

Cochains and Homotopy Theory

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Overview

Cochains and Homotopy Theory

Abstract

The E_∞ algebra structure on the cochain complex of a space contains all the homotopy theoretic information about the space, but for partial information, less structure is needed. I will discuss some ideas and preliminary work in this direction.

Outline

- 1 Distinguishing homotopy types
- 2 Homotopy algebras and operadic algebras
- 3 Formality in characteristic p
- 4 Generalizing AHAH



Distinguishing Homotopy Types

Explanation through examples

Example (1st Semester Algebraic Topology)

Can distinguish $\mathbb{C}P^2$ and S^4 by homology.

Example (2nd Semester Algebraic Topology)

Cannot distinguish $\mathbb{C}P^2$ and $S^2 \vee S^4$ just through homology, but can distinguish them by their cohomology with **cup product**.

Example (3rd or 4th Semester Algebraic Topology)

Can classify homotopy types of all simply connected spaces with homology like $\mathbb{C}P^2$ or $S^2 \vee S^4$ by their cohomology with cup product.



Example (4th Semester Algebraic Topology)

Cannot distinguish $\Sigma\mathbb{C}P^2$ and $S^3 \vee S^5$ through their cohomology just with cup product, but can distinguish them using the Sq^2

Steenrod operation

These are the only 2 homotopy types of simply connected spaces with this homology.

Example (Advanced Graduate Algebraic Topology)

Homotopy types of spaces with homology like $S^n \vee S^{n+r}$ can be distinguished and classified using relations between higher cohomology “operations” implicit in the unstable Adams spectral sequence.



Cohomology and Homotopy Types

Question: Is there some kind of structure of higher operations that distinguishes all simply connected homotopy types?

Answer: Yes.

Question: Well, can you be more specific about the structure?

Answer: Operations generalizing cup and cup-i products

- McClure–Smith, “Multivariable Cochain Operations and Little n -Cubes”, 2003

Fit together into the **sequence operad** \mathcal{S}

Defines an E_∞ algebra structure on C^*X

This structure determined homotopy type for simply connected spaces. Nice spaces X and Y are homotopy equivalent if and only if C^*X and C^*Y are quasi-isomorphic as E_∞ algebras.



E_∞ Algebras and Homotopy Theory

Rational Homotopy Theory

- Serre: Computing rational homotopy groups is “easy”
- Quillen/Sullivan: Commutative differential graded algebras

p -adic Homotopy Theory

- Steenrod operations \implies not commutative

Combining p -adic and rational theory to get integral theory

Key step: André–Quillen cohomology computation



Practical Remarks

Rational Homotopy Theory

- Objects: Rational commutative differential graded algebras
- Homotopy groups are reduced André–Quillen Cohomology groups
- Standard form: Minimal model
- Minimal model is easy to work with
- For “formal” spaces, practical to find minimal model

p -Adic Homotopy Theory

- Objects: E_∞ algebras
- Reduced André–Quillen Cohomology groups are zero
- Standard form: Cofibrant model
- Cofibrant model still very big, not always easy to work with
- For spaces close to $K(\pi, n)$'s, practical to find cofibrant model
- No notion of “formal” space



Further Directions

Explanatory examples show that you do not need to keep track of the whole E_∞ structure to say interesting things about classification of homotopy types.

Approach: Constrain the spaces

Put a constraint on the class of spaces and determine what algebra structure classifies them. For example, limit connectivity, dimension, number and dimension of cells, etc.

Lots of work in this direction by Baues and collaborators

Approach: Weaken the algebraic structure

Look at an algebraic structure weaker than E_∞ and see what information is left.



Algebraic Structures

Example: Steenrod Operations

Look at H^*X as an unstable algebra over the Steenrod algebra.

This is equivalent to looking at C^*X as an H_∞ algebra.

Can formulate this (\pm) in terms of the structure of an algebra over an operad that maps into the McClure–Smith sequence operad.

We should look at operads mapping in to the sequence operad.

Example: Limited Steenrod Operations

McClure–Smith show that sub-operad \mathcal{S}_n coming from first (i.e., last) bunch of Steenrod operations is an E_n operad.

What information is left when we view C^*X as an E_n algebra?



What is left in the E_n Structure?

- The E_{n-1} structure on a homotopy pullback:
Can compute E_{n-1} -structure on $C^*(Y \times_X Z)$ as $\text{Tor}^{C^*X}(C^*Y, C^*Z)$.
- Cohomology of $\Omega^m X$ for $m \leq n$.
- Cohomology of based mapping space X^{M^+} with domain M a framed manifold of dimension $m \leq n$.

Beilinson–Drinfeld / Lurie: “Chiral homology with coefficients in C^*X ”
- For PD complex, $n \geq 2$, String topology BV algebra (?)



Formality in Characteristic Zero

Definition

A commutative differential graded \mathbb{Q} -algebra is *formal* if it is quasi-isomorphic to its cohomology through maps of commutative differential graded algebras.

Examples

- A \mathbb{Q} -CDGA with zero differential is formal
- A \mathbb{Q} -CDGA whose cohomology is a free gr. com. algebra is formal
- A \mathbb{Q} -CDGA whose cohomology is an exterior algebra is formal

Definition

A space is *rationally formal* if its polynomial De Rham complex is a formal \mathbb{Q} -CDGA.



Examples of Rationally Formal Spaces

Spheres

Cohomology is an exterior algebra.

Lie Groups / H-Spaces

Milnor-Moore: Cohomology is a free gr. comm. algebra.

Wedges and Products of Formal Spaces

Smooth Complex Algebraic Varieties

Deligne-Griffiths-Morgan-Sullivan / Morgan

(Mixed) Hodge structure on cohomology gives a (mixed) Hodge structure on the De Rham complex. Limits possibilities for differentials.



Formality for E_∞ Algebras

Definition

An E_∞ algebra is *formal* if it is quasi-isomorphic to its cohomology through maps of E_∞ algebras.

Cohomology of an E_∞ algebra has E_∞ algebra structure from its graded commutative algebra structure.

In characteristic p , cohomology of E_∞ algebras have Steenrod / Dyer-Lashof operations. For commutative algebras, all but the p -th power operation are zero.

For spaces, the zeroth operation is the identity.

The cochain algebra of a space cannot be formal unless the space has contractible components.



E_n Algebras

E_n algebras have operations on $x \in H^*A$

$$\begin{aligned} Sq^m x, Sq^{m-1} x, \dots, Sq^{m-n+1} x & \quad p = 2, |x| = m \\ P^m x, P^{m-1} x, \dots, P^{m-\lfloor (n-1)/2 \rfloor} x & \quad p > 2, |x| = 2m \\ P^m x, P^{m-1} x, \dots, P^{m-\lfloor n/2 \rfloor} x & \quad p > 2, |x| = 2m + 1, n > 1 \end{aligned}$$

For $|x| \geq n$, Sq^0/P^0 not an E_n algebra operation on x .

If X is an $(n-1)$ -connected space, no Sq^0/P^0 operation in E_n structure on cochains



Formal E_n Algebras

Definition

An E_n algebra is *formal* if it is quasi-isomorphic to its graded cohomology ring through maps of E_n algebras.

Cohomology of E_n algebras have $(-n + 1)$ -Poisson structure, but E_n structure bracket is trivial for E_{n+1} algebras.

Which $(n - 1)$ -connected spaces are E_n formal?



Loops and Suspension

Recall: For any space X , Ω^n is an E_n space.

If X is an E_{n-1} -space, ΩX is an E_n space.

Because C^* is contravariant, $C^*\Sigma X$ is “like” loops of C^*X .
(Think $H\mathbb{Z}^{\Sigma X} \cong \Omega H\mathbb{Z}^X$)

Theorem

The E_{n-1} structure on C^*X determines the E_n structure on $C^*\Sigma X$.

Consequence

For any X , $\Sigma^n X$ is E_n formal.

S^n is E_n -formal but not E_{n+1} formal.



Toward Formality

Let X be an n -connected space X of the homotopy type of a finite CW complex.

Conjecture

*After inverting finitely many primes, C^*X is quasi-isomorphic as an E_n algebra to a commutative differential graded algebra.*

Conjecture (Formality)

*If X is rationally formal, then after inverting finitely many primes, C^*X is E_n formal.*



Relationship to AHAH

- Anick, “Hopf Algebras up to Homotopy”, 1989

Theorem (Anick)

Let R be a ring containing $1/m$ for $m < p$ and let X be an r -connected pr -dimensional CW complex. Then the Adams-Hilton model of X with coefficients in R is the universal enveloping algebra of a Lie algebra.

Koszul dual translation (?)

Conjecture

Let R be a ring containing $1/m$ for $m < p$ and let X be an r -connected pr -dimensional simplicial complex. Then $C^(X; R)$ is quasi-isomorphic as an E_2 -algebra to a commutative differential graded algebra.*

For a 2-connected space, after inverting finitely many primes, $C^*(X)$ is quasi-isomorphic as an E_2 -algebra to a commutative differential graded algebra.



Conjecture

Let R be a ring containing $1/m$ for $m < p$ and let X be an r -connected pr -dimensional simplicial complex. Then $C^(X; R)$ is quasi-isomorphic as an E_2 -algebra to a commutative differential graded algebra.*

For an r -reduced simplicial set, p -th tensor power of any reduced cochain is in dimension $\geq p(r + 1)$.

Instead of looking at operad Com , we can look at a truncated commutative algebras: Use Com' with $Com'(k) = 0$ for $k \geq p$.

For $k < p$, $Com'(k)$ is Σ_k -projective.

Can do homotopy theory with Com' algebras.



The McClure–Smith sequence E_2 operad S_2 has

$$\dim S_2(k) = k - 1$$

So if X is r -reduced, the operations on reduced cochains

$$S_2(k) \otimes (\tilde{C}^* X)^{\otimes k} \rightarrow \tilde{C}^* X$$

land in degrees $k(r + 1) - (k - 1) = kr + 1$ and above.

If $\dim X < pr$, $\tilde{C}^* X$ is an algebra over the truncated operad S'_2 with $S'_2(k) = 0$ for $k \geq p$.

We can look at obstruction theory for the S'_2 -structure to extend to a Com' -structure.

Obstruction groups are relative André–Quillen cohomology groups.



The Linearity Hypothesis

Hypothesis. There exists an E_n operad \mathcal{E} that acts on cochain complexes and satisfies the dimension bound

$$\dim \mathcal{E}(k) = (k - 1)(n - 1).$$

Highest chain-level k -ary operation occurs in degree $(k - 1)(n - 1)$.

Notes.

- This is the same degree as highest non-zero homology group.
- The standard E_n operads satisfy this bound for $k = 2$.
- Standard E_1 and E_2 operads satisfy this bound for all n, k .
- Topology: Standard configuration space models and Kontsevich operads satisfy this bound for all n, k .



A Weaker Linearity Hypothesis

At the cost of weakening the conjectures, the hypothesis can be weakened to a linearity hypothesis

$$\dim \mathcal{E}(k) = a(k - 1)(n - 1) \quad \text{for } k \gg 0$$

The little n -cubes operad of spaces has k -th space a non-compact manifold with boundary, dimension $k(n + 1)$.

Hypothesis can be weakened further: Operad does not have to be zero in high dimensions, just act by zero on simplices of a given dimension.



Consequences of the Linearity Hypothesis

Let X be r -reduced dimension d , so $\tilde{C}^* X = 0$ for $* \leq r$ and $* > d$

Look at E_n action.

$$\mathcal{E}(k) \otimes (\tilde{C}^* X)^{\otimes k} \rightarrow \tilde{C}^* X.$$

Left side is non-zero in range $k(r+1) - (k-1)(n-1)$ to kd .

Right side is non-zero in range $r+1$ to d .

$$k(r+1) - (k-1)(n-1) = k(r+1 - (n-1)) + (n-1) = k(r-n+2) + n-1$$

So if $k(r-n+2) + n-1 > d$ the map must be zero.

Limit dimension to $p(r-n+2) - n-2$ or even $p(r-n+2)$.



Generalizing Anick's HAH Theorem

Conjecture

Let R be a ring containing $1/m$ for $m < p$ and let X be an r -connected $p(r-n+2)$ -dimensional simplicial complex. Then $C^(X; R)$ is quasi-isomorphic as an E_n -algebra to a commutative differential graded algebra.*

