Exercises 5, April 11, 2006

Simplicial sets II

1) Let $K_\bullet \in \mathrm{Ob}\ s\underline{Set}$ be a simplicial set, i.e., a contravariant functor

$$K : \underline{\Delta} \to \underline{Set}\,.$$

Recall that

$$\begin{aligned} K_n &= K(n) \\ d_i &= K(\delta^i) \\ s_j &= K(\sigma^j)\,. \end{aligned}$$

Pick any of the simplicial identities

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && ;\, i < j \\ s_i s_j &= s_{j+1} s_i && ;\, i \le j \\ d_i s_j &= s_{j-1} d_i && ;\, i < j \\ d_j s_j &= Id = d_{j+1} s_j && \\ d_i s_j &= s_j d_{i-1} && ;\, i > j+1 \end{aligned}$$

and prove it by proving the dual relation between the δ_i's and the σ^j's.

2 a) Let $A_\bullet \in \mathrm{Ob}\ s\underline{Ab}$ be a simplicial Abelian group. Define $(\hat{A}_*, \hat{d}_*) \in \mathrm{Ob}\ \underline{Ch}$ to be the chain complex with

$$\hat{A}_n = \begin{cases} A_n & n \ge 0 \\ 0 & n < 0 \end{cases}$$

and differential $\hat{d}_n : \hat{A}_n \to \hat{A}_{n-1}$ defined by

$$\hat{d}_n(x) = \sum_{i=0}^{n} (-1)^i d_i(x)\,.$$

Show that $\hat{d}_{n-1}\hat{d}_n = 0$ by explicit computation. (Use the fact that $d_i d_j = d_{j-1} d_i$ when $i < j$.)

2 b) Show that the assignment $A_\bullet \mapsto (\hat{A}_*, \hat{d}_*)$ defines a functor $\chi : s\underline{Ab} \to \underline{Ch}$.

Specifically, given a morphism of simplicial Abelian groups $F : A_\bullet \to B_\bullet$, show that you can define a map of chain complexes

$$\chi(F) : (\hat{A}_*, \hat{d}_*) \to (\hat{B}_*, \hat{d}_*)$$

by letting $\chi(F)_n = F_n : \hat{A}_n \to \hat{B}_n$.

Then, verify that $\chi(id) = id$ and $\chi(F) \circ \chi(G) = \chi(F \circ G)$.

3) Let $K_\bullet \in \text{Ob } s\underline{Gr}$ be a simplicial group (not necessarily Abelian). I.e. K_n is a group for all $n \geq 0$ and $d_i : K_n \to K_{n-1}$ and $s_j : K_n \to K_{n+1}$ are group homomorphisms.

By the following steps, you show that $K_\bullet$ satisfies the Kan extension property, i.e. show that $K_\bullet$ thought of as a simplicial set is a Kan complex:

Let $\{x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\} \subset K_{n-1}$ be $(n-1)$-simplices satisfying $d_i(x_j) = d_{j-1}(x_i)$ for $i \neq k \neq j$ and $i < j$.

3 a) Assume $k > 0$. Let $u_0 = s_0(x_0) \in K_n$. Verify that $d_0(u_0) = x_0$.

3 b) Assume inductively that $u_{r-1} \in K_n$ exists such that $d_i(u_{r-1}) = x_i$ for $i \leq r-1$ and $0 < r \leq k-1$. Let

$$\begin{aligned} y_{r-1} &:= s_r(d_r(u_{r-1})^{-1} x_r) \\ u_r &:= u_{r-1} y_{r-1} \,. \end{aligned}$$

Show by calculation that for $i < r$ we have

$$d_i(y_{r-1}) = e_{n-1}$$

where $e_{n-1} \in K_{n-1}$ is the neutral element in the group.

Then show that $d_i u_r = x_i$ for $i \leq r$.

3 c) Let

$$v_0 := \begin{cases} u_{k-1} & k > 0 \\ e_n & k = 0 \,. \end{cases}$$

Assume inductively that v_{r-1} has been defined such that

$$d_i(v_{r-1}) = x_i \qquad \text{if } i < k \text{ or } i > n-r+1.$$

Then let

$$\begin{aligned} z_{r-1} &:= s_{n-r}(d_{n-r+1}(v_{r-1})^{-1} x_{n-r+1}) \\ v_r &:= v_{r-1} z_{r-1} \,. \end{aligned}$$

Show that $d_i(z_{r-1}) = e_{n-1}$ if $i < k$ or $i > n-r+1$.

Then show that $d_i(v_r) = x_i$ if $i < k$ or $i > n-r$.

Let $y = v_{n-k}$. We have now shown that $d_i(y) = x_i$ for all $i \neq k$, and hence that $K_\bullet$ is a Kan complex when thought of as a simplicial set.