$$H(\Omega S \;\;) = \mathrm{Cotor} \qquad (\mathbb{Z}/p, \mathbb{Z}/p)$$

$$\otimes\ H(\Omega^2 S^{2pn+1})$$

$$= \left[E(\tau_0) \otimes P(\sigma_1)\right] \otimes \left[E(\tau_1, \tau_2, \ldots) \otimes P(\sigma_2, \sigma_3, \ldots)\right].$$

We know $\beta^1 \tau_1 = \sigma_1$.

Apply this to $\Omega^2 S^{2pn+1}$ and get $\beta^1 \tau_2 = \sigma_2$.

Iteration gives $\beta^1 \tau_i = \sigma_i$ for all $i \geq 1$.

Step 3. Prove the James decomposition

$$\Sigma \Omega S^{2n+1} \simeq \bigvee_{k \geq 1} \Sigma S^{2kn}$$

Corollary of the James decomp:

Any map $\bar{f} : X \to \Omega Y$ defined on a subcomplex $X = S^{2n} \cup e^{4n} \cup \cdots \cup e^{2kn} \subseteq \Omega S^{2n+1}$ can be extended to a map $f : \Omega S^{2n+1} \to \Omega Y$.

Proof of the corollary:

The extension problem

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & \Omega Y \\ \Omega S^{2n+1} \arrow[ur, dashed] & \end{tikzcd}$$