \widehat{A}.$$

Then $Rp_{\widehat{A},*}(\mathcal{P})$ is not a sheaf, i.e.

$$Rp_{\widehat{A},*}(\mathcal{P}) \not\simeq R^2 p_{\widehat{A},*}(\mathcal{P}),$$

where $\mathcal{P} := \lambda^*\mathcal{L}$.

Taking **products with abelian varieties** and using **Künneth formula**, we can produce **counterexamples in any dimension**.

Corollary 3. For any $n \geq 2$ and prime $p \geq 3$, there exist a smooth n-fold X and an abelian variety Y of the same dimension defined over a field of characteristic p such that

- X admits a finite map $a : X \to Y$ of degree coprime with p;
- $a_*\omega_X$ is not a GV-sheaf;
- $Rp_{\widehat{Y},*}(\mathcal{P}) \not\simeq R^n p_{\widehat{Y},*}(\mathcal{P})$, where the notation is as in Corollary 2.

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