André–Quillen Towers and Infinite Loop Spaces

Nicholas J. Kuhn

University of Virginia

Lausanne, May, 2011

The big picture

- Infinite loopspace theory.

  This concerns the adjoint functors $\mathcal{T} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{S}$.

  $\mathcal{T}$ = based spaces. $\mathcal{S}$ = spectra (aka S–modules).

- Topological André–Quillen theory.

  This concerns homotopical tools to study how commutative rings and ring spectra are built up from indecomposables, etc.

My main objectives

- Recast part of the TAQ theory.

  It also concerns adjoint functors $\mathcal{A}lg \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{S}$.

  $\mathcal{A}lg$ = augmented, commutative S–algebras.

- Use this to construct filtrations and towers.

  Example Goodwillie tower of $1_{\mathcal{A}lg}$ = Augmentation ideal tower.

- Mix and match.

  Example $X \in \mathcal{S} \Rightarrow \Sigma^\infty_+ \Omega^\infty X \in \mathcal{A}lg$.

- Application to homology of infinite loopspaces.

  Example $K(n)_*(\Omega^\infty X)$ wishes it were a functor of $K(n)_*(X)$.

Σ^∞ and Ω^∞ for $\mathcal{T}$

Recollections . . .

$\mathcal{T}$ is tensored over itself: $K \otimes Z = K \wedge Z$.

$\Sigma^\infty Z$ is the S–module arising from the (pre)spectrum with nth space $S^n \otimes Z = \Sigma^n Z$.

$\Omega^\infty X$ is just the 0th space of X. This is an infinite loop space.

Examples

If $X = H\mathbb{Z}/2$, then $\Omega^\infty X = \mathbb{Z}/2$.

If $X = \Sigma H\mathbb{Z}$, then $\Omega^\infty X = S^1$.

If $X = \Sigma^\infty Z$, then $\Omega^\infty X = \operatorname{hocolim}_n \Omega^n \Sigma^n Z$

The category $\mathcal{A}lg$

S is the sphere spectrum.

$R \in \mathcal{A}lg$ is an 'E_∞–algebra' equipped with $\epsilon : R \to S$.

The fiber $I = I(R)$ is a nonunital commutative S–algebra.

Examples

If $X \in \mathcal{S}$, then $\Sigma^\infty_+ \Omega^\infty X \in \mathcal{A}lg$.

If $X \in \mathcal{S}$, then $\mathbb{P}(X) = \bigvee_{d=0}^{\infty} (X^{\wedge d})_{h\Sigma_d} \in \mathcal{A}lg$.
This is the free algebra generated by X.

If $Z \in \mathcal{T}$, then $D(Z_+) \in \mathcal{A}lg$. ($D =$ Spanier–Whitehead dual)

Σ^∞ and Ω^∞ for $\mathcal{A}lg$

$\mathcal{A}lg$ is also tensored over $\mathcal{T}$; e.g., $S^1 \otimes R$ is the 'bar construction' on R.

$\Sigma^\infty : \mathcal{A}lg \to \mathcal{S}$ is defined by $\Sigma^\infty R = \operatorname{hocolim}_n \Omega^n I(S^n \otimes R)$.

$\Omega^\infty : \mathcal{S} \to \mathcal{A}lg$ is defined by $\Omega^\infty X = S \vee X$, with trivial multiplication.

Thus $\Omega^\infty \Sigma^\infty R = \operatorname{hocolim}_n \Omega^n (S^n \otimes R)$.

Easy examples

If $R = \Sigma^\infty_+ \Omega^\infty X$, then $\Sigma^\infty R$ is the connective cover of X.

If $R = \mathbb{P}(X) = \bigvee_{d=0}^{\infty} X^{\wedge d}_{h\Sigma_d}$, then $\Sigma^\infty R = X$.

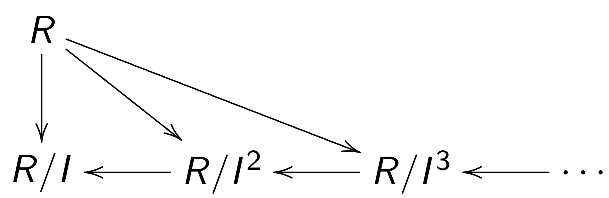
A much less obvious example If $R = D(S^1_+)$, then $\Sigma^\infty R = \Sigma^{-1} H\mathbb{Q}$.

The augmentation ideal tower

Let $\epsilon : R \to \mathbb{Z}$ be an ordinary augmented ring. Always commutative.

$I = \ker \epsilon$, augmentation ideal.

R maps to its augmentation ideal tower



Indecomposables: $Q(R) = I/I^2$. Completion: $\hat{R} = \lim_d R/I^d$.

Example $R = \mathbb{P}(V)$, the free algebra on a free abelian group V.

$Q(R) = V$, and $I^d/I^{d+1} = (V^{\otimes d})_{\Sigma_d}$.

Classic André–Quillen theory

André and Quillen: do this in the world of homotopical algebra.

Work in the category of simplicial commutative rings.

$R_* \to R$, R_* a simplicial resolution built from $\mathbb{P}(V)$'s.

Let $AQ(R) = Q(R_*)$.

Then $AQ_*(R;\mathbb{Z}) = \pi_*(AQ(R))$ is André–Quillen homology with coefficients in the R-module $\mathbb{Z}$.

Topological André–Quillen theory

Goerss, Hopkins, Miller, Robinson, Whitehouse, Richter, McCarthy, and particularly Basterra and Mandell:

Work in the category $\mathcal{A}lg$.

$$Q(R) = \text{pushout}\{* \leftarrow I(R) \wedge I(R) \rightarrow I(R)\}$$

$\tilde{R} \rightarrow R$, $\tilde{R}$ a cofibrant replacement. Cofibrant: built from $\mathbb{P}(X)$'s.

Let $TAQ(R) = Q(\tilde{R})$.

Then $TAQ_*(R; S) = \pi_*(TAQ(R))$ is Topological André–Quillen homology with coefficients in the R-module S.

$TAQ(R) = \Sigma^\infty R$

André and Quillen, and then Basterra and Mandell emphasize that these are *homology theories*.

Thus, for example, $TAQ(S^n \otimes R) \simeq \Sigma^n TAQ(R)$.

Theorem [B-M, 2005] $TAQ(R) \simeq \Sigma^\infty R$.

Sketch proof: With R cofibrant, there is a natural map

$$\Sigma^\infty R = \operatorname*{colim}_n \Omega^n I(S^n \otimes R) \rightarrow \operatorname*{colim}_n \Omega^n TAQ(S^n \otimes R) \simeq TAQ(R).$$

Now check that this map is an equivalence when $R = \mathbb{P}(X)$.

Remark Thus $\Omega^\infty \Sigma^\infty R = S \vee TAQ(R)$.

From $TAQ(R)$ to R via calculus

Goodwillie's calculus gives a tower for $1_{\mathcal{A}lg} : \mathcal{A}lg \to \mathcal{A}lg$.

Theorem For $R \in \mathcal{A}lg$, there is a natural tower

$$\begin{array}{ccccccc} R & & & & & & \\ \downarrow & \searrow & & \searrow & & & \\ P_0(R) & \leftarrow & P_1(R) & \leftarrow & P_2(R) & \leftarrow & \cdots \end{array}$$

such that $P_0(R) = S$, and $D_d(R) = (TAQ(R)^{\wedge d})_{h\Sigma_d}$,

where $D_d(R) = \text{hofib}\{P_d(R) \to P_{d-1}(R)\}$.

Remark This can be viewed as the augmentation ideal tower for R, with

$$I^d(R) = \text{hofib}\{R \to P_{d-1}(R)\} \qquad \text{and} \qquad \hat{R} = \operatorname{holim}_d P_d(R).$$

The easy proof, using that $TAQ(R) = \Sigma^\infty R$:

General calculus theory tells us that

$D_d(R) =$ is the ho-orbits of multilinearization of the dth cross effect cr_d.

In $\mathcal{A}lg$, the coproduct is $\wedge$, and one easily sees $cr_d(R) = I(R)^{\wedge d}$.

Multilinearization: stabilize each $I(R)$, so one gets $TAQ(R)^{\wedge d}$.

Punchline: $D_d(R) = (TAQ(R)^{\wedge d})_{h\Sigma_d}$.

Remark Take the Arone–Ching viewpoint. This can be rewritten as

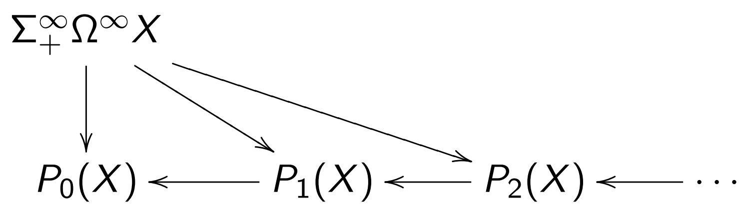
$$D_d(R) = (\text{Com}(d) \wedge TAQ(R)^{\wedge d})_{h\Sigma_d},$$

where $\text{Com}(d) = S$. So $\partial_* 1_{\mathcal{A}lg} = \text{Com}(*)$, the commutative operad.

From X to $\Sigma^\infty\Omega^\infty X$

Specialize to $R = \Sigma^\infty_+\Omega^\infty X$, X connective, and recall that $TAQ(R) = X$.

Theorem For connective $X \in \mathcal{S}$, there is a tower in $\mathcal{A}lg$



with $D_d(X) = (X^{\wedge d})_{h\Sigma_d}$, where $D_d(X) = \text{hofib}\{P_d(X) \to P_{d-1}(X)\}$.

This adds much structure to the tower of $\Sigma^\infty\Omega^\infty : \mathcal{S} \to \mathcal{S}$.

Remark Note that $\partial_*\Sigma^\infty\Omega^\infty = \text{Com}(*)$.

From R to $TAQ(R)$

Less well known ...

Theorem [K, 2003] For all $R \in \mathcal{A}lg$, $TAQ(R)$ has a natural filtration

$$F_1 \to F_2 \to F_3 \to \dots$$

such that $\operatorname{hocolim}_d F_d \simeq TAQ(R)$ and

$$F_d/F_{d-1} \simeq (\text{Lie}(d) \wedge I(R)^{\wedge d})_{h\Sigma_d}.$$

Here Lie$(*)$ is Ching and Arone's geometric Lie co-operad.

Remark Otherwise said, $\Omega^\infty\Sigma^\infty R$ is filtered with F_d/F_{d-1} as above.

Sketch proof:

- For $K \in \mathcal{T}$ and $R \in \mathcal{A}lg$, $K \otimes R$ is naturally filtered with
$$F_d/F_{d-1} = (K^{(d)} \wedge I(R)^{\wedge d})_{h\Sigma_d},$$
where $K^{(d)} = K^{\wedge d}/\text{fat diagonal}$.

  A filtration of this type occurs in a 1969 paper by McCord.
- An observation of Arone-Mahowold (see [Arone-Dwyer]):
$$\operatorname{hocolim}_n \Omega^n S^{n(d)} \simeq \text{Lie}(d).$$

Remark There are emerging new perspectives which explain the Koszul duality we are seeing here in our tower and filtration. (Arone, Ching, K, Behrens, . . .)

From $\Sigma^\infty\Omega^\infty X$ to X

Again specialize to $R = \Sigma^\infty_+\Omega^\infty X$.

Theorem A connective spectrum X has a natural filtration
$$F_1 \to F_2 \to F_3 \to \dots$$
such that $\operatorname{hocolim}_d F_d \simeq X$ and
$$F_d/F_{d-1} \simeq \Sigma^\infty(\text{Lie}(d) \wedge (\Omega^\infty X)^{\wedge d})_{h\Sigma_d}.$$

Some applications and examples of all this . . .

Two parallel splittings

From these towers and filtrations one can deduce:

- With $\mathcal{T} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{S}$, there is a natural splitting

$$\Sigma^\infty\Omega^\infty\Sigma^\infty Z \simeq \bigvee_d (\mathrm{Com}(d) \wedge (\Sigma^\infty Z)^{\wedge d})_{h\Sigma_d}$$

  for all 0–connected $Z \in \mathcal{T}$.

- With $\mathcal{A}lg \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{S}$, there is a natural splitting

$$\Sigma^\infty\Omega^\infty\Sigma^\infty R \simeq \bigvee_d (\mathrm{Lie}(d) \wedge (\Sigma^\infty R)^{\wedge d})_{h\Sigma_d}$$

  for all $R \in \mathcal{A}lg$. (N.B. Already $\Sigma^\infty\Omega^\infty X$ splits for all $X \in \mathcal{S}$.)

Applications to homology isomorphisms

Let h_* be a homology theory, and $f : A \to B$ a map in $\mathcal{A}lg$.

From the filtration of TAQ, we learn

Corollary If $f_* : h_*(A) \xrightarrow{\sim} h_*(B)$ is an isomorphism, then so is

$$f_* : h_*(TAQ(A)) \to h_*(TAQ(B)).$$

From the tower for S–algebras, we learn

Corollary If $f_* : h_*(TAQ(A)) \xrightarrow{\sim} h_*(TAQ(B))$ is an isomorphism, then, for all d, so is

$$f_* : h_*(P_d(A)) \to h_*(P_d(B)).$$

In particular, if $TAQ(A) \to TAQ(B)$ is an equivalence, then so is $\hat{A} \to \hat{B}$.

Interesting examples of the filtration

Example Let $X = \Sigma H\mathbb{Z}$, so $\Omega^\infty X = S^1$. Arone–Dywer:

$$\Sigma^\infty(\mathrm{Lie}(d) \wedge S^d)_{h\Sigma_d} = \Sigma SP^d(S)/SP^{d-1}(S).$$

One can conclude that the filtration of $\Sigma H\mathbb{Z} = \Sigma SP^\infty(S)$ is the symmetric powers of spheres filtration.

Example If $R = D(S^1)_+$, then $I(R) \simeq S^{-1}$, and $TAQ(R) \simeq \Sigma^{-1}H\mathbb{Q}$. Localized at 2, $F_d/F_{d-1} \simeq *$ unless $d = 2^k$, and

$$(\mathrm{Lie}(2^k) \wedge S^{-2^k})_{h\Sigma_{2^k}} = \Sigma^{-1} SP^{2^k}_\Delta(S).$$

Remark Let $\tilde{D}_d(X) = (\mathrm{Lie}(d) \wedge X^{\wedge d})_{h\Sigma_d}$. A 2001 paper of mine has formulae that come close to describing $H^*(\tilde{D}_d(X); \mathbb{Z}/2)$ as a functor of $H^*(X; \mathbb{Z}/2)$, including the action of Steenrod algebra $\mathcal{A}$.

The $K(n)$ homology of infinite loopspaces

Given any homology theory h_*, the tower gives a spectral sequence, natural for $X \in \mathcal{S}$, of the form $E^1_{*,*} = h_*(\mathbb{P}(X)) \Rightarrow h_*(\Omega^\infty X)$.

Theorem [K, 2006] This always collapses when h is a Morava K–theory $K(n)$. Even more, there is a natural monomorphism of $K(n)_*$–Hopf algebra

$$K(n)_*(\mathbb{P}(X)) \to K(n)_*(\Omega^\infty X).$$

Sketch proof: In the $K(n)$–local category, there exists a natural section $\eta_n : X \to L_{K(n)}\Sigma^\infty\Omega^\infty X$ to the evaluation $\Sigma^\infty\Omega^\infty X \to X$. This induces a natural map in $K(n)$–local algebras

$$s_n : L_{K(n)}\mathbb{P}(X) \to L_{K(n)}\Sigma^\infty_+\Omega^\infty X$$

By construction, this is induces an equivalence on TAQ ... and thus on the associated towers. But the tower for $\mathbb{P}(X)$ is clearly trivial.

The mod 2 homology of infinite loopspaces

A very new result . . .

In joint work with Jason McCarty, we have recently figured out the 'generic' differentials in the spectral sequence

$$E^1_{*,*} = H_*(\mathbb{P}(X);\mathbb{Z}/2) \Rightarrow H_*(\Omega^\infty X;\mathbb{Z}/2).$$

This leads to an *unstable* upper bound for $E^\infty_{*,*}$ which is a functor of $H_*(X;\mathbb{Z}/2)$ as an $\mathcal{A}$–module. This algebraic functor involves the derived functors of destablization of $\mathcal{A}$–modules (as studied in the 1980's by Singer, Lannes–Zarati, Goerss).

To hear more, come to my talk in Paris next Wednesday!

Thanks for listening today!

Some references

- G. Arone and M. Ching, *Operads and chain rules for the calculus of functors*, Astérisque, to appear.
- M. Basterra and M. Mandell, *Homology and cohomology of E_∞ ring spectra*, Math. Zeit. **249** (2005), 903–944.
- N. J. Kuhn, *New relationships among loopspaces, symmetric products, and Eilenberg MacLane spaces*, Cohomological Methods in Homotopy Theory (Barcelona, 1998), Progress in Math **196** (2001), 185–216.
- N. J. Kuhn, *The McCord model for the tensor product of a space and a commutative ring spectrum*, Categorical Decomposition Techniques in Algebraic Topology (Isle of Skye, Scotland, 2001), Progress in Math **215** (2003), 213–236.
- N. J. Kuhn, *Localization of André–Quillen–Goodwillie towers, and the periodic homology of infinite loopspaces*, Adv. Math. **201** (2006), 318–378.