André-Quillen (co)Homology, Abelianization and Stabilization

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Outline

- **Introduction**: Homotopical Algebra via Model Categories.
- **Quillen Homology**: Derived functors of *abelianization.*
- **Special Case**: *Abelianization = Indecomposibles*
  Topological André-Quillen (co)Homology.
- **Cohomology via Axioms**: Brown Representability.
  Abelianization = Stabilization
- **Applications.**

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Introduction

Model Categories: (Quillen 1967)

$$\mathcal{C}, \mathcal{W}, \mathcal{F}ib, \mathcal{C}of \rightsquigarrow Ho\mathcal{C} \equiv \mathcal{C}[\mathcal{W}^{-1}]$$

- Axiomatized context that allows for the definition of a homotopy category.
- Provides resolutions of objects and extends the definition of derived functors to non-abelian settings.

$$X \xrightarrow{\sim} f(X) \twoheadrightarrow * \quad \text{fibrant approximation}$$

If $\mathcal{C}$ and $\mathcal{D}$ are model categories and

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

takes fibrations to fibration and preserves weak equivalences between fibrant objects, define its *total right derived functor* by

$$RF : Ho\mathcal{C} \longrightarrow Ho\mathcal{D}$$

$$RF(X) = F(f(X))$$

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$$\emptyset \rightarrowtail c(X) \overset{\sim}{\twoheadrightarrow} X \quad \textit{cofibrant approximation}$$

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Abelian Objects

Definition: An object A of a categoy C is an *abelian object* iff $C(-, A)$ is naturally an abelian group.
If C has enough limits this is equivalent to having maps

$$m : A \times A \longrightarrow A \quad \textit{multiplication}$$

$$\eta : * \longrightarrow A \quad \textit{identity} \quad \text{and } i : A \longrightarrow A \quad \textit{inverse}$$

satisfying the usual axioms.
Examples: Let C_{ab} denote the category of abelian objects in C.

- $$C = sSets \Longrightarrow C_{ab} = sAb.$$
- For R a commutative ring,

$$C = C_R \Longrightarrow (C_R)_{ab} = \{0\}$$

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Commutative R algebras over A

Fix a commutative R algebra A and consider $\mathcal{C}_R/A$

- **Objects:** $\epsilon : B \longrightarrow A$ (R algebras B with an R-algebra map to A)
- **Morphisms:**

$$\begin{array}{ccc} B & \longrightarrow & B' \\ & {\scriptstyle\epsilon}\searrow \quad \swarrow{\scriptstyle\epsilon'} & \\ & A & \end{array}$$

For any A-module M, get and R-algebra over A

$$A \ltimes M = A \oplus M \stackrel{proj_1}{\longrightarrow} A$$

$$(a, m)(a', m') = (aa', am' + a'm)$$

It satisfies that $\mathcal{C}_R/A(B, A \ltimes M) \cong Der_R(B, M) \leftarrow$ abelian group .

We get an equivalence of categories

$$\mathcal{M}od_A \cong (\mathcal{C}_R/A)_{ab}$$

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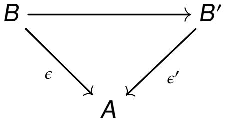

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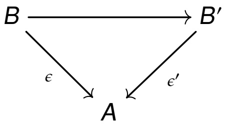

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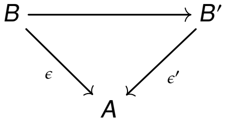

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Abilianization

General Context:

- C a model category
- Assume that subcategory C_{ab} is also a model category with fibrations and weak equivalences as in C
- Assume that the *forgetful functor* has a left adjoint ("abelianization")

$$Ab : C \rightleftarrows C_{ab} : U$$

(Ab, U) give a Quillen adjunction

Definition: Quillen Homology is the total left derived functor of abelianization.

For $B \in C$, $\mathbb{L}Ab(B)$ gives the Quillen Homology of B.

Examples: $C = sSets$ $\quad Ab(X) = \mathbb{Z}[X] \Longrightarrow \mathbb{L}Ab(X) = \mathbb{Z}[X]$ since X is cofibrant.

$\pi_*\mathbb{L}Ab(X) \cong H_*(X)$ usual homology

$C = \mathcal{T}op \qquad Ab(X) = Sp^\infty(X) \Longrightarrow \mathbb{L}Ab(X) = Sp^\infty(cX).$

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Abilianization = Indecomposables

Let B be an object of $\mathcal{C}_R/A$.

Definition: The B-module of *Kähler differentials* is given by the *indecomposables* of the kernel of the multiplication:

$$I(B) = \ker\{B \otimes B \longrightarrow B\} \quad \text{and} \quad \Omega_{B|R} = I(B)/(I(B))^2.$$

There is a universal derivation:

$$d : B \longrightarrow \Omega_{B|R} \quad \text{given by} \quad b \mapsto b \otimes 1 - 1 \otimes b.$$

We get that for any A-module M

$$\mathcal{M}od_A(A \otimes_B \Omega_{B|R}, M) \cong \cong \mathcal{C}_R/A(B, A \ltimes M).$$

Hence, under the identification $\mathcal{M}od_A \cong (\mathcal{C}_R/A)_{ab}$, we see that

$$Ab(B \to A) \cong A \otimes_B \Omega_{B|R}$$

This functors extend to the corresponding simplicial categories and we get back to the setting of model categories.

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There is a universal derivation:

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We get that for any A-module M

$$\mathcal{M}od_A(A \otimes_B \Omega_{B|R}, M) \cong \cong \mathcal{C}_R/A(B, A \ltimes M).$$

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André-Quillen (co)Homology

Definition: Given a commutative R algebra A, consider $Id : A \longrightarrow A$ as a constant simplicial object of $s(\mathcal{C}_R/A)$.

We have that

$$Ab(A \to A) \cong \Omega_{A|R} = I(A)/(I(A))^2 := Q(I(A))$$

Define the *cotangent complex* for A over R as the simplicial A-module

$$\mathbf{L}_{\mathbf{A|R}} := \mathbb{L}Ab(A) \cong Ab(P_\bullet) = A \otimes_{P_\bullet} \Omega_{P_\bullet|R}$$

where $P_\bullet$ is a cofibrant simplicial algebra replacement for A.

Then, for any A module M the *André-Quillen homology* of A with coefficients in M is defined by

$$D_*(A|R; M) := H_*(M \otimes_A \mathbf{L}_{\mathbf{A|R}}),$$

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Properties I:

Think of the cotangent complex $\mathbf{L_{A|R}}$ as a functor on a pair (A, R)

- **Naturality:** Given a *map of pairs*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

get a morphism $B \otimes_A \mathbf{L_{A|R}} \longrightarrow \mathbf{L_{B|S}}$

- **Transitivity Exact Sequence:** Given $R \longrightarrow A \longrightarrow B$. Get a cofiber sequence of simplicial B-modules.

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Properties II:

- **Flat Base Change:** Given a push-out diagram

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  with either ϵ or f flat, the induced morphism

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- $\mathcal{C}_B/B$ commutative R-algebras over and under B.
  Given A define its *augmentaion ideal* by

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The Topological Cotangent Complex

Proposition: Let R be a cofibrant commutative S algebra and $R \longrightarrow B$ a cofibration of commutative S-algebras.
We have the following sequence of Quillen adjunctions:

$$\mathcal{C}_R/B \underset{}{\overset{\wedge_R B}{\rightleftarrows}} \mathcal{C}_B/B \underset{I_B}{\overset{K_B}{\leftrightarrows}} \mathcal{N}_B \underset{Z}{\overset{Q_B}{\rightleftarrows}} \mathcal{M}_B$$

where the middle one is a Quillen equivalence.

$$\text{Hence,} \quad \mathcal{H}o\mathcal{C}_R/B(A, B \ltimes M) \cong \mathcal{H}o\mathcal{M}_B(\mathbb{L}Q_B\mathbb{R}I_B(A \wedge^{\mathbb{L}}_R B), M).$$

We see that "$Ab(R \to A \to B)$" $= \mathbb{L}Q_B\mathbb{R}I_B(A \wedge^{\mathbb{L}}_R B)$
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$$\mathbb{L}Ab^B_R(A) := \mathbb{L}Q_B\mathbb{R}I_B(A \wedge_R B)$$

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We see that $\quad$ "$Ab(R \to A \to B)$" $= \mathbb{L}Q_B\mathbb{R}I_B(A \wedge_R^{\mathbb{L}} B)$
So, for a cofibrant R-algebra A and a cofibration $A \longrightarrow B$ we define.

$$\mathbb{L}Ab_R^B(A) := \mathbb{L}Q_B\mathbb{R}I_B(A \wedge_R B)$$

Taking $R = B$, we have that $\mathbb{L}Ab_B^B(A) = \mathbb{L}Q_B\mathbb{R}I_B(A)$.
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One checks that in this context $\quad \mathbb{L}Ab_R^B(A) \cong B \wedge_A TAQ(A, R)$.

The Topological Cotangent Complex

Proposition: Let R be a cofibrant commutative S algebra and $R \longrightarrow B$ a cofibration of commutative S-algebras.
We have the following sequence of Quillen adjunctions:

$$\mathcal{C}_R/B \underset{}{\overset{\wedge_R B}{\rightleftarrows}} \mathcal{C}_B/B \underset{I_B}{\overset{K_B}{\leftrightarrows}} \mathcal{N}_B \underset{Z}{\overset{Q_B}{\rightleftarrows}} \mathcal{M}_B$$

where the middle one is a Quillen equivalence.

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Topological André-Quillen (co) Homology

- R be a cofibrant commutative S-algebra
- $R \longrightarrow B$ a cofibration of commutative S-algebras.

Definition: Given a pair $A \longrightarrow X$ in C_R/B and a B-module M, define

$$D_*^R(X, A; M) := \pi_*(\mathbb{L}Ab_A^B(X) \wedge_B M)$$

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When $A \longrightarrow X$ is not cofibrant, we find a cofibrant replacement $A' \longrightarrow X'$ and take $\mathbb{L}Ab_A^B(X) = B \wedge_{X'} TAQ(X', A')$
Note that for cofibrant objects under B i.e. pairs with $A = B$ we have that

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TAQ satisfies analogous properties as in the algebra setting.

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Eilemberg-Steenrod Axioms for Cohomology

Work with Mike Mandell.

Definition:A cohomology theory on a model category $\mathcal{C}$ consists of a contravariant functor h^* from the category of pairs to the category of graded abelian groups together with natural transformations of abelian groups $\delta^n : h^n(A,\emptyset) \longrightarrow h^{n+1}(X,A)$ for all n, satisfying the following axioms:

- (*Homotopy*) $(X,A) \xrightarrow{\sim} (Y,B) \rightsquigarrow h^*(Y,B) \xrightarrow{\cong} h^*(X,A)$
- (*Exactness*) The following sequence is exact

$$\cdots \longrightarrow h^n(X,A) \longrightarrow h^n(X,\emptyset) \longrightarrow h^n(A,\emptyset) \xrightarrow{\delta^n} h^{n+1}(X,A) \longrightarrow \cdots$$

- (*Excision*) If A is cofibrant and Y is the pushout of $A \rightarrowtail B$ and $A \rightarrowtail X$, then $(X,A) \longrightarrow (Y,B)$ induces $h^*(Y,B) \xrightarrow{\cong} h^*(X,A)$.
- (*Product*) If $\{X_\alpha\}$ is a set of cofibrant objects and X is the coproduct, then the natural map

$$h^* \longrightarrow \prod h^*(X_\alpha)$$

is an isomorphism.

TAQ is a cohomology theory

Given, $R \longrightarrow A \longrightarrow X \longrightarrow B$ in $\mathcal{C}_R/B$, the transitivity exact sequence for $R \longrightarrow A \longrightarrow X$ gives a cofiber sequence of B-modules

$$\mathbb{L}Ab^B_R(A) \longrightarrow \mathbb{L}Ab^B_R(X) \longrightarrow \mathbb{L}Ab^B_A(X).$$

So for a B-module M we get the connecting homomorphism and exactness from the long exact sequence

$$\cdots D^n(X,A;M) \longrightarrow D^n(X,R;M) \longrightarrow D^n(X,A;M) \xrightarrow{\delta} D^{n+1}(X,A;M) \longrightarrow \cdots$$

Excition and the product axiom follow from flat base change.
So, for each B module M, $D^*(-,-;M)$ gives a cohomology theory on $\mathcal{C}_R/B$

Theorem 1: (B, Mandell) Every cohomology theory on $\mathcal{C}_R/B$ is TAQ-cohomology with coefficients in some B-module.

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Reduced Theories

Definition A reduced cohomology theory on a pointed category $\mathcal{C}$ consists of a contravariant functor h^* from the homotopy category $\mathcal{H}o\mathcal{C}$ to the category of abelian groups together with natural isomorphisms of abelian groups

$$\sigma : h^n(X) \longrightarrow h^{n+1}(\Sigma_{\mathcal{C}} X) \textit{ suspension isomorphism}$$

for all n satisfying the following axioms:

- (*Exactness*) If $X \longrightarrow Y \longrightarrow Z$ is part of a cofiber sequence, then,

  $$h^n(Z) \longrightarrow h^n(Y) \longrightarrow h^n(X)$$

  is exact for all n
- (*Product*) If $\{X_\alpha\}$ is a set of cofibrant objects and X is the coproduct, then the natural map $h^*(X) \longrightarrow \prod h^*(X_\alpha)$ is an isomorphism.

Remark: When the final object B on a model category $\mathcal{C}$ is cofibrant, there is an equivalence between the category of cohomology theories on $\mathcal{C}$ and the category of reduced cohomologies on $\mathcal{C} \backslash B$

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Brown Representability

Proposition: Let B be a cofibrant commutative S-algebra. Reduced cohomology theories on $\mathcal{C}_B/B$ are representable i.e., for each n there exists an object X_{h^n} in $\mathcal{H}o\mathcal{C}_B/B$ and a natural isomorphism of functors $h(-) \cong \mathcal{H}o\mathcal{C}_B/B(-, X_{h^n})$
Then, the suspension isomorphism

$$\begin{aligned} \mathcal{H}o\mathcal{C}_B/B(-, X_{h^n}) \quad \cong h^n(-) \longrightarrow h^{n+1}(\Sigma_{\mathcal{C}}-) \quad &\cong \mathcal{H}o\mathcal{C}_B/B(\Sigma_{\mathcal{C}}-, X_{h^{n+1}}) \\ &\cong \mathcal{H}o\mathcal{C}_B/B(-, \Omega X_{h^{n+1}}). \end{aligned}$$

induces (by the Yoneda lemma) an isomorphism in $\mathcal{H}o\mathcal{C}_B/B$

$$X_{h^n} \xrightarrow{\cong} \Omega X_{h^{n+1}}$$

i.e. $\{X_{h^n}\}$ assambles to some sort of *spectrum*. We call it an *Omega weak spectrum*.

Spectra

Proposition: The category of reduced cohomology theories on $\mathcal{C}_B/B$ is equivalent to the category of Omega weak spectra in $\mathcal{C}_B/B$.

TAQ with coefficients in the B-module M is represented by $\{B \ltimes \Sigma^n M\}$.
To prove Theorem 1 want to show that this assignment gives an equivalence between the homotopy category of B-modules and the category of Omega weak spectra in $\mathcal{C}_B/B$

For $\mathcal{C}$ one of $\mathcal{M}_B, \mathcal{N}_B$ or $\mathcal{C}_B/B$ set up a model category of spectra where

- a weak equivalence is a map that induces an isomorphism on homotopy groups $\pi_q \underline{X} = Colim\ \pi_{q+n} X_n$.
- the fibrant objects are the Omega spectra so that every Omega weak spectrum can be rectified to a cofibrant Omega spectrum.
- on Omega spectra, $\underline{X} \xrightarrow{\sim} \underline{Y} \iff X_0 \xrightarrow{\sim} Y_0$.
- we have a Quillen adjunction

$$\Sigma^\infty_{\mathcal{C}} : \mathcal{C} : \rightleftarrows \mathcal{S}(\mathcal{C}) : (-)_0$$

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Stabilization

We have the following sequence of Quillen adjunctions:

$$\mathcal{M}_B \underset{(-)_0}{\overset{\Sigma^\infty_C}{\rightleftarrows}} \mathcal{S}p(\mathcal{M}_B) \underset{U}{\overset{\mathbb{N}}{\rightleftarrows}} \mathcal{S}p(\mathcal{N}_B) \underset{I}{\overset{K}{\rightleftarrows}} \mathcal{S}p(C_B/B)$$

that the first and last are Quillen equivalences follows easily. That the middle one is an equivalence takes some work and represents the main difference between the topology context and the algebra context.

This shows that the stable category of C_B/B is equivalent to the homotopy category of B-modules.

$$\mathcal{H}o\mathcal{S}p(C_B/B) \cong \mathcal{H}o\mathcal{M}_B$$

We show that under this equivalence,

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Calculations I

We can understand $TAQ(B, R)$ if we can calculate $\mathbb{L}Ab_B^B(X)$ with $X = \Sigma_C^n(B \wedge_R B)$ for some $n > 0$.

Example

To calculate $D^*_{H\mathbb{F}_2}(H\mathbb{F}_2, S; H\mathbb{F}_2)$, note that $\Sigma_C(H\mathbb{F}_2 \wedge H\mathbb{F}_2) \simeq THH^S(H\mathbb{F}_2)$ which, by Bokstedt, has homotopy a polynomial algebra on a generator of degree 2. So, $\Sigma_C THH^S(H\mathbb{F}_2)$ has homotopy an exterior algebra on a generator x in degree 3.

We must have that

$$\Sigma_C THH^S(H\mathbb{F}_2) \simeq H\mathbb{F}_2 \ltimes \Sigma^3 H\mathbb{F}_2$$

an *square zero* extension.

Its *TAQ*-cohomology is given by $\{Sq^I x\}$ where the sequence $I = (s_1, \cdots, s_r)$ is Steenrod admissible and $s_r > 3$.

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If A is augmented over B we have

$$TAQ(A,B) = \mathbb{L}Q_A \mathbb{R}I_A(A \wedge_B A) \cong \mathbf{L}Ab_B^B(A) \wedge_B A.$$

Under *stabilization* the B-module $\mathbf{L}Ab_B^B(A) \rightsquigarrow \Sigma_{\mathcal{C}}^{\infty}(B)$.

For an E_∞ space X, we have that $\Sigma^\infty X_+$ is a commutative S algebra augmented over S.

Theorem: Let $\underline{X}$ denote the spectrum associated to X.

$$TAQ(\Sigma^\infty X_+, S) \cong \underline{X} \wedge \Sigma^\infty X_+$$

Corollary: Let $\alpha : X \longrightarrow BF$ be a map of E_∞ spaces, M the associated commutative S-algebra Thom spectrum. Then $TAQ(M,S) \cong \underline{X} \wedge M$

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Maria Basterra

University of New Hampshire

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The End

THANK YOU!