Lecture 3. Computation of the mod p homology of $\Omega^2 S^{2n+1}$, p = an odd prime

Coproducts in differential Cotor

1) coproducts in $\mathrm{Cotor}^{A_1,\dots,A_{n-1}}(X_1,\dots,X_n)$ exist because of the fact that the Eilenberg–Zilber map is a natural transformation of differential coalgebras
$$\nabla : X \otimes Y \to X \times Y.$$

$\therefore$ These coproducts are compatible with the the Eilenberg–Moore chain models. In detail, suppose E is the pullback of the homotopy pullback diagram

$$\begin{tikzcd}[column sep=small]
X_1 \arrow[dr] & & X_2 \arrow[dl] \arrow[dr] & & \cdots & & X_n \arrow[dl] \\
& A_1 & & A_2 & \cdots & A_{n-1} &
\end{tikzcd}$$

and assume HE is torsion free and $A_1,\dots,A_{n-1}$ are all 1-reduced simplicial sets. Write $\mathrm{Cotor}^{A}(X) = \mathrm{Cotor}^{A_1,\dots,A_{n-1}}(X_1,\dots,X_n)$. Then the compatibility of the coproducts is expressed by

$$\begin{tikzcd}
HE \arrow[r, "\Delta"] \arrow[d, "\cong"] & H(E\times E) \arrow[d, "\cong"] & H(E\otimes E) \arrow[l, "\nabla"'] \arrow[d, "\cong"] & HE\otimes HE \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
\mathrm{Cotor}^{A}(X) \arrow[r, "\Delta"] & \mathrm{Cotor}^{A\times A}(X\times X) & \mathrm{Cotor}^{A\otimes A}(X\otimes X) \arrow[l, "\nabla"'] & \mathrm{Cotor}^{A}(X)\otimes \mathrm{Cotor}^{A}(X) \arrow[l]
\end{tikzcd}$$