Step 4. The proof of Serre's conjecture

Assume X s. conn, $\bar{H}_*(X; \mathbb{Z}/p) = 0$ for $* >> 0$
and $\bar{H}_*(X; \mathbb{Z}/p) \neq 0$ for some $*$,
Suppose p-tors $\pi_* X = 0$ for $* >> 0$

Can assume X is localized at p, $\pi_* X$ fin gen over $\mathbb{Z}_{(p)}$

Serre's theorem: $\pi_* X = 0$ for $* >> 0$ is false

$\therefore$ $\exists\, n$ such that p-tors $\pi_i X = 0$ for $i \geq n$
$\pi_n X$ has a split $\mathbb{Z}_{(p)}$ summand.

Let $W = \widetilde{\left(\Omega^{n-2} X\right)_0}$ where: L_0 = basepoint component
$\tilde{L}_0$ = univ. cover

W is a simply connected H-space, $\pi_2 W = \pi_n X$,

$$\begin{tikzcd} \exists \quad \mathbb{Z}_{(p)} \arrow[r] \arrow[dr, "1"'] & \pi_2 W \arrow[d] \\ & \mathbb{Z}_{(p)} \end{tikzcd}$$

$\pi_* W$ torsion free $\forall *$

Claim: $[B\mathbb{Z}/p, W]_* = 0$

Proof: unique path lifting $\Rightarrow$

$[B\mathbb{Z}/p, W]_* = [B\mathbb{Z}/p, \Omega^{n-2} X]_{\mathbb{Z}/p \to 0 \in \pi_1}$