Méthode simpliciale en algèbres de Lie

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Outline

We consider the question of how a given graded connected Lie algebra Λ (over $\mathbb{Q}$) can be realized as the homology of a differential graded Lie algebra (DGL) L_*.

Strategy

- Try to realize a free simplicial resolution $G_\bullet$ of Λ as a DGLA-resolution $V_\bullet$ (of the putative DGLA L_*).
- If this can be done, use Bousfield-Friedlander SS:

$$E^2_{s,t} = \pi_s H_t V_\bullet = \pi_s G_\bullet = \begin{cases} \Lambda & s = 0 \\ 0 & s > 0 \end{cases} \Rightarrow H_{s+t} \| V_\bullet \|$$

Construct such a $V_\bullet$ by induction on the simplicial dimension. At each stage, we can do one of two things:

- Use Dwyer-Kan-Stover obstruction theory, in terms of the André-Quillen cohomology of Λ.
- Use higher homotopy operations as obstructions to rectifying a finite directed diagram.

Goals

- To show that these two approaches are the same.
- To show how vanishing obstructions permit rectification.

Dictionary to generalization:

DG Lie algebras	$\mathbb{Q}$-homotopy theory	Homotopy theory
DGLA $(L_*, \partial) \in \mathcal{DGL}$	$X_{\mathbb{Q}} \in \mathcal{T}op_{\mathbb{Q}}$	Space $X \in \mathcal{T}op$
$H_*L_* = [\mathcal{L}\langle x^k \rangle, L_*]$	$\pi_k \Omega X_{\mathbb{Q}}$	$\pi_* X$
GLA $\Lambda = H_*L_* \in \mathcal{GL}$	GLA $\Lambda = \pi_* \Omega X_{\mathbb{Q}}$	Π-algebra $\pi_* X$
Free GLA $\mathcal{L}\langle x^{n_i} \rangle_{i \in I}$	$\pi_* \bigvee_{i \in I} S^{n_i}_{\mathbb{Q}}$	$\pi_* \bigvee_{i \in I} S^{n_i}$
$(\mathcal{L}\langle x^{n_i} \rangle_{i \in I}, \partial = 0)$?	$\bigvee_{i \in I} S^{n_i}_{\mathbb{Q}}$	$\bigvee_{i \in I} S^{n_i}$

Simplicial constructions

For a simplicial object $X_\bullet$:

- The n-th *Moore chains* objects is $\mathbf{C}_n X_\bullet := \cap_{i=1}^{n} \mathrm{Ker}\{d_i : X_n \to X_{n-1}\}$, with differential $\partial_n := (d_0)|_{\mathbf{C}_n X_\bullet} : \mathbf{C}_n X_\bullet \to \mathbf{C}_{n-1} X_\bullet$
- The n-th *Moore cycles* objects is $\mathbf{Z}_n X_\bullet := \cap_{i=0}^{n} \mathrm{Ker}\{d_i : X_n \to X_{n-1}\}$.
- The n-th *matching object* is $\mathbf{M}_n X_\bullet := \{(x_0, \dots, x_n) \in (X_{n-1})^{n+1} \mid d_i x_j = d_{j-1} x_i \;\; 0 \le i < j \le n\}$.
  All face maps on X_n factor through $\delta_n : X_n \to \mathbf{M}_n X$.
  $X_\bullet$ is *Reedy fibrant* if each δ_n is a fibration.
- Dually, the n-th *latching object* is $L_n X_\bullet := \coprod_{0 \le i \le n-1} X_{n-1} / \sim$, where $[s_J x]_i \sim [s_I x]_j \;\Leftrightarrow\; s_i s_J = s_j s_I$.
  All degeneracies to X_n factor through $\sigma_n : L_n X_\bullet \to X_n$.
- An n-th *CW basis* object is $\overline{X}_n$, equipped with an *attaching map* $\bar{d}_0^{X_n} : \overline{X}_n \to X_{n-1}$, such that $X_n = \overline{X}_n \amalg L_n X_\bullet$, $(d_0)|_{\overline{X}_n} = \bar{d}_0^{X_n}$ and $(d_i)|_{\overline{X}_n} = 0$ for $i \ge 1$.
  Note that $\bar{d}_0^{X_n}$ factors through $\mathbf{Z}_{n-1} X_\bullet \subseteq \mathbf{C}_{n-1} X_\bullet$.

∞-commutative diagrams

Assume given a simplicial resolution $G_\bullet \to \Lambda$, with free GLA CW basis $(\overline{G}_k)_{k=0}^\infty$, and let $\tau_{n+1}V_\bullet$ be an strict $(n+1)$-truncated simplicial DGLA realizing $G_\bullet$ through dim $n+1$.

Choose some free DGLA $\overline{V}_{n+2}$ realizing $\overline{G}_{n+2}$, with attaching map $\bar{d}_0^{V_{n+2}} : \overline{V}_{n+2} \to V_{n+1}$ realizing $\bar{d}_0^{G_{n+2}} : \overline{G}_{n+2} \to G_{n+1}$ up to homotopy (possible, since $\overline{G}_{n+2}$ is free).

We get a *lax* $(n+2)$-truncated simplicial DGLA $\tilde{V}_\bullet^{\langle n+2\rangle}$, with $d_i^{n+1} \circ \bar{d}_0^{V_{n+2}} \sim 0$ $(i \geq 0)$ only up to homotopy.

If we can choose nullhomotopies $\eta_i : d_i^{n+1} \circ \bar{d}_0^{V_{n+2}} \sim 0$, relative homotopies $\eta_{i,j} : d_i \circ \eta_j \sim d_{j-1} \circ \eta_i$ $(i < j)$, and so on, we say that we have made $\tilde{V}_\bullet^{\langle n+2\rangle}$ ∞*-homotopy commutative*.

Theorem (Boardman-Vogt, Dwyer-Kan-Smith, Chachólski-Scherer)

An ∞-homotopy commutative diagram can be rectified.

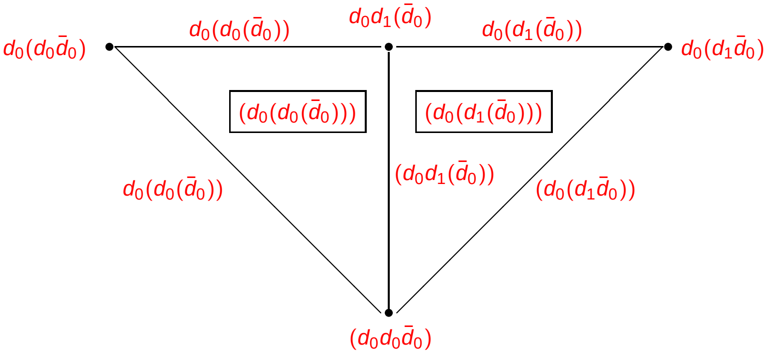
Idea: For each $\phi : \mathbf{n+2} \to \mathbf{k}$ in Δ^{op}, we use the simplicial enrichment in $\mathcal{DGL}$ to assemble the higher homotopies into a map $\psi_\phi : \mathrm{Cone}(\mathcal{P}^{n-k+1}) \to \mathrm{map}(\overline{V}_{n+2}, V_k)$.

Permutohedra

Here $\mathcal{P}^m$ is the $(m-1)$-dimensional permutohedron, whose vertices correspond to permutations on $(1,\ldots,m)$.

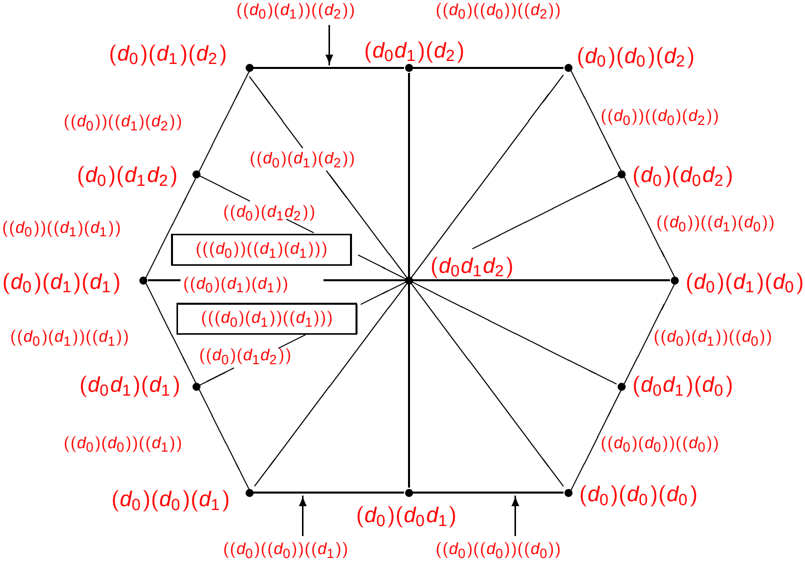
$\text{Cone}(\mathcal{P}^m)$ is the cone on its standard triangulation.

Example ($m=1$):
The 1-permutohedron is an interval (subdivided in the triangulation), so $\text{Cone}(\mathcal{P}^1)$ has two 2-simplices:



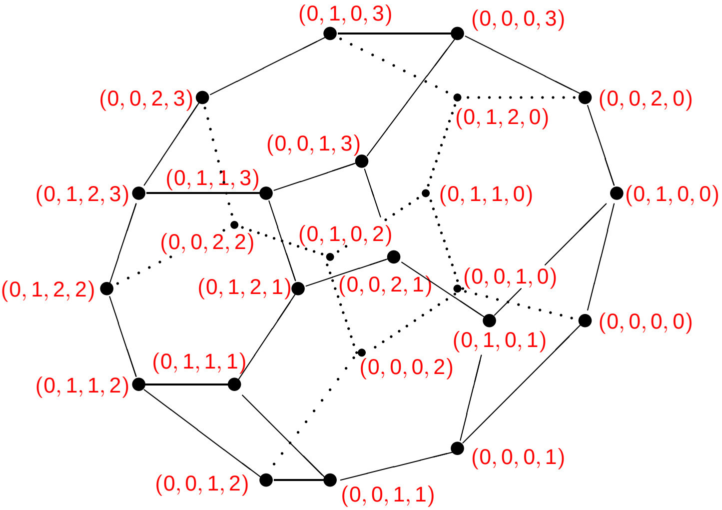
The 2-dimensional permutohedron

For $m = 3$:



The 3-dimensional permutohedron

For $m = 4$:



Higher homotopy operations

The permutohedron $\mathcal{P}^m$ is a convex polytope, whose boundary consists of products of lower-dimensional permutohedra.
Thus the pointed maps $\psi_\phi : \mathrm{Cone}(\mathcal{P}^{n-k}) \to \mathrm{map}(\overline{V}_{n+2}, V_{k-1})$ fit together to form $\psi'_{\phi'} : \partial\,\mathrm{Cone}(\mathcal{P}^{n-k+1}) \to \mathrm{map}(\overline{V}_{n+2}, V_{k-2})$.

Fact: Its adjoint $\widetilde{\psi}' : \Sigma^{n-k+1}\overline{V}_{n+2} \to V_{k-2}$ is null-homotopic iff the ψ_ϕ's extend to $\psi_{\phi'} : \mathrm{Cone}(\mathcal{P}^{n-k+1}) \to \mathrm{map}(\overline{V}_{n+2}, V_{k-2})$.
If this happens at each stage, we obtain a map (from the wedge over all composite face maps $\phi : \mathbf{n+1} \to \mathbf{0}$ in Δ^{op}):

$$\Psi : \bigvee_{\phi} \Sigma^n \overline{V}_{n+2} \to V_0 \,.$$

Definition: The $(n+1)$*-st order homotopy operation* associated to $\tilde{V}_\bullet^{\langle n+2\rangle}$ is the set $\langle\langle \tilde{V}_\bullet^{\langle n+2\rangle} \rangle\rangle \subseteq [\bigvee_\phi \Sigma^n \overline{V}_{n+2},\ V_0]$ of all such Ψ.

Theorem

The higher homotopy operation $\langle\langle \tilde{V}_\bullet^{\langle n+2\rangle} \rangle\rangle$ vanishes (that is, contains 0) if and only if $\tau_{n+1}V_\bullet$ extends to $\tau_{n+2}V_\bullet$ realizing $G_\bullet$ through dim $n+2$.

André-Quillen cohomology

Definition: For $X \in \mathcal{C} = \mathcal{T}op$, $\mathcal{GL}$, or $\mathcal{DGL}$, $\Lambda = \pi_* X$, and any Λ-module M, $\exists$ Eilenberg-Mac Lane objects $E_\Lambda(M, n) \in s\mathcal{C}$, and the n-th André-Quillen cohomology group of X is $H^n_{\mathrm{AQ}}(X; M) = [W_\bullet, E_\Lambda(M, n)]_{s\mathcal{C}/\Lambda}$, (for $W_\bullet$ a resolution of X).

Fact: When $\mathcal{C}$ is "algebraic" (=with an underlying group structure) we calculate $H^*_{\mathrm{AQ}}(G_\bullet; K)$ via its Moore cochains $\mathrm{Hom}(\mathbf{C}_* G_\bullet, K)$; if $G_\bullet$ has free CW basis $(\overline{G}_n)_{n=0}^\infty$, its normalized chains are isomorphic to $\mathrm{Hom}(\overline{G}_*, K)$.

Definition: for any GLA Λ and $n > 0$, $\Omega^n \Lambda$ is the graded Λ-module given by $(\Omega^n \Lambda)_i = \Lambda_{n+i}$

Recall The n-th *Postnikov section* $\mathbf{P}^n W_\bullet$ of a Reedy fibrant simplicial (D)GLA $W_\bullet$ is its $(n+1)$-coskeleton, with $(\mathbf{P}^n W_\bullet)_i = W_i$ for $i \leq n$, and $(\mathbf{P}^n W_\bullet)_{n+1} = \mathbf{M}_{n+1} W_\bullet$.

Lemma: The n-th k-invariant of $W_\bullet \in s\mathcal{GL}$ is the class $k_n \in H^{n+2}_{\mathrm{AQ}}(\mathbf{P}^n X_\bullet; \pi_{n+1} W_\bullet)$ sending $\sigma \in W_{n+1}$ to α_σ in $\pi_n W_\bullet$, represented by matching set $(d_0\sigma, \ldots, d_{n+1}\sigma)$ in $\mathbf{M}_{n+1} W_\bullet$.

Cohomology obstructions

Definition: An $(n-1)$-*semi-Postnikov section* for a GLA Λ is a simplicial DGLA $V_\bullet^{\langle n-1\rangle}$ with $V_\bullet^{\langle n-1\rangle} \simeq \mathbf{P}^{n-1}V_\bullet^{\langle n-1\rangle}$ such that

$$\pi_k H_* V_\bullet^{\langle n\rangle} \cong \begin{cases} \Lambda & \text{for } k=0, \\ \Omega^n\Lambda & \text{for } k=n+1, \\ 0 & \text{otherwise}. \end{cases} \qquad (1)$$

Example: If $W_\bullet$ realizes $G_\bullet$ through simplicial dimension $n+1$, then $\mathrm{csk}_n W_\bullet = \mathbf{P}^{n-1}W_\bullet$ is an $(n-1)$-semi-Postnikov section for Λ.

Theorem: An $(n-1)$-semi-Postnikov section $V_\bullet^{\langle n-1\rangle}$ extends to an n-semi-Postnikov section $V_\bullet^{\langle n\rangle}$ iff the simplicial GLA $H_*V_\bullet^{\langle n-1\rangle}$ has trivial n-th k-invariant.

Remark: By (1), we have a w.e. $f : G_\bullet \simeq \tilde{P}^n H_* V_\bullet^{\langle n-1\rangle}$, mapping G_{n+1} to the matching set $(d_0^{V_{n+1}}, \ldots, d_n^{V_{n+1}})$ in $\mathbf{M}_{n+1}V_\bullet^{\langle n-1\rangle}$.
Thus by the Lemma, k_n "is" $d_0^{V_{n+1}} \circ \bar{d}_0^{\overline{V}_{n+2}} : \overline{V}_{n+2} \to \mathbf{Z}_n V_\bullet^{\langle n\rangle}$.

Lemma (Stover)

For any fibrant simplicial DGLA $W_\bullet$, the inclusion induces an isomorphism on Moore chains $H_*\mathbf{C}_k W_\bullet \simeq \mathbf{C}_k H_* W_\bullet$.

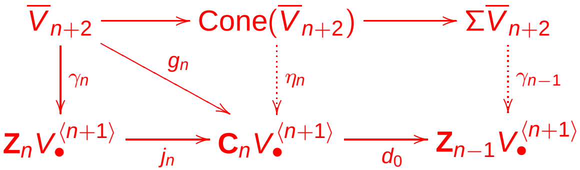
Question: How does $\gamma_n := d_0^{V_{n+1}} \bar{d}_0^{\overline{V}_{n+2}} : \overline{V}_{n+2} \to \mathbf{Z}_n V_\bullet^{\langle n \rangle}$ represent a collection of classes in $\Omega^n \Lambda$?

Idea: Since $H_* V_0^{\langle n \rangle} = G_0$ maps onto Λ, it is enough to find a map $\Sigma^n \overline{V}_{n+2} \to V_0^{\langle n \rangle}$, as follows:

The algebraic attaching map $\bar{d}_0^{\overline{G}_{n+2}}$ lands in $\mathbf{Z}_{n+1} G_\bullet$, but we cannot guarantee that the DGLA realization $\bar{d}_0^{\overline{V}_{n+2}}$ lands in $\mathbf{Z}_{n+1} V_\bullet^{\langle n \rangle}$ (which would mean that $\tilde{V}_\bullet^{\langle n+2 \rangle}$ realizes $\tau_{n+2} G_\bullet$). However, by Stover's Lemma $\bar{d}_0^{\overline{V}_{n+2}}$ can be chosen to land in $\mathbf{C}_{n+1} V_\bullet^{\langle n \rangle}$, so $d_i^{n+1} \bar{d}_0^{\overline{V}_{n+2}} = 0$ on the nose for $i \geq 1$.

Ladder diagrams

Consider the solid commutative diagram:



- By the Lemma we can choose a nullhomotopy η_n for $g_n := j_n \circ \gamma_n$
- Since $d_0^{V_n} d_0^{V_{n+1}} \bar{d}_0^{\overline{V}_{n+2}} = 0$, η_n induces a map γ_{n-1} from the suspension $\Sigma\overline{V}_{n+2} \cong \mathrm{Cone}(\overline{V}_{n+2})/\overline{V}_{n+2}$.
- $G_\bullet$ is acyclic, so $d_0 : \mathbf{C}_n[\Sigma\overline{V}_{n+2}, V_\bullet^{\langle n+1\rangle}] \to \mathbf{Z}_{n-1}[\Sigma\overline{V}_{n+2}, V_\bullet^{\langle n+1\rangle}]$ is onto. So by the Lemma $\exists\alpha : \Sigma\overline{V}_{n+2} \to \mathbf{C}_n V_\bullet^{\langle n+1\rangle}$ with $d_0\alpha = -\gamma_{n-1}$.
- Replacing η_n by $\eta_n \top \alpha$ makes the new $j_{n-1} \circ \gamma_{n-1}$ nullhomotopic.

- Continue inductively for all $m > 0$ to:

$$
\begin{array}{ccccc}
\Sigma^m \overline{V}_{n+2} & \longrightarrow & \mathrm{Cone}(\Sigma^m \overline{V}_{n+2}) & \longrightarrow & \Sigma^{m+1} \overline{V}_{n+2} \\
\downarrow{\scriptstyle \gamma_m} & \searrow^{g_m} & \vdots{\scriptstyle \eta_m} & & \vdots{\scriptstyle \gamma_{m-1}} \\
\mathbf{Z}_m V_\bullet^{\langle n+1 \rangle} & \xrightarrow[j_m]{} & \mathbf{C}_m V_\bullet^{\langle n+1 \rangle} & \xrightarrow[d_0]{} & \mathbf{Z}_{m-1} V_\bullet^{\langle n+1 \rangle}
\end{array}
$$

yields $\gamma_0 : \Sigma^n \overline{V}_{n+2} \to V_0^{\langle n \rangle}$.

- Composing with the augmentation $\varepsilon : H_0 V_0^{\langle n \rangle} = G_0 \to \Lambda$ and taking adjoints yields the required map $\overline{G}_{n+2} \to \Omega^n \Lambda$.

Summary: The above choices (starting with $\bar{d}_0 = \bar{d}_0^{\overline{V}_{n+2}}$) yield

- A 1-nullhomotopy $\eta_n : d_0 \bar{d}_0 \sim 0$, with $d_i \bar{d}_0 = 0$ for $i \geq 1$.
- A 2-nullhomotopy $\eta_{n-1} : d_0 \eta_n \sim 0$, with $d_i \eta_{n-1} = 0$ for $i \geq 1$.
- A 3-nullhomotopy $\eta_{n-2} : d_0 \eta_{n-1} \sim 0$, and so on.

Minimal values and the comparison homomorphism

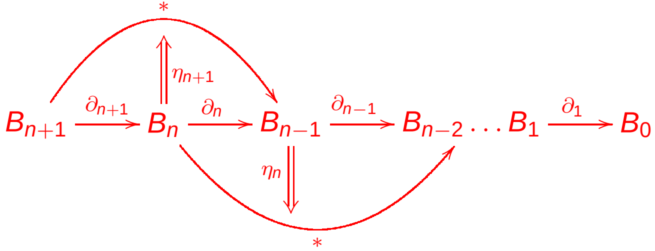
Definition: From $(\eta_{n-i})_{i=0}^{n}$ (and all other homotopies trivial), we obtain a *minimal value* of the $(n+1)$-st order higher homotopy operation $\langle\langle \tilde{V}_{\bullet}^{\langle n+2\rangle} \rangle\rangle$ which is zero on all wedge summands but $\phi = d_0 d_0 \dots \bar{d}_0$.

Definition: For $\varepsilon : G_{\bullet} \to \Lambda$ and $\tilde{V}_{\bullet}^{\langle n+2\rangle}$ as above, the *comparison homomorphism* $\Phi : [\Sigma^n \overline{V}_{n+2}, V_0] \to H_{AQ}^{n+2}(\Lambda; \Omega^n \Lambda)$ is the composite of

$$\begin{aligned}[\Sigma^n \overline{V}_{n+2}, V_0] &\cong [\overline{V}_{n+2}, \Omega^n V_0] \cong \mathrm{Hom}_{\mathcal{GL}}(H_* \overline{V}_{n+2}, H_* V_0) \\ &\cong \mathrm{Hom}_{\mathcal{GL}}(\overline{G}_{n+2}, \Omega^n G_0) \xrightarrow{\varepsilon_*} \mathrm{Hom}_{\mathcal{GL}}(\overline{G}_{n+2}, \Omega^n \Lambda) \\ &\twoheadrightarrow H_{AQ}^{n+2}(\Lambda; \Omega^n \Lambda)\end{aligned}$$

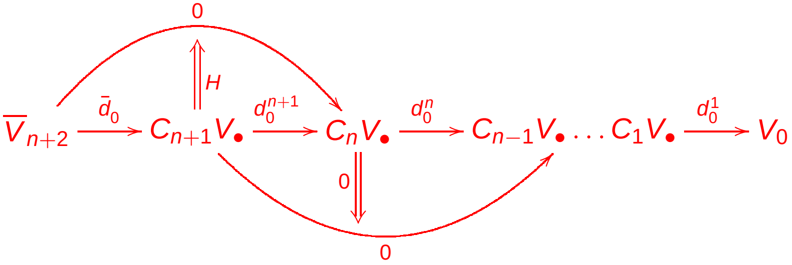
Fact: Φ takes a minimal value of $\langle\langle \tilde{V}_{\bullet}^{\langle n+2\rangle} \rangle\rangle$ to the obstruction $k_n \in H_{AQ}^{n+2}(\Lambda; \Omega^n \Lambda)$.

Definition: A *long Toda bracket* is the higher homotopy operation $\langle\langle B_* \rangle\rangle$ for a *higher-order chain complex* B_*:



B_* is *fibrant* if each $\partial'_k : B_k \to Z_{k-1}B_* := \operatorname{Ker} \partial_{k-1}$ is a fibration.

Example: Minimal values as above are long Toda brackets for:



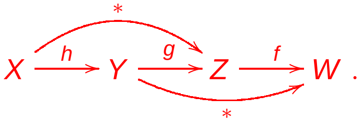
Proposition

A fibrant higher chain x B_* is rectifiable (without changing objects) iff $\langle\langle B_* \rangle\rangle$ vanishes.

Proof:

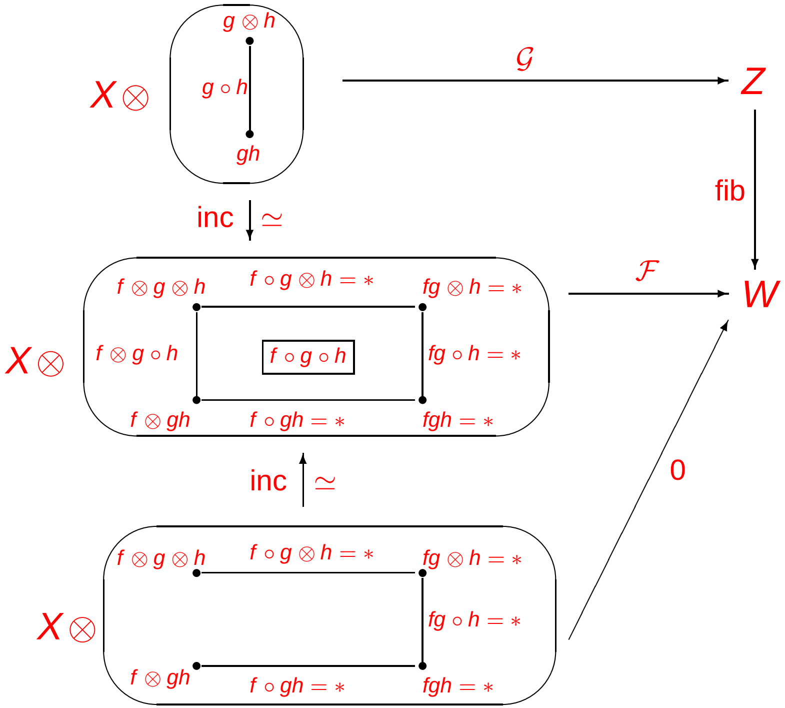
By induction on n we may assume that B_* has been rectified from B_n down.

For simplicity, consider the usual Toda diagram:



We use cubical notation (as for W-construction):

$f \times g$ is (cubical) composition, fg is chosen representative for composite, and $f \circ g$ is homotopy $f \otimes g$ and fg.


g ⊗ h
g ∘ h
gh
X ⊗
𝒢
Z
fib
inc ≃
f ⊗ g ⊗ h
f ∘ g ⊗ h = ∗
fg ⊗ h = ∗
X ⊗
f ⊗ g ∘ h
f ∘ g ∘ h
fg ∘ h = ∗
f ⊗ gh
f ∘ gh = ∗
fgh = ∗
ℱ
W
inc ≃
0
X ⊗
f ⊗ g ⊗ h
f ∘ g ⊗ h = ∗
fg ⊗ h = ∗
fg ∘ h = ∗
f ⊗ gh
f ∘ gh = ∗
fgh = ∗

Difference obstructions

Assume we have two different $(n+2)$-realizations $V_\bullet^{\langle a\rangle}$ and $V_\bullet^{\langle b\rangle}$ of $G_\bullet \to \Lambda$, with same $(n+1)$-coskeleton $W_\bullet$, and Postnikov fibrations $p_{n+1}^{(t)} : V_\bullet^{\langle t\rangle} \to W_\bullet$, determined by the attaching maps $\bar{d}_0^t : \overline{V}_{n+2} \to \mathbf{Z}_{n+1}V_\bullet^{\langle t\rangle} = \mathbf{Z}_{n+1}W_\bullet$ $(t = a, b)$.

Fact: This is equivalent to choosing sections $s^{(t)} : \widetilde{B}\Lambda \to H_*W_\bullet \simeq \widetilde{E}(\Omega^{n+1}\Lambda, n+2)$ $(t = a, b)$.

Theorem: The *difference obstruction* $\delta_n = [s^{(a)} - s^{(b)}]$ vanishes in $H_{\mathrm{AQ}}^{n+2}(\Lambda, \Omega^{n+1}\Lambda)$ iff $V_\bullet^{\langle a\rangle} \simeq V_\bullet^{\langle b\rangle}$ $(\mathrm{rel}\, W_\bullet)$.

Fact: δ_n is represented by $\bar{\delta} := \bar{d}_0^a - \bar{d}_0^b : \overline{V}_{n+2} \to \mathbf{Z}_{n+1}W_\bullet$.

Idea: Define a lax $(n+3)$-truncated simp. CW object $\tilde{Y}_\bullet$ by $\tau_{n+1}\tilde{Y}_\bullet = \tau_{n+1}W_\bullet$, $Y_{n+2} = Z_{n+1}W_\bullet$, and $\overline{Y}_{n+3} = \overline{V}_{n+2}$, with attaching map $\bar{\delta}$.

Get: Higher operation $\langle\langle \tilde{Y}_\bullet \rangle\rangle$ with minimal value in $[\Sigma^{n+1}\overline{V}_{n+2}, V_0]$ corresponding to δ_n, and:

$\langle\langle \tilde{Y}_\bullet \rangle\rangle$ vanishes $\Leftrightarrow$ $\tilde{Y}_\bullet$ rectifiable $\Leftrightarrow$ $V_\bullet^{\langle a\rangle} \simeq V_\bullet^{\langle b\rangle}$ $(\mathrm{rel}\, W_\bullet)$.

Interpreting Postnikov sections

Question: if $V_\bullet$ realizes $G_\bullet$, and so $X := \|V_\bullet\|$ realizes Λ, what does $\mathbf{P}^n V_\bullet$ tell us about X?

Theorem

An n-semi-Postnikov section for Λ determines the n-stem of X: that is, the n-windows $\mathbf{P}^{n+k} X\langle k-1\rangle$ $(k \geq 0)$

Remark: A special feature of DGLAs: unlike spaces, each GLA Λ has a prefered *coformal model* $L_* = (\Lambda, 0)$. For free GLAs, these DGLAs are cofibrant.

Thus $G_\bullet$ always has a prefered DGLA realization $V_\bullet$, in which all higher homotopy operations vanish, so the same is true for $X = \|V_\bullet\|$. Note that $V_\bullet$ is not usually Reedy fibrant, so this is not visible in Postnikov version.

However, we can use the comparison homomorphism to get:

Fact: if $W_\bullet$ is another realization of $G_\bullet$ with $Y = \|W_\bullet\|$ and $\mathbf{P}^n W_\bullet \simeq \mathbf{P}^n V_\bullet$, the n-stem of Y is coformal, and so has vanishing higher homotopy operations.