Lecture 3. Miller's theorem and the Zabrodsky Lemma

Recall: Hopf: $\exists\ \eta : S^3 \to S^2$ such that $\eta \neq 0$.
$\pi_3 S^2 = \mathbb{Z} \neq 0$

Serre: X simply connected finite complex such that $\pi_i X = 0$ for $i >> 0 \implies X$ contractible

Goal: give a new proof of Serre's theorem via Miller's theorem, Zabrodsky's lemma, and exotic localization

a) Miller's theorem: X simply connected, $H_*(X; \mathbb{Z}/p) = 0$ for $* >> 0 \implies$
$map_*(B\mathbb{Z}/p, X) \simeq_{wk} *$
$(\iff map(B\mathbb{Z}/p, X) \simeq_{wk} X)$

i.e. X is $* \to B\mathbb{Z}/p$ local

b) Zabrodsky's lemma: $F \to E \xrightarrow{\pi} B$ fibre bundle, B CW complex with connected fibre F. Then
$map(F, X) \simeq_{wk} X \overset{(*)}{\implies} map(E, X) \xleftarrow{\simeq_{wk}} map(B, X)$

Proof: Let $C \subseteq B$ be a maximal subcomplex of B such that $(*)$ is true for $\pi^{-1}(C) \to C$

If $C \neq B$, $\exists$ cell e such that $C \subsetneq C \cup e \subseteq B$
WLOG: $C \cap e = S$ = sphere