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The completion theorem in K-theory for proper actions of a discrete group

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Abstract

We prove a version of the Atiyah–Segal completion theorem for proper actions of an infinite discrete group G. More precisely, for any finite proper G-CW-complex X, $K^*(EG \times_G X)$ is the completion of $K^*_G(X)$ with respect to a certain ideal. We also show, for such G and X, that $K_G(X)$ can be defined as the Grothendieck group of the monoid of G-vector bundles over X.

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Let G be any discrete group. For such G, a *G-CW-complex* is a CW-complex with G-action which permutes the cells such that an element $g \in G$ sends a cell to itself only by the identity map. A G-CW-complex X is *proper* if all of its isotropy subgroups have finite order, and is *finite* if it is made up of finitely many orbits of cells. A *G-CW-pair* is a pair of G-spaces (X, A), where X is a G-CW-complex and A is a G-invariant subcomplex.

The main results of this paper are Theorems 3.2 and 4.3 below. The first says that equivariant K-theory $K^*_G(-)$ can be defined on the category of finite proper G-CW-pairs using (finite dimensional) G-vector bundles, in the sense that this does define an equivariant cohomology theory. In particular, for any X, $K_G(X)$ is just the Grothendieck group of the monoid of G-vector bundles over X.

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The second theorem is an extension of the Atiyah–Segal completion theorem to this situation. It says that for any finite proper G-CW-complex X, $K^*(EG \times_G X)$ is the completion of $K_G^*(X)$ with respect to a certain ideal. In particular, when the universal proper G-space $E_{\mathscr{F}\mathscr{I}\mathscr{N}}(G)$ ($= \underline{E}G$ in the notation of Baum and Connes [7]) has the homotopy type of a finite G-CW-complex, then this completion is taken with respect to the augmentation ideal of $K_G(E_{\mathscr{F}\mathscr{I}\mathscr{N}}(G))$. For example, when $X = E_{\mathscr{F}\mathscr{I}\mathscr{N}}(G)$, Theorem 4.3 implies that $K^*(BG)$ is the completion of $K_G^*(E_{\mathscr{F}\mathscr{I}\mathscr{N}}(G))$ with respect to the augmentation ideal in $K_G(E_{\mathscr{F}\mathscr{I}\mathscr{N}}(G))$.

There are two ways in which the proofs of these theorems, when G is infinite and discrete, diverge from the usual proofs for finite group actions. First, since the category of spaces with proper G-action does not contain cones or suspensions (fixed points are not allowed), we need to find other ways to define $K_G(X, A)$ and $K_G^{-n}(X)$. This is easily handled. A more crucial difference is that special constructions are needed, carried out in Section 2, to get around the lack of "sufficiently many" product bundles. This second difficulty is illustrated by the fact that both of these theorems fail in general when G is a positive dimensional noncompact Lie group. This is discussed in detail, with examples, in Section 5. Examples which show that $K_G(-)$ defined using G-vector bundles is not an equivariant cohomology theory in this situation were originally due to Phillips [15], who instead defined $K_G(-)$ using infinite dimensional G-vector bundles with Hilbert space fibers (see also [17]).

In a separate paper, we will construct an equivariant cohomology theory $K_G^*(-)$ for arbitrary (not necessarily proper) G-CW-complexes using spectra. More precisely, this will be done using $\mathfrak{Or}(G)$-spectra: contravariant functors from the orbit category of G to spectra. We will also construct an equivariant Chern character for proper G-CW-complexes which takes values in equivariant Bredon cohomology, and which is rationally an isomorphism for finite proper G-CW-complexes.

Let $\mathbb{K}_G^*(X)$ be the Grothendieck group of (finite-dimensional) G-vector bundles over X. There is a natural transformation $\varphi_G: \mathbb{K}_G^*(X) \to K_G^*(X)$, which is an isomorphism for finite proper G-CW-complexes. In the nonequivariant case, this is well known to be an isomorphism for any finite dimensional CW-complex X (since any map $X \to BU$ factors through some $BU(n)$). But even for finite $G \neq 1$, Example 3.11 below shows that $\mathbb{K}_G^*$ is *not* a cohomology theory on the category of all finite dimensional proper G-CW-pairs.

The paper is organized as follows:

1. G-vector bundles over proper G-CW-complexes,
2. Constructions of G-vector bundles,
3. Equivariant K-theory for finite proper G-CW-complexes,
4. The completion theorem,
5. Proper actions of Lie groups,

   References.

1. G-vector bundles over proper G-CW-complexes

Throughout this section G is a Lie group. We collect here some basic facts about G-vector bundles over proper G-CW-complexes.

A *G-CW-complex* X is a space with G-action, which is filtered by its "skeleta" $X^{(n)}$, such that X has the weak topology as the union of the $X^{(n)}$, and such that each $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching orbits of cells $G/H_i \times D^n$ via attaching maps $G/H_i \times S^{n-1} \to X^{(n-1)}$. (Here $X^{(-1)} = \emptyset$.) When G is discrete, a G-CW-complex can be thought of as a CW-complex with G-action which permutes the cells, such that an element $g \in G$ sends a cell to itself only by the identity map. Note that the orbit space of a G-CW-complex inherits the structure of an (ordinary) CW-complex. For more details about G-CW-complexes, see, e.g., [9, Sections II.1 and II.2] or [13, Sections I.1 and I.2].

A G-CW-complex X is *finite* if it is made up of finitely many orbits of cells $G/H \times D^n$, or equivalently if X/G is a finite CW-complex. A G-CW-complex X will be called *proper* if all of its isotropy subgroups are compact. (For G-CW-complexes, this is equivalent to the various definitions of proper actions which have been given in more general situations.)

A *G-vector bundle* over a G-CW-complex X consists of a (complex) vector bundle $p: E \to X$, together with a G-action on E such that p is G-equivariant and each $g \in G$ acts on E and X via a bundle isomorphism. We let $E|_x$ denote the fiber over a point $x \in X$. A *map* of G-vector bundles from $p: E \to X$ to $p': E' \to X'$ is just a map $(\bar{f}, f)$ of vector bundles, such that $\bar{f}: E \to E'$ and $f: X \to X'$ are G-equivariant. Here, we assume only that $\bar{f}$ restricts to a linear map $E|_x \to E'|_{f(x)}$ for each $x \in X$. We call $(\bar{f}, f)$ a *strong map* if $\bar{f}$ restricts to a linear isomorphism $E|_x \xrightarrow{\cong} E'|_{f(x)}$ for each $x \in X$. This is clearly equivalent to the condition that $p: E \to X$ is isomorphic to the pullback of $p': E' \to X'$ over f.

Most of the properties of G-vector bundles over G-CW-complexes we need will be easy consequences of the following elementary lemma.

Lemma 1.1. (a) *Any G-vector bundle over an orbit of cells $G/H \times D^n$ is isomorphic to $G \times_H (V \times D^n)$ for some H-representation V.*

(b) *For any G-CW-complex X, both X and X/G are paracompact.*

(c) *Fix a G-vector bundle $p: E \to X$ over a G-CW-complex X. Let $X^{(n)}$ be the n-skeleton of X, and set $E_n = p^{-1}(X^{(n)})$. Then the squares*

$$\begin{tikzcd} \coprod_{j\in J} G/H_j \times S^{n-1} \arrow[r, "\coprod_{j\in J} q_j"] \arrow[d] & X^{(n-1)} \arrow[d] \\ \coprod_{j\in J} G/H_j \times D^n \arrow[r, "\coprod_{j\in J} Q_j"] & X^{(n)} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \coprod_{j\in J} G \times_{H_j} (V_j \times S^{n-1}) \arrow[r, "\coprod_{j\in J} \bar{q}_j"] \arrow[d] & E_{n-1} \arrow[d] \\ \coprod_{j\in J} G \times_{H_j} (V_j \times D^n) \arrow[r, "\coprod_{j\in J} \bar{Q}_j"] & E_n \end{tikzcd}$$

are pushout squares for each n. Also, X and E have the weak topology with respect to the subspaces $X^{(n)}$ and E_n, respectively. More generally, if $\{X_i\}_{i \in I}$ is any set of subcomplexes which cover X, then X and E have the weak topology with respect to the subspaces X_i and $p^{-1}(X_i)$, respectively.

(d) *For any G-CW-pair (X, A), there is a neighborhood W of A in X, which can be chosen to be closed or open, such that A is an equivariant strong deformation retract of W.*

Proof. (a) Note that for any G-map $p: X \to G/H$, the canonical map $G \times_H p^{-1}(eH) \to X$ is a G-homeomorphism. (This will be used frequently throughout the paper.) It thus suffices to show that any H-vector bundle over D^n is isomorphic to the product bundle $V \times D^n$ for some H-representation V, and this follows from [3, Proposition 1.6.2].

(c) The pushout square for $X^{(n)}$, and the fact that X has the weak topology with respect to its skeleta, follow from the definition of a G-CW-complex. In particular, a function $X \to Y$ (for any space Y) is continuous if and only if its composite with each equivariant cell $G/H \times D^n \to X$ is continuous; and from this one sees immediately that X has the weak topology with respect to any covering set of subcomplexes. The G-pushout property for E_n follows the pushout property for $X^{(n)}$, together with (a) and [13, Lemma 1.26].

We now claim for *any* X, any vector bundle $p: E \to X$, and any covering $X = \bigcup_{i \in I} X_i$ by closed subspaces, that E has the weak topology with respect to its subsets $p^{-1}(X_i)$ if X has the weak topology with respect to the X_i. Upon restricting to a neighborhood of any given $x \in X$, this is reduced to the case where E is a product bundle; and the result then follows easily since the fibers are locally compact.

(b) Given an open covering $\mathscr{U}$ of X or of X/G, a partition of unity subordinate to $\mathscr{U}$ can be constructed by applying Zorn's lemma to the set of such partitions of unity over subcomplexes of X (and using (c) above). For more details (in the case of a nonequivariant CW-complex), see [14, Theorem II.4.2].

(d) For each n, one easily constructs a collar neighborhood V_n of $X^{(n)}$ in $X^{(n+1)}$ (open or closed), together with an equivariant deformation retraction $\rho_n : V_n \to X^{(n)}$, which restricts to a deformation retraction of $\rho_n^{-1}(B)$ to B for any $B \subseteq X^{(n)}$. Now set

$$W_{-1} = A, \quad W_n = A \cup \rho_n^{-1}(W_{n-1} \cap X^{(n)}) \quad (\text{all } n \geqslant 0), \quad \text{and} \quad W = \bigcup_{n=0}^{\infty} W_n,$$

let $r_n : W_n \to W_{n-1}$ be the identity on A and ρ_n on $W_n \backslash A$, and let $r: W \to A$ be the composite of the r_n. $\square$

The next three results, which list some of the standard properties of G-vector bundles, are easy consequences of Lemma 1.1. We begin with homotopy invariance. As usual, I denotes the unit interval [0,1].

Theorem 1.2. *Let X be a proper G-CW-complex, let $p: E \to X \times I$ be a G-vector bundle, and set $E_0 = E|_{X \times 0}$, regarded as a G-vector bundle over X. Then there is an isomorphism $\rho: E \xrightarrow{\cong} E_0 \times I$ of G-vector bundles, which is the identity on E_0 and covers the identity on $X \times I$. If, in addition, $A \subseteq X$ is any G-invariant subcomplex, then ρ can be chosen to extend any given isomorphism $\rho_A : E|_{A \times I} \xrightarrow{\cong} E_0|_{A \times I}$.*

Proof. Using Lemma 1.1(c), this is quickly reduced to the case where $(\dot{X}, A) = (G/H \times D^n, G/H \times S^{n-1})$. By Lemma 1.1(a), $E = G \times_H (V \times D^n \times I)$. Thus, ρ_A is equivalent to a map $\rho' : S^{n-1} \times I \to \mathrm{Aut}_H(V)$ which sends $S^{n-1} \times 0$ to the identity, and this can be extended to $D^n \times I$ since $S^{n-1} \to D^n$ is a cofibration. $\square$

The proof of the next two lemmas is similiar to that of Theorem 1.2.

Lemma 1.3. *Let (X, A) be a proper G-CW-pair, and let E and E' be G-vector bundles over X. Then any map $f: E'|_A \to E|_A$ of G-vector bundles over A extends to a map $\bar{f}: E' \to E$ of G-vector bundles over X.*

Proof. Via Lemma 1.1, it suffices to prove this when $(X, A) = (G/H \times D^n, G/H \times S^{n-1})$, $E = G \times_H (V \times D^n)$, and $E' = G \times_H (V' \times D^n)$. A map $E' \to E$ of G-vector bundles thus corresponds to an H-map $D^n \to \mathrm{Hom}_H(V', V)$, and any map over A extends to a map over X since $\mathrm{Hom}_H(V', V)$ is contractible. $\square$

Lemma 1.4. *Let (X, A) be a proper G-CW-pair, and let E be a G-vector bundle over X. Then any G-invariant Hermitian metric of $E|_A$ extends to a G-invariant Hermitian metric on E.*

Proof. Again, it suffices to prove this when $(X, A) = (G/H \times D^n, G/H \times S^{n-1})$, and $E = G \times_H (V \times D^n)$. A Hermitian metric over X then corresponds to a map $D^n \to \mathrm{Herm}_H(V)$ (the space of H-invariant Hermitian metrics over V); and any such map on S^{n-1} can be extended to one on D^n since $\mathrm{Herm}_H(V)$ is convex (and hence contractible). $\square$

To finish the section, we check that a pushout of G-vector bundles is a G-vector bundle over the pushout of the base spaces. This will, of course, be used to prove excision in Section 3.

Lemma 1.5. *Let $\varphi : (X_1, X_0) \to (X, X_2)$ be a map of G-CW-pairs, set $\varphi_0 = \varphi|_{X_0}$, and assume that $X \cong X_2 \cup_{\varphi_0} X_1$. Let $p_1 : E_1 \to X_1$ and $p_2 : E_2 \to X_2$ be G-vector bundles, let $\bar{\varphi}_0 : E_1|_{X_0} \to E_2$ be a strong map covering φ_0, and set $E = E_2 \cup_{\bar{\varphi}_0} E_1$. Then $p = p_1 \cup p_2 : E \to X$ is a G-vector bundle over X.*

Proof. The only problem is to show that $p : E \to X$ is locally trivial (in a non-equivariant sense). Since E_1 is locally trivial, so is $E|_{X \setminus X_2} \cong E_1|_{X_1 \setminus X_0}$. So it remains to find a neighborhood of X_2 over which E is locally trivial. Choose a closed neighborhood W_1 of X_0 in X_1 for which there is a strong deformation retraction $r : W_1 \to X_0$ (Lemma 1.1(d)). By the homotopy invariance for nonequivariant vector bundles over paracompact spaces (cf. [10, Corollary 3.4.5]), r is covered by a strong map of vector bundles $\bar{r} : E_1|_{W_1} \to E_0$ which extends $\bar{i}_1$. Set $W = X_2 \cup_{\varphi_0} W_1$. Then $\bar{r}$ extends, via the pushout, to a strong map of vector bundles $E|_W \to E_2$ which extends $\bar{i}_2$, and hence $E|_W$ is locally trivial. $\square$

Let $p : E \to B$ be a G-vector bundle over a proper G-CW-complex. Each orbit $Gx \subseteq X$ has a G-invariant neighborhood U_x such that Gx is an equivariant retract of U_x, and the neighborhood can in fact be chosen such that the retraction is covered by a strong map $E|_{U_x} \to E|_{Gx}$. There is thus a G-covering $\mathscr{U}$ of X such that each $E|_U$ (for $U \in \mathscr{U}$) is "trivial" in the sense that it is the pullback of a bundle over a (proper) orbit. Also, since X/G is paracompact by Lemma 1.1(b), $\mathscr{U}$ is G-numerable in the sense that there is a locally finite partition of unity $\{t_U | U \in \mathscr{U}\}$ by G-invariant functions t_U with $\mathrm{supp}(t_U) \subseteq U$. Hence our notion of G-vector bundles agrees with that of tom Dieck in [9, Section I.9]. For these same reasons, the results of this section can easily be extended to proper G-spaces which have paracompact quotients (any such space has tubes, i.e., equivariant neighborhood retracts of orbits).

2. Constructions of *G*-vector bundles

The main result in this section is Theorem 2.6. Given a discrete group G, a G-CW-complex X, and a family $\{V_H\}$ of representations of the isotropy subgroups in X, we would like to be able to construct a G-vector bundle $E \to X$ whose fiber over any $x \in X$ is isomorphic to V_{G_x}. This is in general not possible, even for finite G, for reasons discussed at the end of the section. What we show here is that we can do this, assuming certain conditions on X and the V_H, but only after replacing the V_H by some iterated direct sum $(V_H)^k$, or by some iterated tensor product $(V_H)^{\otimes k}$. These bundles are the crucial ingredients in the proof that G-vector bundles define an equivariant cohomology theory (Theorem 3.2), and the proof of the completion theorem (Theorem 4.3).

Throughout the first part of the section, G and Γ will denote arbitrary Lie groups. A *family* $\mathscr{F}$ of subgroups of G is a set of (closed) subgroups of G which is closed under conjugation. We will need to work with some classifying spaces and universal spaces: first for (proper) G-actions and then for bundles.

Definition 2.1. For any family $\mathscr{F}$ of subgroups of G, let $\mathscr{E}_{\mathscr{F}}(G)$ denote the topological category whose objects are the pairs $(G/H, gH)$ for $H \in \mathscr{F}$ and $g \in G$, and where $\mathrm{Mor}((G/H, gH),(G/K, g'K))$ is the set of G-maps $G/H \to G/K$ which send gH to $g'K$ (a set of cardinality at most one). Let $E_{\mathscr{F}}(G)$ be the realization of the nerve of $\mathscr{E}_{\mathscr{F}}(G)$, considered as a G-CW-complex:

$$E_{\mathscr{F}}(G) = \left(\coprod_{n=0}^{\infty} \coprod_{G/H_0 \to \cdots \to G/H_n} G/H_0 \times \Delta^n \right) \Big/ \sim ,$$

where the identifications are those induced by the obvious face and degeneracy maps.

As usual, we are assuming that $E_{\mathscr{F}}(G)$ has the weak topology with respect to its cellular structure.

Lemma 2.2. *Fix a Lie group G and a family $\mathscr{F}$ of subgroups of G.*

(a) *For any $K \in \mathscr{F}$, $(E_{\mathscr{F}}(G))^K$ is contractible.*

(b) *Let (X, A) be any G-CW-pair such that $G_x \in \mathscr{F}$ for all $x \in X$. Then any G-map $f_A{:}A \to E_{\mathscr{F}}(G)$, extends to a G-map $f_X{:}X \to E_{\mathscr{F}}(G)$, and any two such extensions are G-homotopic relative A.*

Proof. (a) For any $K \subseteq G$, $(E_{\mathscr{F}}(G))^K$ is the nerve of the full subcategory of $\mathscr{E}_{\mathscr{F}}(G)$ with objects those $(G/H, gH)$ such that $K \subseteq gHg^{-1}$. And if $K \in \mathscr{F}$, then this category has the initial object $(G/K, eK)$.

(b) This follows immediately from point (a) (see [13, Proposition 2.3 p. 35]). □

A *G-equivariant Γ-bundle* (or (G,Γ)-bundle for short) consists of a Γ-principal bundle $p{:}E \to X$, together with left G-actions on E and X, such that p is G-equivariant, and such that the left G-action and the right Γ-action on E commute. We let $\mathrm{Bdl}_{G,\Gamma}X$ denote the set of isomorphism classes of (G,Γ)-bundles over the G-space X.

One natural example of this is the case $\Gamma = U(n)$. A $(G, U(n))$-bundle $E \to X$ is just the principal bundle associated with the G-vector bundle $E \times_{U(n)} \mathbb{C}^n \to X$. Similarly, a (G,Σ_n)-bundle is the principal bundle associated with a G-equivariant n-sheeted covering space. In the constructions below, we will have to consider (G,Γ)-bundles for certain finite subgroups $\Gamma \subseteq U(n)$.

Now fix a family $\mathscr{F}$ of compact subgroups of G. For each $H \in \mathscr{F}$, set $\mathrm{Rep}_{\Gamma}(H) = \mathrm{Hom}(H, \Gamma)/\mathrm{Inn}(\Gamma)$; i.e., the set of conjugacy classes of homomorphisms from H to Γ. For example, $\mathrm{Rep}_{U(n)}(H)$ is the set of isomorphism classes of n-dimensional complex representations of H, and

$\mathrm{Rep}_{\Sigma_n}(H)$ is the set of isomorphism classes of H-sets of order n. Note that for any H and Γ, there are natural bijections

$$\mathrm{Rep}_\Gamma(H) \cong \mathrm{Bdl}_{H,\Gamma}(\mathrm{pt}) \cong \mathrm{Bdl}_{G,\Gamma}(G/H). \tag{2.3}$$

We need a way to specify the isomorphism types of the fibers of a (G,Γ)-bundle. Suppose we are given an element

$$\mathbf{A} = (\alpha_H) \in \varprojlim_{H \in \mathscr{F}} \mathrm{Rep}_\Gamma(H) \subseteq \prod_{H \in \mathscr{F}} \mathrm{Rep}_\Gamma(H),$$

where the limit is taken with respect to all homomorphisms induced by inclusions and conjugation in G. This is equivalent to an element in $\varprojlim_{\mathfrak{Or}_{\mathscr{F}}(G)} \mathrm{Bdl}_{G,\Gamma}(-)$, where $\mathrm{Bdl}_{G,\Gamma}(-)$ is considered as a contravariant functor (via pullback) on the orbit category $\mathfrak{Or}_{\mathscr{F}}(G)$. If X is a G-space all of whose isotropy subgroups lie in $\mathscr{F}$, then we define a $(G,\mathbf{A})$-*bundle* over X to be a (G,Γ)-bundle such that the fiber over any point $x \in X$ is isomorphic to (Γ, α_{G_x}), regarded as a (G_x, Γ)-bundle over a point (see (2.3)). When $\Gamma = U(n)$, this corresponds to those G-vector bundles whose fibers are isomorphic to certain given representations of the isotropy subgroups.

We want to define classifying spaces for (G,Γ)-bundles and for $(G,\mathbf{A})$-bundles. In fact, these are just the universal $(G \times \Gamma)$-CW-complexes with respect to appropriate families.

Definition 2.3. Let $\mathscr{F}$ be a family of compact subgroups of G. Define

$$E_{\mathscr{F}}(G,\Gamma) = E_{\mathscr{F}_\Gamma}(G \times \Gamma) \quad \text{and} \quad B_{\mathscr{F}}(G,\Gamma) = E_{\mathscr{F}}(G,\Gamma)/\Gamma,$$

where

$$\mathscr{F}_\Gamma = \{H \subseteq G \times \Gamma \mid \mathrm{pr}_1(H) \in \mathscr{F},\ H \cap (1 \times \Gamma) = 1\}.$$

For any element

$$\mathbf{A} = (\alpha_H) \in \varprojlim_{H \in \mathscr{F}} \mathrm{Rep}_\Gamma(H) \subseteq \prod_{H \in \mathscr{F}} \mathrm{Rep}_\Gamma(H),$$

define

$$\mathscr{F}_{\mathbf{A}} = \{H \subseteq G \times \Gamma \mid H = \text{graph}\ (\alpha : K \to \Gamma),\ \text{some}\ K \in \mathscr{F},\ \text{some}\ \alpha\ \text{conjugate to}\ \alpha_K\},$$

and set

$$E_{\mathscr{F}}(G,\mathbf{A}) = E_{\mathscr{F}_{\mathbf{A}}}(G \times \Gamma) \quad \text{and} \quad B_{\mathscr{F}}(G,\mathbf{A}) = E_{\mathscr{F}}(G,\mathbf{A})/\Gamma.$$

In the above situation, if $E \xrightarrow{p} X$ is any (G,Γ)-bundle, where X is a proper G-CW-complex all of whose isotropy subgroups lie in $\mathscr{F}$, then E is a proper $(G \times \Gamma)$-CW-complex all of whose isotropy subgroups lie in $\mathscr{F}_\Gamma$. Conversely, if E is any proper $(G \times \Gamma)$-CW-complex all of whose isotropy subgroups lie in $\mathscr{F}_\Gamma$, then E/Γ is a proper G-CW-complex all of whose isotropy subgroups lie in $\mathscr{F}$, and the projection $E \to E/\Gamma$ is a (G,Γ)-bundle. Similarly, for any $\mathbf{A}$, there is a correspondence between $(G,\mathbf{A})$-bundles and $(G\times\Gamma)$-CW-complexes all of whose isotropy subgroups lie in $\mathscr{F}_{\mathbf{A}}$. This leads to the following:

Lemma 2.4. *Fix a family $\mathscr{F}$ of compact subgroups of G, and an element*

$$\mathbf{A}=(\alpha_H)\in \varprojlim_{H\in\mathscr{F}} \mathrm{Rep}_\Gamma(H),$$

where the limit is taken with respect to inclusions and conjugation in G. Then the following hold:

(a) *For each $H\in\mathscr{F}$, let $C_\Gamma(\alpha_H)$ denote the centralizer of the image of $\alpha_H\colon H\to\Gamma$ (well defined up to conjugacy). Then there is a homotopy equivalence*

$$(B_{\mathscr{F}}(G,\mathbf{A}))^H \simeq BC_\Gamma(\alpha_H),$$

which is natural with respect to maps induced by homomorphisms $\Gamma\to\Gamma'$.

(b) *The $(G,\mathbf{A})$-bundle*

$$E_{\mathscr{F}}(G,\mathbf{A})\to B_{\mathscr{F}}(G,\mathbf{A})$$

is the universal $(G,\mathbf{A})$-bundle in that it defines, via pullbacks, a bijection

$$[X,B_{\mathscr{F}}(G,\mathbf{A})]_G\to \mathrm{Bdl}_{G,\mathbf{A}}(X),$$

for any proper G-CW-complex X all of whose isotropy subgroups are in $\mathscr{F}$.

Proof. (a) Fix H, and write $C=C_\Gamma(\alpha_H)$ for short. Consider the $(G,\mathbf{A})$-bundle

$$G\times_H(EC\times_C\Gamma)\to G/H\times BC,$$

where H acts on $EC\times_C\Gamma$ via $h(x,\gamma)=(x,\alpha_H(h)\gamma)$. The classifying map for this bundle restricts to a map

$$BC\to(B_{\mathscr{F}}(G,\mathbf{A}))^H.$$

Similarly, the restriction of the universal $(G,\mathbf{A})$-bundle over $(B_{\mathscr{F}}(G,\mathbf{A}))^H$ is an (H,α_H)-bundle over a space with trivial H-action, and hence has structure group $C=C_\Gamma(\alpha_H)$. It is thus classified by a map

$$(B_{\mathscr{F}}(G,\mathbf{A}))^H\to BC,$$

and the above two maps are homotopy inverses by the universal properties of the spaces.

(b) This follows immediately from Lemma 2.2(b). □

We now assume, throughout the rest of the section, that G is discrete. We need to construct maps to the classifying spaces defined in Definition 2.3. The obstructions to doing so lie in certain Bredon cohomology groups.

Let (X,A) be any G-CW-pair such that the isotropy group of each point in $X\backslash A$ lies in $\mathscr{F}$. For each $n\geqslant 0$, let

$$\underline{C}_n(X,A)\colon \mathfrak{Or}_{\mathscr{F}}(G)\to\mathfrak{Ab}$$

denote the contravariant functor which sends G/H to $C_n(X^H, A^H)$. Here, $C_n(X^H, A^H)$ is the free abelian group with one generator for each n-cell in $X^H \backslash A^H$. For any contravariant functor $M : \mathfrak{Or}_{\mathcal{F}}(G) \to \mathfrak{Ab}$, $\mathrm{Hom}_{\mathfrak{Or}_{\mathcal{F}}(G)}(\underline{C}_n(X, A), M)$ is the direct sum of one copy of $M(G/H)$ for each orbit $G/H \times D^n$ of n-cells in $X \backslash A$. In particular, $\underline{C}_n(X, A)$ is projective in the category $\mathfrak{Or}_{\mathcal{F}}(G)$-$\mathfrak{mod}$ of contravariant functors $\mathfrak{Or}_{\mathcal{F}}(G) \to \mathfrak{Ab}$. The Bredon cohomology groups $H^*_G(X, A; M)$ are thus the homology groups of the cochain complex

$$0 \to \mathrm{Hom}_{\mathfrak{Or}_{\mathcal{F}}(G)}(\underline{C}_0(X, A), M) \xrightarrow{\delta} \mathrm{Hom}_{\mathfrak{Or}_{\mathcal{F}}(G)}(\underline{C}_1(X, A), M) \xrightarrow{\delta} \mathrm{Hom}_{\mathfrak{Or}_{\mathcal{F}}(G)}(\underline{C}_2(X, A), M) \xrightarrow{\delta} \cdots.$$

Lemma 2.5. *Assume that G is discrete. Fix a family $\mathcal{F}$ of finite subgroups of G, a finite group Γ, and a system of representations*

$$\mathbf{A} = (\alpha_H)_{H \in \mathcal{F}} \in \varprojlim_{H \in \mathcal{F}} \mathrm{Rep}_\Gamma(H).$$

Set $B = B_{\mathcal{F}}(G, \mathbf{A})$, and let

$$\beta_{\mathbf{A}} : B \to E_{\mathcal{F}}(G)$$

be any G-map. (This exists and is unique up to G-homotopy by Lemma 2.2(b), since all isotropy subgroups for B lie in $\mathcal{F}$.) Let Z denote the mapping cylinder of $\beta_{\mathbf{A}}$. Let $M : \mathfrak{Or}_{\mathcal{F}}(G) \to \mathfrak{Ab}$ be any contravariant functor. Then for each $n \geqslant 0$,

$$|\Gamma|^n \cdot H^n_G(Z, B; M) = 0.$$

Proof. There is a cohomology spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathfrak{Or}_{\mathcal{F}}(G)}(\underline{H}_q(Z, B), M) \;\Rightarrow\; H^{p+q}_G(Z, B; M),$$

where $\underline{H}_p(Z, B)$ denotes the functor $\mathfrak{Or}_{\mathcal{F}}(G) \to \mathfrak{Ab}$ which assigns to G/H the abelian group $H_p(Z^H, B^H)$. It is induced by the double complex $\mathrm{Hom}_{\mathfrak{Or}_{\mathcal{F}}(G)}(\underline{C}_q(Z, B), I_p)$, where $\{I_p\}$ is any injective resolution in $\mathfrak{Or}_{\mathcal{F}}(G)$-$\mathfrak{mod}$ of M. We have just seen that the $\underline{C}_q(Z, B)$ are all projective in $\mathfrak{Or}_{\mathcal{F}}(G)$-$\mathfrak{mod}$. This category does have enough injectives by, e.g., [19, Example 2.3.13].

Since $Z^H \simeq (E_{\mathcal{F}}(G))^H$ is contractible by Lemma 2.2(a), we conclude from Lemma 2.4(a) that

$$H_q(Z^H, B^H) \cong \tilde{H}_{q-1}(B^H) \cong \tilde{H}_{q-1}(C_\Gamma(\alpha_H)).$$

In particular, since $C_\Gamma(\alpha_H) \subseteq \Gamma$, this shows that

$$|\Gamma| \cdot H_*(Z^H, B^H) = 0.$$

So $|\Gamma|$ annihilates all terms in the above spectral sequence, and hence (since $E_2^{p,0} = 0$) $|\Gamma|^n$ annihilates $H^p_G(Z, B; M)$. $\square$

Given any $\mathbf{A} = (\alpha_H) \in \varprojlim_{H \in \mathcal{F}} \mathrm{Rep}_\Gamma(H)$, and any homomorphism $\rho : \Gamma \to U(n)$, there is a natural map

$$\rho_* : B_{\mathcal{F}}(G, \mathbf{A}) \to B_{\mathcal{F}}(G, \rho \circ \mathbf{A}) \quad \text{where } \rho \circ \mathbf{A} = (\rho \circ \alpha_H) \in \varprojlim_{H \in \mathcal{F}} \mathrm{Rep}_{U(n)}(H);$$

and ρ_* commutes with the maps $\beta_{\mathbf{A}}$ and $\beta_{\rho \circ \mathbf{A}}$ to $E_{\mathcal{F}}(G)$.

Theorem 2.6. *Assume that G is discrete. Fix any family $\mathscr{F}$ of finite subgroups of G, and let*

$$\mathbf{V} = (V_H) \in \varprojlim_{H \in \mathscr{F}} \mathrm{Rep}_{U(n)}(H)$$

be any system of compatible n-dimensional representations. Assume that there is a finite group Γ, a system

$$\mathbf{A} = (\alpha_H) \in \varprojlim_{H \in \mathscr{F}} \mathrm{Rep}_{\Gamma}(H),$$

and a homomorphism $\rho : \Gamma \to U(n)$ such that $\mathbf{V} = \rho \circ \mathbf{A}$. Then for any $d > 0$ there is an integer $k = k(d) > 0$, such that for any d-dimensional G-CW-complex X all of whose isotropy subgroups lie in $\mathscr{F}$, there are G-vector bundles $E, E' \to X$ such that the fibers $E|_x$ and $E'|_x$ over each point $x \in X$ are isomorphic as G_x-representations to $(V_{G_x})^k$ and $(V_{G_x})^{\otimes k}$, respectively.

Proof. We only treat the case of direct sums here; the tensor product case is analogous. By the universal property of $E_{\mathscr{F}}(G)$ (Lemma 2.2(b)), it suffices to prove this when $X = E_{\mathscr{F}}(G)^{(d)}$ (the d-skeleton).

Write $B = B_{\mathscr{F}}(G, \mathbf{A})$ and $B'_k = B_{\mathscr{F}}(G, \mathbf{V}^k)$ for short (any $k \geqslant 1$), and let Z be the mapping cylinder of $\beta_{\mathbf{A}}: B \to E_{\mathscr{F}}(G)$. We must construct, for some k, a map $Z^{(d)} \to B'_k$; and we will do so by extending the map $\rho^k_*: B \to B'_k$. By Lemma 2.4(a), $(B'_k)^H \simeq B\,\mathrm{Aut}_H(V^k_H)$ for each $H \in \mathscr{F}$, and is in particular a product of $BU(m)$'s and hence simply connected. So there is no obstruction to extending $\rho^1_* : B \to B'_1$ to a map $f_2 : B \cup Z^{(2)} \to B'_1$.

Assume inductively that $f_{d-1} : B \cup Z^{(d-1)} \to B'_r$ has been constructed, (where $r = k(d-1)$). We now apply standard equivariant obstruction theory. For each $H \in \mathscr{F}$, let

$$c^d(f_{d-1})(G/H) : C_d(Z^H, B^H) \to \pi_{d-1}((B'_r)^H)$$

be the map which sends each generator, corresponding to a d-cell σ in $Z^H \backslash B^H$, to the element $f_{d-1}(\partial\sigma)$. This is well defined independently of the basepoint, since $(B'_r)^H$ is simply connected. By naturality, this defines an element

$$c^d(f_{d-1}) \in C^d_G(Z, B; \underline{\pi}_{d-1}(B'_r)) = \mathrm{Hom}_{\mathfrak{Or}_{\mathscr{F}}(G)}(\underline{C}_d(Z, B), \underline{\pi}_{d-1}(B'_r)),$$

and f_{d-1} can be extended to a map $B \cup Z^{(d)} \to B'_r$ if and only if $c^d(f_{d-1}) = 0$. Furthermore, $c^d(f_{d-1})$ is a cocycle by [20, Theorem V.5.6], and hence defines an element

$$o^d(f_{d-1}) \in H^d_G(Z, B; \underline{\pi}_{d-1}(B'_r)).$$

Finally, for any $\tilde{c} \in C^{d-1}_G(Z, B; \underline{\pi}_{d-1}(B'_r))$, there is a map $f' : B \cup Z^{(d-1)} \to B'_r$ such that f' agrees with f_{d-1} on $B \cup Z^{(d-2)}$, and such that $c^d(f') - c^d(f_{d-1}) = \delta(\tilde{c})$ (as in [20, Theorem V.5.6′]). So if $o^d(f_{d-1}) = 0$, then $f_{d-1}|B \cup Z^{(d-2)}$ can be extended to $B \cup Z^{(d)}$. For more details, see [20, Section V.5], and also [8, Section II.1] (where equivariant obstruction theory is developed for actions of a finite group).

By Lemma 2.5, $H^d_G(Z,B;\underline{\pi}_{d-1}(B'_r))$ has exponent $|\Gamma|^d$. Furthermore, by Lemma 2.4(a) again, $\underline{\pi}_{d-1}(B'_r)$ is the functor $G/\bar{H}\mapsto\pi_{d-1}(B\,\mathrm{Aut}_H(V^r_H))$. Write $m=|\Gamma|^d$ for short, and consider the homomorphisms

$$\pi_{d-1}(B\,\mathrm{Aut}_H(V^r_H))\underset{\text{diag}}{\overset{i_1,\dots,i_m}{\rightrightarrows}}(\pi_{d-1}(B\,\mathrm{Aut}_H(V^r_H)))^m\xrightarrow{B\oplus_*}\pi_{d-1}(B\,\mathrm{Aut}_H(V^{mr}_H)).$$

These are all homomorphisms of functors on $\mathfrak{Or}_{\mathscr{F}}(G)$. Also, $B\oplus_*\circ i_s=B\oplus_*\circ i_1$ for all s, since the corresponding maps between spaces differ by conjugation by an element of $\mathrm{Aut}_H(V^{mr}_H)$ (and the automorphism group is connected). Since diag $=\sum_{s=1}^m i_s$, it follows that $(B\oplus_*)\circ$diag factors through multiplication by m, and hence that the induced map

$$H^d_G(Z,B;\underline{\pi}_{d-1}(B'_r))\overset{\Delta^m_*}{\to}H^d_G(Z,B;\underline{\pi}_{d-1}(B'_{mr}))$$

is zero. Here, $\Delta^m:U(rn)\to U(mrn)$ denotes the diagonal inclusion. We can thus extend $\Delta^m_*\circ f_{d-1}|B\cup Z^{(d-2)}$ to a map

$$f_d:B\cup Z^{(d)}\to B'_{mr}=B_{\mathscr{F}}(G,\mathbf{V}^{mr}).$$

Set $k=k(d)=mr$; the pullback to $E_{\mathscr{F}}(G)^{(d)}\subseteq Z^{(d)}$ of the $(G,\mathbf{V}^k)$-vector bundle

$$E_{\mathscr{F}}(G,\mathbf{V}^k)\times_{U(nk)}\mathbb{C}^{nk}\to B_{\mathscr{F}}(G,\mathbf{V}^k)$$

now has the desired properties. □

As a first consequence of Theorem 2.6, we show the following result, which will be needed when proving excision for equivariant K-theory defined via G-vector bundles.

Corollary 2.7. *Assume that G is discrete, and let X be any finite dimensional proper G-CW-complex whose isotropy subgroups have bounded order. Then there is a G-vector bundle $E\to X$ such that for each $x\in X$, the fiber $E|_x$ is a multiple of the regular representation of G_x.*

Proof. Let $\mathscr{F}$ be the family of isotropy subgroups in X, and let n be the least common multiple of their orders. For each $H\in\mathscr{F}$, let V_H be the free complex H-representation of dimension n, and let $\alpha_H:H\to\Sigma_n$ be a homomorphism corresponding to a free H-set of order n. Then

$$\mathbf{V}=(V_H)\in\varprojlim_{H\in\mathscr{F}}\mathrm{Rep}_{U(n)}(H)\quad\text{and}\quad\mathbf{A}=(\alpha_H)\in\varprojlim_{H\in\mathscr{F}}\mathrm{Rep}_{\Sigma_n}(H),$$

and these satisfy the hypotheses of Theorem 2.6 (with $\Gamma=\Sigma_n$). So by the theorem, there is some k, and a G-vector bundle $E\to X$, such that for each $x\in X$, $E|_x\cong V^k_{G_x}$ is a multiple of the regular representation of G_x. □

An H-representation V will be called *p'-free* if for any subgroup $K\subseteq H$ of order prime to p, $V|K$ is a multiple of the regular representation of K. This is equivalent to the condition that the character of any element $h\in H$ not of p-power order is zero. The next result, a second consequence of Theorem 2.6, is the main technical ingredient in our extension of the completion theorem of

Atiyah and Segal from finite groups and compact spaces to arbitrary discrete groups and finite proper G-CW-complexes.

Corollary 2.**8.** *Assume that G is discrete, and let X be any finite dimensional proper G-CW-complex whose isotropy subgroups have bounded order. Then for any prime p, there is a G-vector bundle $E \to X$ of dimension prime to p, such that for each $x \in X$, $E|_x$ is p'-free as a G_x-representation.*

Proof. Let $\mathscr{F}$ be the family of isotropy subgroups in X, and let m be the least common multiple of the $|H|$ for $H \in \mathscr{F}$. For each $H \in \mathscr{F}$, let $\alpha_H : H \to \Sigma_m$ be the homomorphism corresponding to any free action of H on $\{1, \dots, m\}$. The α_H clearly form an element

$$\mathbf{A} = (\alpha_H) \in \varprojlim_{H \in \mathscr{F}} \mathrm{Rep}_{\Sigma_m}(H).$$

Set $n = |\Sigma_m / \mathrm{Syl}_p(\Sigma_m)|$, let $\rho : \Sigma_m \to U(n)$ be the corresponding permutation representation, and (for each H) let V_H be the n-dimensional representation defined by $\rho \circ \alpha_H$. By Theorem 2.6, there is $k > 0$ and a G-bundle $E \to X$, such that the fiber $E|_x$ over any point $x \in X$ is isomorphic to $(V_{G_x})^{\otimes k}$. This bundle has dimension n^k, which is prime to p. Furthermore, for each $H \in \mathscr{F}$, and each subgroup $K \subseteq H$ of order prime to p, $(V_H)|_K$ is a free $\mathbb{C}[K]$-module by construction, and so the same holds for $(V_H^{\otimes k})|_K$. In other words, $E|_x$ is p'-free as a G_x-representation for each x, and so E has all of the required properties. This finishes the proof of Corollary 2.8. □

In view of Theorem 2.6, the following question arises. Let X be a G-CW-complex. Given a compatible family $\{V_H\}$ of representations of the isotropy subgroups of X, is there a G-vector bundle $E \to X$ such that $E|_x \cong V_{G_x}$ as G_x-representations for each $x \in X$? Here, "compatible" means that if $\alpha(K) \subseteq H$, where α is an inner automorphism of G, then $(\alpha^* V_H)|_K \cong V_K$. This question can also be posed more generally, requiring different representations on different components of fixed point sets.

It is in fact easy to find counterexamples to this question, even in the case where G is finite. Fix a finite group G and a normal subgroup $H \lhd G$, and set $I_H = \mathrm{Ker}[R(G) \xrightarrow{\mathrm{res}} R(H)]$. By a theorem of Jackowski [11, Theorem 5.1 and Example 5.5], the pro-rings $\{K_G(E(G/H)^{(n)})\}_{n \geqslant 1}$ and $\{R(G)/(I_H)^n\}_{n \geqslant 1}$ are isomorphic. In particular, for n sufficiently large, the fibers of any G-vector bundle over $E(G/H)^{(n)}$, considered as H-representations, can always be extended to virtual G-representations. On the other hand, any G/H-invariant H-representation V_H defines a "compatible family" of representations of the isotropy subgroups of $E(G/H)$. It is not hard to find examples of G and H where $\mathrm{Im}[R(G) \to R(H)] \subsetneqq R(H)^{G/H}$, and hence of a compatible family which cannot be the fibers of a G-vector bundle over the finite G-CW-complex $E(G/H)^{(n)}$.

What we really would like to find is an example of an infinite discrete group G, such that $E_{\mathscr{FIN}}(G)$ has the homotopy type of a finite G-CW-complex, and for which not every compatible family $\{V_H\}$ of representations of the finite subgroups can be realized as a G-vector bundle over $E_{\mathscr{FIN}}(G)$ (not even stably). Presumably such examples exist, but we have so far been unable to find any.

3. Equivariant K-theory for finite proper G-CW-complexes

The main result in this section is that when G is discrete, G-vector bundles define a $\mathbb{Z}/2$-graded multiplicative cohomology theory $K_G^*(-)$ on the category of finite proper G-CW-complexes. This is summarized in Theorem 3.2 below.

The assumption here that G is discrete is essential, even in the case of finite proper CW-complexes. So this will be assumed throughout most of the section. The problems arising in the case of positive dimensional Lie groups will be discussed in Section 5 below.

The usual way to define $K_G(X, A)$, when G is finite, is as the reduced K-theory of the mapping cone of the inclusion of A in X (cf. [3] or [18]). That approach is not possible here, since the mapping cone of a map of proper G-CW-complexes has a G-fixed point, and hence is not proper if G is not compact. For the same reason, we are unable to use suspensions in this situation to define the groups $K_G^{-n}(X, A)$. Instead, we make the following definitions:

Definition 3.1. For any Lie group G and any proper G-CW-complex X, let $\mathbb{K}_G(X) = \mathbb{K}_G^0(X)$ be the Grothendieck group of the monoid of isomorphism classes of G-vector bundles over X. Define $\mathbb{K}_G^{-n}(X)$, for all $n > 0$, by setting

$$\mathbb{K}_G^{-n}(X) = \operatorname{Ker}[\mathbb{K}_G(X \times S^n) \xrightarrow{\text{incl}^*} \mathbb{K}_G(X)].$$

For any proper G-CW-pair (X, A), set

$$\mathbb{K}_G^{-n}(X, A) = \operatorname{Ker}[\mathbb{K}_G^{-n}(X \cup_A X) \xrightarrow{i_2^*} \mathbb{K}_G^{-n}(X)].$$

When G is discrete and (X, A) is a *finite* proper G-CW-pair, write

$$K_G(X, A) = \mathbb{K}_G(X, A) \quad \text{and} \quad K_G^{-n}(X, A) = \mathbb{K}_G^{-n}(X, A).$$

The pullback construction makes $\mathbb{K}_G^{-n}(-)$ and $K_G^{-n}(-)$ into contravariant functors on the categories of proper, or finite proper, G-CW-pairs.

Note that we get a natural isomorphism

$$\mathrm{pr}_X^* \oplus i : K_G^0(X) \oplus K_G^{-n}(X) \xrightarrow{\cong} K_G^0(X \times S^n),$$

where i is the inclusion and pr_X the projection. We can now state the main theorem in this section.

Theorem 3.2. *For any discrete group G, the groups $K_G^{-n}(X, A)$ extend to a $\mathbb{Z}/2$-graded multiplicative equivariant cohomology theory on the category of finite proper G-CW-pairs. In particular, $K_G^*(-)$ is* *a homotopy invariant contravariant functor, satisfies excision, and there is an exact sequence*

$$\begin{tikzcd} K_G^0(X,A) \arrow[r] & K_G^0(X) \arrow[r] & K_G^0(A) \arrow[d, "\delta^0"] \\ K_G^1(A) \arrow[u, "\delta^1"] & K_G^1(X) \arrow[l] \arrow[r] & K_G^1(X,A) \end{tikzcd}$$

of $K_G^*(X)$-modules for any finite proper G-CW-pair (X,A). For any pushout $X = X_1 \cup_A X_2$ where (X_1, A) is a finite proper G-CW-pair, all maps in the induced Mayer–Vietoris sequence are $K_G^*(X)$-linear. For finite subgroups $H \subseteq G$, there are natural isomorphisms $K_G^0(G/H) \cong R(H)$, and $K_G^1(G/H) = 0$. If G is finite and X is compact, this construction agrees with the classicial definition.

The proof of Theorem 3.2 will occupy most of the rest of the section. We first show some of the properties of $\mathbb{K}_G^*(-)$ which hold for any Lie group G and any proper G-CW-complex X, beginning with homotopy invariance.

Lemma 3.3 (Homotopy invariance). *Let G be a Lie group. If $f_0, f_1 : (X,A) \to (Y,B)$ are G-homotopic G-maps between proper G-CW-pairs, then*

$$f_0^* = f_1^* : \mathbb{K}_G^{-n}(Y,B) \to \mathbb{K}_G^{-n}(X,A)$$

for all $n \geqslant 0$.

Proof. When $n = 0$ and $A = B = \emptyset$, this follows immediately from Theorem 1.2. The general case then follows from the definition of $\mathbb{K}_G^{-n}(X,A)$. □

We next note the following relation between equivariant K-theory for different groups.

Lemma 3.4 (Induction). *Let $H \subseteq G$ be an inclusion of Lie groups and let (X,A) be a proper H-CW-pair. Then $G \times_H (X,A)$ is a proper G-CW-pair, and there are isomorphisms*

$$i_H^G : \mathbb{K}_H^{-n}(X,A) \xrightarrow{\cong} \mathbb{K}_G^{-n}(G \times_H (X,A))$$

(*for all $n \geqslant 0$*) *defined by sending $[E]$ to $[G \times_H E]$.*

Proof. This is clear when $A = \emptyset$. When $A \neq \emptyset$, it follows since

$$G \times_H (X \cup_A X) \cong (G \times_H X) \cup_{G \times_H A} (G \times_H X). \qquad \square$$

The next two lemmas are also very elementary.

Lemma 3.5 (Free quotients). *Let G be a Lie group. Let (X,A) be a proper G-CW-pair for which the normal subgroup $H \lhd G$ acts freely on X. Then the projection* pr: $X \to X/H$ *induces an isomorphism*

$$\mathrm{pr}^*: \mathbb{K}_{G/H}^{-n}(X/H, A/H) \xrightarrow{\cong} \mathbb{K}_G^{-n}(X,A).$$

Proof. This is quickly reduced to the case $n = 0$ and $A = \emptyset$, for which the inverse of the above map is defined by sending $[E] \in \mathbb{K}_G(X)$ to $[E/H] \in \mathbb{K}_{G/H}(X/H)$. $\square$

Lemma 3.6. *Let G be a Lie group, and let (X, A) be a proper G-CW-pair. Suppose that $X = \coprod_{i\in I} X_i$, the disjoint union of open G-invariant subspaces X_i (for any index set I), and set $A_i = A \cap X_i$. Then there is a natural isomorphism*

$$\mathbb{K}_G^{-n}(X, A) \xrightarrow{\cong} \prod_{i\in I} \mathbb{K}_G^{-n}(X_i, A_i)$$

induced by the inclusions of the components.

We now assume, throughout (most of) the rest of the section, that G is a discrete group. When proving excision and constructing the exact sequences for equivariant K-theory, we need to know when a G-vector bundle E_0 over a G-subspace $A \subseteq X$ can be embedded as a summand of some bundle E over X. Suppose for simplicity that X/G is compact. If G is a finite group (or a compact Lie group), then it is easy to find E, since any G-vector bundle over a compact G-CW-complex is a summand of some product bundle. This is no longer the case when G is not compact, and instead of product bundles we will construct E using the bundles constructed in Corollary 2.7.

Note that the following lemma does *not* hold when G is a noncompact Lie group of positive dimension, even in the special case where X is finite. In fact, Phillips [15, Section 9] has shown that in this situation, equivariant K-theory defined via (finite-dimensional) G-vector bundles need not be an equivariant cohomology theory. We will discuss this in more detail in Section 5.

Lemma 3.7. *Assume G is discrete, let $\varphi : X \to Y$ be an equivariant map between finite proper G-CW-complexes, and let $E' \to X$ be a G-vector bundle. Then there is a G-vector bundle $E \to Y$ such that E' is a summand of φ^*E.*

Proof. Let m be the maximum dimension of any fiber of E'. By Corollary 2.7, there is a G-vector bundle $F \to Y$ such that each fiber $F|_y$ is a multiple of the regular G_y-representation. After possibly replacing F by some iterated direct sum with itself, we can assume that for each $x \in X$, $(\varphi^*F)|_x \cong F|_{\varphi(x)}$ contains at least m copies of the regular representation of G_x; and hence that there is a G_x-linear injection of $E'|_x$ into $(\varphi^*F)|_x$. This extends to a monomorphism of G-vector bundles from $E'|_{Gx}$ into $(\varphi^*F)|_{Gx}$, which by Lemma 1.3 extends to a bundle map $f_x : E' \to \varphi^*F$ covering the identity on X. In particular, f_x is a monomorphism over some open G-invariant neighborhood U_x of Gx in X.

Since X/G is compact, we can choose $x_1, \dots, x_n \in X$ such that X is covered by the sets $U_{x_1}, \dots, U_{x_n}$. The sum of the f_{x_i} is then a monomorphism f: $E' \to \varphi^*(F^n)$ of bundles covering the identity on X. The image of f is a G-invariant subbundle of $\varphi^*(F^n)$ (cf. [3, Lemma 1.3.1]). And via a Hermitian metric on F, it is seen to be a G-vector bundle summand. $\square$

The Mayer–Vietoris exact sequence follows as a consequence of Lemma 3.7.

Lemma 3.8 (Mayer–Vietoris sequence). *Assume G is discrete. Let*

$$\begin{array}{ccc} A & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

be a pushout square of finite proper G-CW-complexes, where i_1 and j_2 are inclusions of subcomplexes. Then there is a natural exact sequence, infinite to the left

$$\ldots \xrightarrow{d^{-n-1}} K_G^{-n}(X) \xrightarrow{j_1^* \oplus j_2^*} K_G^{-n}(X_1) \oplus K_G^{-n}(X_2) \xrightarrow{i_1^* - i_2^*} K_G^{-n}(A) \xrightarrow{d^{-n}}$$

$$\ldots \to K_G^{-1}(A) \xrightarrow{d^{-1}} K_G^0(X) \xrightarrow{j_1^* \oplus j_2^*} K_G^0(X_1) \oplus K_G^0(X_2) \xrightarrow{i_1^* - i_2^*} K_G^0(A). \tag{1}$$

Proof. We first show that the sequence

$$K_G(X) \xrightarrow{j_1^* \oplus j_2^*} K_G(X_1) \oplus K_G(X_2) \xrightarrow{i_1^* - i_2^*} K_G(A) \tag{2}$$

is exact, and hence that sequence (1) is exact at $K_G^{-n}(X_1) \oplus K_G^{-n}(X_2)$ for all n. Clearly the composite in (2) is zero. So fix an element $(\alpha_1, \alpha_2) \in \operatorname{Ker}(i_1^* - i_2^*)$. By Lemma 3.7, we can add an element of the form $([j_1^* E'], [j_2^* E'])$ for some G-vector bundle $E' \to X$, and arrange that $\alpha_1 = [E_1]$ and $\alpha_2 = [E_2]$ for some pair of G-vector bundles $E_k \to X_k$. Then $i_1^* E_1$ and $i_2^* E_2$ are stably isomorphic, and after adding the restrictions of another bundle over X (Lemma 3.7 again), we can arrange that $i_1^* E_1 \cong i_2^* E_2$. Lemma 1.5 now applies to show that there is a G-vector bundle E over X such that $j_k^* E \cong E_k$ ($k = 1,2$), and hence that $([E_1],[E_2]) \in \operatorname{Im}(j_1^* \oplus j_2^*)$.

Assume now that A is a retract of X_1. We claim that in this case,

$$\operatorname{Ker}[K_G(X) \xrightarrow{j_2^*} K_G(X_2)] \underset{\cong}{\xrightarrow{j_1^*}} \operatorname{Ker}[K_G(X_1) \xrightarrow{i_1^*} K_G(A)] \tag{3}$$

is an isomorphism. It is surjective by the exactness of (2). So fix an element $[E] - [E'] \in \operatorname{Ker}(j_1^* \oplus j_2^*)$. To simplify the notation, we write $E|_{X_1} = j_1^* E$, $E|_A = i_2^* j_2^* E$, etc. (But we are *not* assuming that j_1 and i_2 are injective.) Let $p_1 : X_1 \to A$ be a retraction, and let $p : X \to X_2$ be its extension to X. Using Lemma 3.7, we can arrange that $E|_{X_k} \cong E'|_{X_k}$ for $k = 1,2$. Upon applying Lemma 3.7 to the retraction $p : X \to X_2$, we obtain a G-vector bundle $F' \to X_2$ such that E' is a summand of $p^* F'$. Upon stabilizing again, we can assume that $E' \cong p^* F'$, and hence that $F' \cong E'|_{X_2}$ and $E'|_{X_1} \cong p_1^*(F'|_A) \cong p_1^*(E'|_A)$. Fix isomorphisms $\psi_k : E|_{X_k} \to E'|_{X_k}$ covering Id_{X_k}. The automorphism $(\psi_2|_A) \circ (\psi_1|_A)^{-1}$ of $E'|_A$ pulls back, under p_1, to an automorphism φ of $E'|_{X_1}$; and by replacing ψ_1 by $\varphi \circ \psi_1$ we can arrange that $\psi_1|_A = \psi_2|_A$. Then $\psi_1 \cup \psi_2$ is an isomorphism from E to E', and this proves (3).

We now return to the general case. For each $n \geqslant 1$,

$$K_G^{-n}(A) = \mathrm{Ker}\,[K_G(A \times S^n) \to K_G(A)]$$

$$\cong \mathrm{Ker}\,[K_G(X \cup_{A \times \bullet} (A \times S^n)) \xrightarrow{\mathrm{incl}^*} K_G(X)] \quad \text{(by (3))}$$

$$\cong \mathrm{Ker}\,[K_G((X_1 \times D^n) \cup_{A \times S^{n-1}} (X_2 \times D^n)) \xrightarrow{(-,\bullet)^*} K_G(X)], \quad \text{(hty. invar.)}$$

the last step since $((X_1 \times \bullet) \cup (A \times D^n))$ is a strong deformation retract of $X_1 \times D^n$. Define $d^{-n}\colon K_G^{-n}(A) \to K_G^{-n+1}(X)$ to be the homomorphism which makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & K_G^{-n}(A) \arrow[r] \arrow[d, "d^{-n}"'] & K_G((X_1 \times D^n) \cup_{A \times S^{n-1}} (X_2 \times D_n)) \arrow[r, "(-,\bullet)^*"] \arrow[d, "\mathrm{incl}^*"'] & K_G(X) \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\
0 \arrow[r] & K_G^{-n+1}(X) \arrow[r] & K_G(X \times S^{n-1}) \arrow[r, "(-,\bullet)^*"] & K_G(X) \arrow[r] & 0
\end{tikzcd}$$

We have already shown that sequence (1) is exact at $K_G^{-n}(X_1) \oplus K_G^{-n}(X_2)$ for all n. To see its exactness at $K_G^{-n+1}(X)$ and $K_G^{-n}(A)$ (for any $n \geqslant 1$), apply the exactness of (2) to the following split inclusion of pushout squares:

$$\begin{tikzcd}
X_1 \amalg X_2 \arrow[r] \arrow[d] & X_1 \amalg X_2 \arrow[d] & & (X_1 \amalg X_2) \times S^{n-1} \arrow[r] \arrow[d] & (X_1 \times D^n) \amalg (X_2 \times D^n) \arrow[d] \\
X \arrow[r] \arrow[d] & X \arrow[d] \arrow[rr, "\mathrm{incl}"] & & X \times S^{n-1} \arrow[r] \arrow[d] & (X_1 \times D^n) \cup_{A \times S^{n-1}} (X_2 \times D^n) \arrow[d] \\
X \arrow[r] & X & & X \arrow[r] & (X_1 \times S^n) \cup_{A \times \bullet} (X_2 \times S^n)
\end{tikzcd}$$

The upper pair of squares induces a split surjection of exact sequences whose kernel yields the exactness of (1) at $K_G^{-n+1}(X)$. And since

$$\mathrm{Ker}\,[K_G((X_1 \times S^n) \cup_{A \times \bullet} (X_2 \times S^n)) \to K_G(X)]$$

$$\cong \mathrm{Ker}\,[K_G((X_1 \times S^n) \amalg (X_2 \times S^n)) \to K_G(X_1 \amalg X_2)] \cong K_G^{-n}(X_1) \oplus K_G^{-n}(X_2)$$

by (3), the lower pair of squares induces a split surjection of exact sequences whose kernel yields the exactness of (1) at $K_G^{-n}(A)$. □

Excision, and the long exact sequence for a pair, follow as immediate consequences of the Mayer–Vietoris sequence.

Lemma 3.9 (Excision). *Assume G is discrete. Let*

$$\varphi : (X, A) \to (Y, B)$$

be a map of finite proper G-CW-pairs, such that $Y \cong B \cup_{\varphi|A} X$. *Then*

$$\varphi^*: K_G^{-n}(Y,B) \to K_G^{-n}(X,A)$$

is an isomorphism for all $n \geqslant 0$.

Proof. For each n, the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X \cup_A X \arrow[r] & Y \cup_B Y \end{tikzcd}$$

is a pushout, and X is a retract of $X \cup_A X$. So its Mayer–Vietoris sequence splits into short exact sequences

$$0 \to K_G^{-n}(Y \cup_B Y) \to K_G^{-n}(X \cup_A X) \oplus K_G^{-n}(Y) \to K_G^{-n}(X) \to 0.$$

And hence $K_G^{-n}(Y,B) \cong K_G^{-n}(X,A)$. $\square$

Lemma 3.10 (Exactness)**.** *Assume G is discrete, and let* (X,A) *be a finite proper G-CW-pair. Then the following sequence, extending infinitely far to the left, is natural and exact*:

$$\ldots \xrightarrow{\delta^{-n-1}} K_G^{-n}(X,A) \xrightarrow{i^*} K_G^{-n}(X) \xrightarrow{j^*} K_G^{-n}(A) \xrightarrow{\delta^{-n}} K_G^{-n+1}(X,A) \xrightarrow{i^*}$$

$$\ldots \xrightarrow{\delta^{-1}} K_G^0(X,A) \xrightarrow{i^*} K_G^0(X) \xrightarrow{j^*} K_G^0(A).$$

Proof. This follows immediately from the Mayer–Vietoris sequence for the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "i_2"'] \\ X \arrow[r, "i_1"] & X \cup_A X \end{tikzcd}$$

In the nonequivariant case, $\mathbb{K}(X) \cong K(X)$ for any finite dimensional CW-complex X: since any map $X \to BU$ factors through some $BU(n)$. The following example shows that this is no longer true in the equivariant case, even for actions of finite groups: the Mayer–Vietoris sequence need not be exact in this situation.

Example 3.11. Fix any finite group $G \neq 1$. Define $X = (G \times \mathbb{R})/\sim$, where $(g,n) \sim (1,n)$ for any $g \in G$ and any $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, set $A_n = (G \times [n-\frac{1}{2}, n+\frac{1}{2}])/(G \times \{n\})$. Set $X_1 = \coprod_{n \in \mathbb{Z}} A_{2n}$, $X_2 = \coprod_{n \in \mathbb{Z}} A_{2n+1}$, and $X_0 = X_1 \cap X_2$. Let $i_k : X_k \to X$ and $j_k : X_0 \to X_k$ $(k = 1,2)$

denote the inclusions. Then the sequence

$$\mathbb{K}_G(X) \xrightarrow{(i_1^*, i_2^*)} \mathbb{K}_G(X_1) \oplus \mathbb{K}_G(X_2) \xrightarrow{j_1^* - j_2^*} \mathbb{K}_G(X_0)$$

is not exact.

Proof. For each n, $K_G(A_n) \cong R(G)$ (each A_n is equivariantly contractible); and the kernel of the restriction map $K_G(A_n) \to K_G(A_n \cap X_0)$ is (under this identification) the augmentation ideal $IR(G)$. Choose representations V_n, W_n (all $n \in \mathbb{Z}$) such that $\dim(V_n) = \dim(W_n)$, $\mathrm{Hom}(V_n, W_n) = 0$, and $\{\dim(V_n)\}$ is unbounded. Then the element

$$(\{[V_{2n} \times A_{2n}] - [W_{2n} \times A_{2n}]\}_{n \in \mathbb{Z}}, \{[V_{2n+1} \times A_{2n+1}] - [W_{2n+1} \times A_{2n+1}]\}_{n \in \mathbb{Z}})$$

lies in $\mathrm{Ker}(j_1^* - j_2^*)$, but not in $\mathrm{Im}(i_1^*, i_2^*)$. $\square$

We now consider products on $K_G^*(X)$ and on $K_G^*(X, A)$. For any proper G-CW-complex X, tensor product of G-vector bundles makes $\mathbb{K}_G(X)$ into a commutative ring, and all induced maps $f^*: \mathbb{K}_G(Y) \to \mathbb{K}_G(X)$ are ring homomorphisms. For each $n, m \geqslant 0$,

$$\begin{aligned} K_G^{-n-m}(X) &\cong \mathrm{Ker}[K_G^{-m}(X \times S^n) \to K_G^{-m}(X)] \\ &= \mathrm{Ker}[K_G(X \times S^n \times S^m) \to K_G(X \times S^n) \oplus K_G(X \times S^m)], \end{aligned}$$

where the first isomorphism follows from the usual Mayer–Vietoris sequences. Hence

$$K_G(X \times S^n) \otimes K_G(X \times S^m) \xrightarrow{\mathrm{mult} \circ (p_1^* \otimes p_2^*)} K_G(X \times S^n \times S^m)$$

restricts to a homomorphism

$$K_G^{-n}(X) \otimes K_G^{-m}(X) \to K_G^{-m-n}(X).$$

By applying the above definition with $n = 0$ or $m = 0$, the multiplicative identity for $K_G(X)$ is seen to be an identity for $K_G^*(X)$. Associativity of the graded product is clear, and graded commutativity follows upon showing (using a Mayer–Vietoris sequence) that composition with a degree -1 map $S^n \to S^n$ induces multiplication by -1 on $K^{-n}(X)$. This product thus makes $K_G^*(X)$ into a ring. Clearly, $f^*: K_G^*(Y) \to K_G^*(X)$ is a ring homomorphism for any G-map $f: X \to Y$.

For a finite proper G-CW-pair (X, A), $K_G^*(X \cup_A X) \to K_G^*(X)$ is a split surjection and ring homomorphism (and split by a ring homomorphism), and so its kernel is a $K_G^*(X)$-module. For any $X = X_1 \cup_A X_2$, where (X_1, A) is a finite proper G-CW-pair, the boundary map in the corresponding Mayer–Vietoris sequence is $K_G^*(X)$-linear, since it is defined via a certain map between spaces which commutes with their (split) projections onto X. And hence the boundary maps in the long exact sequence for a pair (X, A) are $K_G^*(X)$-linear, since they are defined to be the boundary maps of a certain Mayer–Vietoris sequence all of whose spaces map to X.

It remains to prove Bott periodicity in this situation. Recall that $\tilde{K}(S^2) \overset{\mathrm{def}}{=} \mathrm{Ker}[K(S^2) \to K(\mathrm{pt})] \cong \mathbb{Z}$, and is generated by the *Bott element* $B \in \tilde{K}(S^2)$: the element $[S^2 \times \mathbb{C}] - [H] \in \tilde{K}(S^2)$, where H is the canonical complex line bundle over $S^2 = \mathbb{CP}^1$. For any finite proper G-CW-complex X, there is an obvious pairing

$$K_G^{-n}(X) \otimes \tilde{K}(S^2) \xrightarrow{\otimes} \operatorname{Ker}[K_G^{-n}(X \times S^2) \to K_G^{-n}(X \times \mathrm{pt})] \cong K_G^{-n-2}(X),$$

induced by (external) tensor product of bundles. Evaluation at the Bott element now defines a homomorphism

$$b = b(X): K_G^{-n}(X) \to K_G^{-n-2}(X),$$

which by construction is natural in X. And this is then extends to a homomorphism

$$b = b^*(X, A): K_G^{-n}(X, A) \to K_G^{-n-2}(X, A)$$

defined for any finite proper G-CW-pair (X, A) and all $n \geqslant 0$.

Theorem 3.12 (Equivariant Bott periodicity). *Assume G is discrete. Then the Bott homomorphism*

$$b = b(X, A): K_G^{-n}(X, A) \to K^{-n-2}(X, A)$$

is an isomorphism for any discrete group G and any finite proper G-CW-pair (X, A) (*and all* $n \geqslant 0$).

Proof. Assume first that $X = Y \cup_\varphi (G/H \times D^m)$, where $H \subseteq G$ is finite and $\varphi: G/H \times S^{m-1} \to Y$ is a G-map; and assume inductively that $b(Y)$ is an isomorphism. Since

$$K_G^{-n}(G/H \times S^{m-1}) \cong K_H^{-n}(S^{m-1}) \quad \text{and} \quad K_G^{-n}(G/H \times D^m) \cong K_H^{-n}(D^m),$$

the Bott homomorphisms $b(G/H \times S^{m-1})$ and $b(G/H \times D^m)$ are isomorphisms by the equivariant Bott periodicity theorem for actions of finite groups [4, Theorem 4.3]. The Bott map is natural, and compatible with the various boundary operators in the Mayer–Vietoris sequence (in nonpositive degrees) for Y, X, $G/H \times S^{m-1}$, and $G/H \times D^m$; and so $b(X)$ is an isomorphism by the 5-lemma. The proof that $b(X, A)$ is an isomorphism for an arbitrary proper finite G-CW-pair (X, A) now follows immediately from the definitions of the relative groups. □

We are now ready to prove the main theorem. Define, for all $n \in \mathbb{Z}$,

$$K_G^n(X, A) = \begin{cases} K_G^0(X, A) & \text{if } n \text{ is even,} \\ K_G^{-1}(X, A) & \text{if } n \text{ is odd.} \end{cases}$$

For any finite proper G-CW-pair (X, A), define the boundary operator $\delta^n: K_G^n(A) \to K_G^{n+1}(X, A)$ to be $\delta: K_G^{-1}(A) \to K_G^0(X, A)$ if n is odd, and to be the composite

$$K_G^0(A) \underset{\cong}{\xrightarrow{b}} K_G^{-2}(A) \xrightarrow{\delta^{-2}} K_G^{-1}(X, A)$$

if n is even.

Proof of Theorem 3.2. We have already proven excision (Lemma 3.9) and homotopy invariance (Lemma 3.3). The long exact sequence of a pair follows from that in negative degrees (Lemma 3.10), and the fact that the Bott map is natural and commutes with the boundary operators δ^{-n}. The same holds for the product structure which comes from that on $K_G^{-n}(X, A)$. For any $X = X_1 \cup_A X_2$, the boundary map in the corresponding Mayer–Vietoris sequence is $K_G^*(X)$-linear, since it is defined via a certain map between spaces which commutes with their (split) projections onto X. And hence the boundary maps in the long exact sequence for a pair (X, A) are $K_G^*(X)$-linear, since they are defined to be the boundary maps of a certain Mayer–Vietoris sequence all of whose spaces map to X. The other claims are immediate. $\square$

We next consider the Thom isomorphism theorem for proper actions of infinite discrete groups. This first requires a slight detour. The Thom class of a G-vector bundle E is an element in $K_G(D(E), S(E))$, where $S(E) \subseteq D(E)$ denote the unit sphere and disk bundles in E (with respect to some G-invariant metric). This is most easily defined in terms of a chain complex of vector bundles over $D(E)$, and we must first explain how such a chain complex determines an element in K-theory.

A *G-vector bundle chain complex* over a proper G-CW-pair (X, A) is a finite dimensional chain complex (C_*, c_*) of G-vector bundles over X whose restriction to A is acyclic. In other words, for some $N > 0$,

$$0 \to C_N \xrightarrow{c_N} C_{N-1} \xrightarrow{c_{N-1}} \cdots \xrightarrow{c_3} C_2 \xrightarrow{c_2} C_1 \xrightarrow{c_1} C_0 \to 0$$

is a sequence of G-vector bundles and bundle maps, such that $c_{n-1} \circ c_n = 0$ for all n, and such that restriction to the fibers over any $x \in A$ is exact. When G is compact, the monoid of G-vector bundle chain complexes over (X, A), modulo an appropriate submonoid, is isomorphic to $K_G(X, A)$ by a theorem of Segal [18, Proposition 3.1]. In a later paper, we will prove this in our present setting, for proper actions of infinite discrete groups. But for now, all we need to know is that any such complex defines an element of $K_G(X, A)$ in a natural (functorial) way.

Fix a G-vector bundle chain complex (C_*, c_*) over (X, A). For each n, set $C'_n = \mathrm{Im}(c_{n+1}|_A) = \mathrm{Ker}(c_n|_A)$. Each $C'_n \subseteq C_n$ is a G-invariant subbundle: this follows by induction on n, since the kernel of the surjection $C_n|_A \xrightarrow{c_n} C'_{n-1}$ is a subbundle (cf. [12, Theorems 5.13 and 6.3]). Let $C''_n \subseteq C_n|_A$ be any G-invariant complementary bundle to C'_n; defined, for example, using a G-invariant Hermitian metric on C_n. Thus, for each n, c_n sends C''_n isomorphically to C'_{n-1}. Set $C_{\mathrm{odd}} = \bigoplus_{n \in \mathbb{Z}} C_{2n+1}$ and $C_{\mathrm{ev}} = \bigoplus_{n \in \mathbb{Z}} C_{2n}$, let $f_C : C_{\mathrm{odd}}|_A \to C_{\mathrm{ev}}|_A$ be the sum of the isomorphisms

$$C'_{2n+1} \xrightarrow[\cong]{(c_{2n+2})^{-1}} C''_{2n+2} \quad \text{and} \quad C''_{2n+1} \xrightarrow[\cong]{c_{2n+1}} C'_{2n}.$$

Finally, define

$$[C_*, c_*] = [C_{\mathrm{odd}} \cup_{f_C} C_{\mathrm{ev}}] - [C_{\mathrm{ev}} \cup_{\mathrm{Id}} C_{\mathrm{ev}}] \in \mathrm{Ker}[K_G(X \cup_A X) \xrightarrow{i_2^*} K_G(X)] = K_G(X, A).$$

This is independent of the choice of C''_n, since there is an affine structure on the space of all complementary bundles (and hence a homotopy between any two of them).

Now let $p : E \to X$ be an n-dimensional G-vector bundle over a proper G-CW-complex X, and set $p_D = p|D(E)$. Consider the cochain complex of G-vector bundles $(\Lambda^k p_D^* E, \delta)$ over $(D(E), S(E))$, which over any $v \in D(E)$ takes the form

$$0 \to \Lambda^0 E_{p(v)} \xrightarrow{\wedge v} \Lambda^1 E_{p(v)} \xrightarrow{\wedge v} \Lambda^2 E_{p(v)} \xrightarrow{\wedge v} \dots \xrightarrow{\wedge v} \Lambda^n E_{p(v)} \to 0.$$

Here, $\wedge v$ denotes the exterior product with the element $v \in E_{p(v)}$. One easily checks that this sequence is exact for all v not in the zero section of E.

There is a technical problem here: $D(E)$ and $S(E)$ do not have natural structures as G-CW-complexes, and so $K_G^*(D(E), S(E))$ is not defined in Definition 3.1. It is not difficult, however, to modify the definitions (and the proof of Theorem 3.2) to include this case: either by showing that $(D(E), S(E))$ has the G-homotopy type of a finite proper G-CW-pair, or via a more general definition of equivariant cellular complexes, or by constructing $K_G^*(-)$ as an equivariant cohomology theory for all proper G-spaces with compact quotient. This last approach will be taken by the authors in a later, more technical, paper. For now, we just assume that equivariant K-theory has been defined, in some way or other, for disk and sphere bundles of G-vector bundles over finite proper G-CW-complexes.

Definition 3.13. For any G-vector bundle E over X, the *Thom class* of E is the element

$$\lambda_E \in K_G^0(D(E), S(E)),$$

defined to be the class of the cochain complex $(\Lambda^*(p_D^* E), \delta)$ over $(D(E), S(E))$ as defined above. The *Thom homomorphism* is the composite

$$T_E : K_G^*(X) \underset{\cong}{\xrightarrow{p_D^*}} K_G^*(D(E)) \xrightarrow{\cdot \lambda_E} K_G^*(D(E), S(E)),$$

where the second map is multiplication with the Thom class.

Theorem 3.14 (Thom isomorphism theorem). *Assume G is discrete. Then for any G-vector bundle $p : E \to X$ over a finite proper G-CW-complex X, the Thom homomorphism*

$$T_E : K_G^*(X) \xrightarrow{\cong} K_G^*(D(E), S(E))$$

is an isomorphism.

Proof. Assume first that $X = G/H \times Y$, where $Y = S^{n-1}$ or D^n, and where $E|_Y \cong V \times Y$ for some H-representation V. Then

$$K_G^n(X) \cong K_G^n(G/H \times Y) \cong K_H^n(Y);$$

and

$$\begin{aligned} K_G^n(D(E), S(E)) &\cong K_G^n(G \times_H (D(V) \times Y), G \times_H (S(V) \times Y)) \\ &\cong K_H^n(D(V) \times Y, S(V) \times Y) \end{aligned}$$

(the last step by Lemma 3.4). So in this case, T_E is an isomorphism by the Thom isomorphism theorem for actions of finite groups [4, Theorem 4.3].

Now assume that $X = Y \cup_\varphi (G/H \times D^n)$, where H is finite, $\varphi : G/H \times S^{n-1} \to Y$ is a G-map, and $T_{E|_Y}$ is an isomorphism. There is a relative Mayer–Vietoris sequence involving the groups $K_G^*(D(E|_A), S(E|_A))$ for $A = X$, Y, $G/H \times D^n$, and $G/H \times S^{n-1}$: this follows immediately from the usual Mayer–Vietoris sequence and our definition of the relative groups. Since all maps in both Mayer–Vietoris sequences — for X and for $(D(E), S(E))$ — are $K_G^*(X)$-linear (Lemma 3.8), they commute with the Thom homomorphisms. So T_E is an isomorphism by the five-lemma. $\square$

So far, we have worked entirely with complex K-theory. To finish the section, we note that the results of this section all hold in the real case as well. Define $KO_G(X)$, for any discrete G and any finite proper G-CW-complex X, to be the Grothendieck group of real G-vector bundles over X, and extend this to a functor $KO_G^{-n}(X, A)$ as in Definition 3.1. The key to proving the exactness properties of K_G was Lemma 3.7 (given $\varphi : X \to Y$, any bundle over X is contained in the pullback of a bundle over Y); and this automatically holds in the real case using the forgetful and induction functors between complex and real G-vector bundles. Bott periodicity still holds, but with period eight: just as in the complex case, this reduces to the Bott periodicity theorem for $KO_G(-)$ when G is a finite group, which was shown in [4, Theorem 6.1]. We thus get:

Theorem 3.15. *For any discrete group G, the groups $KO_G^{-n}(X, A)$ extend to a multiplicative equivariant cohomology theory on the category of finite proper G-CW-pairs. In particular, $KO_G^*(-)$ is a homotopy invariant contravariant functor, satisfies excision, and there are exact sequences*

$$\cdots \to KO_G^n(X, A) \to KO_G^n(X) \to KO_G^n(A) \to KO_G^{n+1}(X, A) \to \cdots .$$

There are natural Bott periodicity isomorphisms $KO_G^n(X) \underset{\cong}{\xrightarrow{b(X)}} KO_G^{n-8}(X)$. For any finite subgroup $H \subseteq G$, there are natural isomorphisms $KO_G^*(G/H) \cong KO_H^*(\mathrm{pt})$, and in particular $KO_G^0(G/H) \cong RO(H)$.

4. The completion theorem

Given a discrete group G, we prove here a completion theorem, analogous to that of Atiyah and Segal [6] for actions of compact Lie groups. We show that for any finite proper G-CW-complex X, $K^*(EG \times_G X)$ is the completion of $K_G^*(X)$ with respect to a certain ideal. When the universal space $\mathrm{E}_{\mathscr{FIN}}(G)$ (see Definition 2.1) for the family $\mathscr{FIN}$ of finite subgroups of G has the G-homotopy type of a finite-dimensional G-CW-complex, and there is an upper bound on the order of finite subgroups of G, then the ideal in question is that generated by the augmentation ideal of $K_G(E_{\mathscr{FIN}}(G))$.

In fact, as in the theorem of Atiyah and Segal, we prove an isomorphism not just of inverse limits, but also of inverse systems. This has the advantage that it gives a stronger result (Theorem 4.3), and it easily implies that $\{(EG \times_G X)^{(n)}\}$ satisfies the Mittag-Leffler condition and has vanishing $\varprojlim^1$. It is also needed in the proof of Theorem 4.3, which is carried out by induction over the number of cells using a version of the five-lemma.

We first fix our notation for handling pro-groups (by which we always mean pro-abelian groups). For the definitions in full generality, see [6, Section 2]. For simplicity, all pro-groups dealt with here will be indexed by the nonnegative (or positive) integers. We write (G, α) for the inverse system

$$\xrightarrow{\alpha_3} G_3 \xrightarrow{\alpha_2} G_2 \xrightarrow{\alpha_1} G_1 \xrightarrow{\alpha_0} G_0,$$

and also write $\alpha_i^j = \alpha_i \circ \cdots \circ \alpha_{j-1} : G_j \to G_i$ for $j \geqslant i$ ($\alpha_i^i = \mathrm{Id}_{G_i}$). For the purposes here, it will suffice (and greatly simplify the notation) to work with "strict" pro-homomorphisms: homomorphisms $f:(G,\alpha) \to (H,\beta)$ such that $f_i: G_i \to H_i$, and $\beta_{i-1} \circ f_i = f_{i-1} \circ \alpha_{i-1}$, for all i. Kernels and cokernels of strict homomorphisms are defined in the obvious way.

A pro-group will be called *pro-trivial* if for each $i \geqslant 0$, there is some $j \geqslant i$ such that $\alpha_i^j = 0$. A strict homomorphism $f:(G,\alpha) \to (H,\beta)$ is a *isomorphism of pro-groups* if and only if $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are both pro-trivial, or, equivalently, for each $i \geqslant 0$ there is some $j \geqslant i$ such that $\mathrm{Im}(\beta_i^j) \subseteq \mathrm{Im}(f_i)$ and $\mathrm{Ker}(f_j) \subseteq \mathrm{Ker}(\alpha_i^j)$. A sequence of strict homomorphisms

$$(G,\alpha) \xrightarrow{f} (G',\alpha') \xrightarrow{f'} (G'',\alpha'')$$

will be called *exact* if $f_i{}' \circ f_i = 0$ for each i, and if the pro-group $\{\mathrm{Ker}(f_i{}')/(f_i)\}_{i \geqslant 0}$ is pro-trivial. The following result will be needed.

Lemma 4.1. *Fix any commutative noetherian ring A, and any ideal $I \subseteq A$. Then for any exact sequence $M' \to M \to M''$* of *finitely generated A-modules, the sequence*

$$\{M'/I^nM'\} \to \{M/I^nM\} \to \{M''/I^nM''\}$$

of pro-groups (pro-A-modules) is exact.

Proof. It suffices to prove this when the sequence is short exact. Regard M' as a submodule of M, and consider the exact sequence

$$0 \to \left\{\frac{(I^nM) \cap M'}{I^nM'}\right\} \to \{M'/I^nM'\} \to \{M/I^nM\} \to \{M''/I^nM''\} \to 0.$$

By [5, Theorem 10.11, p. 107], the filtrations $\{(I^nM) \cap M'\}$ and $\{I^nM'\}$ of M' have "bounded difference": i.e., there exists $k > 0$ such that $(I^{n+k}M) \cap M' \subseteq I^nM'$ for all n. The first term in the above exact sequence is thus pro-trivial, and so the remaining terms define a short exact sequence of pro-groups. □

To avoid ambiguity, for any proper G-CW-complex X, the *augmentation ideal* $I\mathbb{K}_G(X) \subseteq \mathbb{K}_G(X)$ is defined to be the set of elements represented by virtual G-vector bundles of dimension zero on all connected components. In other words,

$$I\mathbb{K}_G(X) = \mathrm{Ker}\left[\mathbb{K}_G(X) \xrightarrow{\dim} \prod_{\pi_0(X)/G} \mathbb{Z}\right],$$

where the ring homomorphism dim sends $[E]$ to the map from $\pi_0(X)/G \to \mathbb{Z}$ which assigns to the G-orbit through a path component $C \subset X$ the dimension of the fiber E_x for any point $x \in C$. Given a G-map $f: X \to Y$, the ring homomorphism $f^*: \mathbb{K}_G(Y) \to \mathbb{K}_G(X)$ induces a map

$f^*: I\mathbb{K}_G(Y) \to I\mathbb{K}_G(X)$. Similarly, when working with ordinary nonequivariant K-theory, we define

$$IK^*(X) = \mathrm{Ker}\left[K^*(X) \xrightarrow{\mathrm{Res}} \prod_{\pi_0(X)} K^*(\mathrm{pt}) \right].$$

Lemma 4.2. *Let X be a CW-complex of dimension $n-1$. Then any n-fold product of elements in $IK^*(X)$ is zero.*

Proof. Write $X = Y \cup A$, where Y and A are closed subsets, Y contains $X^{(n-2)}$ as a deformation retract, and A is a disjoint union of $(n-1)$-disks. Fix elements $v_1, v_2, \ldots, v_n \in IK^*(X)$. We can assume by induction that $v_1 \cdots v_{n-1}$ vanishes after restricting to Y, and hence that it is the image of an element $u \in K^*(X, Y)$. Also, v_n clearly vanishes after restricting to A, and hence is the image of an element $v \in K^*(X, A)$. Their product is thus the image in $K^*(X)$ of the element $uv \in K^*(X, Y \cup A) = 0$ (cf. [12, Section 5.8]), and so $v_1 \cdots v_n = 0$. □

Fix any finite proper G-CW-complex X, and any map $f: X \to L$ to a finite dimensional proper G-CW-complex L whose isotropy subgroups have bounded order. Regard $K_G^*(X)$ as a module over the ring $K_G(L)$. Set $I = IK_G(L)$. For any $n \geqslant 0$, the composite

$$I^n \cdot K_G^*(X) \subseteq K_G^*(X) \xrightarrow{\mathrm{proj}^*} \mathbb{K}_G^*(EG \times X) \to \mathbb{K}^*(EG \times_G X)$$

$$\xrightarrow{\mathrm{res}} \mathbb{K}^*((EG \times_G X)^{(n-1)}) \xrightarrow{\cong} K^*((EG \times_G X)^{(n-1)})$$

is zero, since the image is contained in $IK^*((EG \times_G X)^{(n-1)})^n = 0$ which vanishes by Lemma 4.2. (Recall that $\mathbb{K}^*(Y) \cong K^*(Y)$ for finite dimensional Y, since any map $Y \to BU$ factors through some $BU(n)$.) This thus defines a homomorphism of pro-groups

$$\lambda^{X,f}: \{K_G^*(X)/I^n \cdot K_G^*(X)\}_{n \geqslant 1} \to \{K^*((EG \times_G X)^{(n-1)})\}_{n \geqslant 1}.$$

As usual, $(-)_{\hat{I}}$ denotes completion with respect to an ideal I.

Theorem 4.3 (Completion theorem). *Let G be a discrete group. Fix a finite proper G-CW-complex X, a finite dimensional proper G-CW-complex L whose isotropy subgroups have bounded order, and a G-map $f: X \to L$. Regard $K_G^*(X)$ as a module over $\mathbb{K}_G(L)$, and let $\mathrm{I} = \mathrm{I}\mathbb{K}_G(L)$ be the augmentation ideal. Then*

$$\lambda^{X,f}: \{K_G^*(X)/I^n \cdot K_G^*(X)\}_{n \geqslant 1} \to \{K^*((EG \times_G X)^{(n-1)})\}_{n \geqslant 1}.$$

is an isomorphism of pro-groups. Also, the inverse system $\{K^*((EG \times_G X)^{(n)})\}$ satisfies the Mittag–Leffler condition. In particular,

$$\lim_{\leftarrow}{}^1 K^*((EG \times_G X)^{(n)}) = 0,$$

and $\lambda^{X,f}$ induces an isomorphism

$$K_G^*(X)_{\hat{I}} \xrightarrow{\cong} K^*(EG \times_G X) \cong \lim_{\leftarrow} K^*((EG \times_G X)^{(n)}).$$

Proof. Assume that $\lambda^{X,f}$ is an isomorphism. Then the system $\{K^*((EG\times_G X)^{(n)})\}_{n\geqslant 1}$ satisfies the Mittag–Leffler condition because $\{K^*_G(X)/I^n\}$ does. In particular, $\varprojlim^1 K^*((EG\times_G X)^{(n)})=0$, and so $K^*(EG\times_G X)\cong\varprojlim K^*((EG\times_G X)^{(n)})$ (cf. [6, Proposition 4.1]).

It remains to show that $\lambda^{X,f}$ is an isomorphism.

Step 1: Assume first that $X=G/H$, for some finite subgroup $H\subseteq G$. Then the following diagram commutes

$$\begin{tikzcd} \mathbb{K}_G(L) \arrow[r, "f^*"] \arrow[d, "\mathrm{ev}_{f(eH)}"'] & K^*_G(G/H) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{ev}_{eH}"', "\cong"] & K^*(EG\times_G G/H) \arrow[d, "q"', "\cong"] \\ R(H) \arrow[r, "\cong"'] & K^*_H(*) \arrow[r, "\mathrm{pr}_2"] & K^*(BH), \end{tikzcd}$$

where $\mathrm{ev}_{f(eH)}$ sends the class of a G-vector bundle $E\to L$ to the class of the fiber $E|_{f(eH)}$ considered as an H-representation and the other maps are the obvious ones. Also, pr_2 induces an isomorphism of pro-groups

$$\{K^*_H/IR(H)^n\cdot K^*_H(*)\}_{n\geqslant 1}\to\{K^*((BH)^{(n-1)})\}_{n\geqslant 1}$$

by the theorem of Atiyah and Segal [6] (where $IR(H)$ denotes the augmentation ideal of $R(H)$). We want to show that pr_1 induces an isomorphism of pro-groups

$$\{K^*_G(G/H)/I^n\cdot K^*_G(G/H)\}_{n\geqslant 1}\to\{K^*((EG\times_G G/H)^{(n-1)})\}_{n\geqslant 1}.$$

So we must show that for some k, $IR(H)^k\subseteq I'\overset{\mathrm{def}}{=}\mathrm{ev}_{f(eH)}(I)$.

This means showing that the ideal $IR(H)/I'$ is nilpotent; or equivalently (since $R(H)$ is noetherian) that it is contained in all prime ideals of $R(H)/I'$ (cf. [5, Proposition 1.8]). In other words, we must show that every prime ideal of $R(H)$ which contains I' also contains $IR(H)$. Fix any prime ideal $\mathfrak{P}\subseteq R(H)$ which does *not* contain $IR(H)$. Set $\zeta=\exp(2\pi i/|H|)$, and $A=\mathbb{Z}[\zeta]$. By a result of Atiyah [2, Lemma 6.2], there is a prime ideal $\mathfrak{p}\subseteq A$ and an element $s\in H$ such that

$$\mathfrak{P}=\{v\in R(G)\mid\chi_v(s)\in\mathfrak{p}\}.$$

Also, $s\neq 1$ since $\mathfrak{P}\not\supseteq IR(H)$. Set $p=\mathrm{char}(A/\mathfrak{p})$ (possibly $p=0$). By [2, Lemma 6.3], we can assume that s has order prime to p.

By Corollary 2.8 (or Corollary 2.7 if $p=0$), there is a G-vector bundle $E\to L$ such that $p\dim(E)$, and such that $(E|_x)|_{\langle s\rangle}$ is a multiple of the regular representation of $\langle s\rangle$. In particular, $\chi_{E|x}(s)=0$. Set $k=\dim(E)$, and $v=[\mathbb{C}^k]-[E|_x]\in R(H)$. Then $([\mathbb{C}^k\times L]-[E])\in I$, so $v\in I'$. Also, $\chi_v(s)=k\notin\mathfrak{p}$, so $v\notin\mathfrak{P}$, and thus $\mathfrak{P}\not\supseteq I'$.

Step 2: We now prove the theorem by induction over the dimension of X and the number of cells in a given dimension. It holds when $\dim(X)=0$ by Step 1. So assume that $\dim(X)=m>0$. Write $X=Y\cup_\varphi(G/H\times D^m)$, for some attaching map $\varphi: G/H\times S^{m-1}\to Y$, where $\lambda^{Y,L}$ is an isomorphism.

Consider the Mayer–Vietoris sequence of Lemma 3.8:

$$\to K^*_G(X) \to K^*_G(Y) \oplus K^*_G(G/H \times D^m) \to 3K^*_G(G/H \times S^{m-1}) \to$$

for X as a pushout of Y and $G/H \times D^m \simeq G/H$ over $G/H \times S^{m-1}$. All terms in this sequence are $K_G(X)$-modules and all homomorphisms $K_G(X)$-linear, and the $\mathbb{K}_G(L)$-module structure on each term is induced from the $K_G(X)$-module structure. So if we let $I' \subseteq K_G(X)$ be the ideal generated by the image of I, then dividing out by $(I')^n$ is the same as dividing out by I^n for all terms. In addition, $K_G(X)$ is noetherian (in fact, a finitely generated abelian group), and so the Mayer–Vietoris sequence above induces an exact sequence of pro-groups

$$\begin{aligned} &\to \{K^*_G(X)/I^n\}_{n \geqslant 1} \to \{K^*_G(Y)/I^n \oplus K^*_G(G/H \times D^m)/I^n\}_{n \geqslant 1} \\ &\qquad \to \{K^*_G(G/H \times S^{m-1})/I^n\}_{n \geqslant 1} \to \end{aligned}$$

by Lemma 4.1. (We write here M/I^n for M/I^nM for short.) Since the obvious strict map of progroups

$$\{K^*((EG \times_G X)^{(n-1)})\}_{n \geqslant 1} \to \{K^*(EG^{(n-1)} \times_G X)\}_{n \geqslant 1}$$

is an isomorphism of progroups, the various long exact Mayer–Vietoris sequences of the pushouts $EG^{(n-1)} \times_G X = EG^{(n-1)} \times_G Y \cup_{\mathrm{Id} \times_G \varphi} (EG^{(n-1)} \times_G (G/H \times D^m))$ yield a long exact sequence of progroups

$$\begin{aligned} &\to \{K^*((EG \times_G X)^{(n-1)})\}_{n \geqslant 1} \\ &\to \{K^*((EG \times_G Y)^{(n-1)}) \oplus K^*((EG \times_G (G/H \times D^m))^{(n-1)})\}_{n \geqslant 1} \\ &\to \{K^*((EG \times_G (G/H \times S^{m-1}))^{(n-1)})\}_{n \geqslant 1} \to \end{aligned}$$

The five-lemma for pro-groups (whose proof is essentially the same as that of the usual 5-lemma), together with the induction hypothesis applied to Y and $G/H \times S^{m-1}$, and Step 1 applied to $G/H \times D^m \simeq_G G/H$, now proves that $\lambda^{X,f}$ is an isomorphism of pro-groups. □

As one immediate consequence of Theorem 4.3, we get:

Theorem 4.4. *Let $E_{\mathcal{FIN}}(G)$ be the universal space for the family $\mathcal{FIN}$ of finite subgroups of G (introduced in Definition 2.1). Set $I = IK_G(E_{\mathcal{FIN}}(G))$.*

(a) *If $E_{\mathcal{FIN}}(G)$ has the G-homotopy type of a finite dimensional G-CW-complex and there is an upper bound on the orders of the finite order subgroups of G, then for any finite proper G-CW-complex X,*

$$K^*(EG \times_G X) \cong K^*_G(X)_{\hat{I}}.$$

(b) *If $E_{\mathcal{FIN}}(G)$ has the G-homotopy type of a finite G-CW-complex, then*

$$K^*(BG) \cong K^*_G(E_{\mathcal{FIN}}(G))_{\hat{I}}.$$

Note that when G is finite, $E_{\mathcal{FIN}}(G) \simeq *$ and $K_G(E_{\mathcal{FIN}}(G)) \cong R(G)$. So in this case, Theorem 4.4 is exactly the theorem of Atiyah and Segal. If G is torsion free, then any proper G-action is free, and so Theorems 4.3 and 4.4 follow immediately from Lemma 4.2.

Notice that the formulation in Theorem 4.4 does not apply to all discrete groups G, but only to those with bounded torsion and for which $E_{\mathscr{FIN}}(G)$ is finite dimensional. One of the interesting features of Theorem 4.3 is that one does not complete with respect to one single canonical ideal of $K_G(X)$, but rather an ideal which depends on the choice of another space L. Thus, different choices of ideals yield the same result. In an attempt to give an intrinsic choice of ideal, we give the following third formulation of the completion theorem.

Theorem 4.5. *Let X be any finite proper G-complex. Define*

$$S = \{x \in IK_G(X) \mid \mathrm{res}_{X^H}(x) \in \mathrm{Im}[R(H) \to K_H(X^H)],\ \textit{all finite } H \subseteq G\}.$$

Here, $R(H) \to K_H(X^H)$ sends a representation to the product bundle. Let I be the ideal generated by S. Then

$$K^*(EG \times_G X) \cong K^*_G(X)_{\hat{I}}.$$

Proof. Let $\mathscr{F}$ be the family of subgroups of isotropy subgroups of X, and let L be the $(\dim(X)+1)$-skeleton of $E_{\mathscr{F}}(G)$. Choose any G-map $f\colon X \to L$ (unique up to homotopy by Lemma 2.2(b)). Let $J \subseteq K_G(X)$ be the ideal generated by $f^*(I\mathbb{K}_G(L))$. Then $J \subseteq I \subseteq IK_G(X)$. So

$$\{K^*((EG \times_G X)^{(n)})\}_{n\geqslant 1} \cong \{K^*_G(X)/J^n \cdot K^*_G(X)\}_{n\geqslant 1} \cong \{K^*_G(X)/IK_G(X)^n \cdot K^*_G(X)\}_{n\geqslant 1},$$

as inverse systems by Theorem 4.3, and so they are all isomorphic to the inverse system

$$\{K^*_G(X)/I^n \cdot K^*_G(X)\}_{n\geqslant 1}.$$

The result now follows upon taking inverse limits. □

The completion theorem, in the above forms, also holds for $KO_G(X)$, as described in Theorem 3.15. This can be proven in the same way as Theorem 4.3 above, but the classifying spaces for real G-vector bundles do not have simply connected fixed point sets, and hence a more complicated form of the obstruction theory used in Section 2 is needed. Instead of including those details here, we will prove this result in a different way in a later paper, using an equivariant version of the Chern character.

5. Proper actions of Lie groups

Recall (Definition 3.1) that we write $\mathbb{K}_G^{-n}(-)$ to denote the graded functor defined via G-vector bundles, also when G is a positive dimensional Lie group. Certain properties of $\mathbb{K}_G^{-n}(-)$, such as homotopy invariance (Lemma 3.3), were shown to hold in this generality. It was our proof of excision, and of the long exact sequence for a proper G-CW-pair, which required the assumption that G is discrete. In order to help explain exactly what goes wrong for Lie groups, we now exhibit an explicit example of a group G for which $\mathbb{K}_G^{-n}$ is not a cohomology theory, and for which the completion theorem fails: even after replacing $\mathbb{K}_G^{-n}$ by the "correct" equivariant K-theory.

Phillips [15] has constructed an equivariant cohomology theory $K^*_G(-)$, for any second countable locally compact group G, on the category of proper locally compact G-spaces. This is done using infinite dimensional G-vector bundles with Hilbert space fibers. When G is a Lie group (or in any situation where G-vector bundles are defined), there is an obvious natural transformation

$$\varphi_G(X): \mathbb{K}_G^*(X) \to K_G^*(X)$$

for proper G-CW-complexes. Phillips [15, Example 9.11] also constructs G and X for which not all elements of $K_G^0(X)$ are represented by (finite-dimensional) G-vector bundles over X; i.e., for which $\varphi_G(X)$ is not surjective. In these terms, what we have shown in Section 3 is that $\varphi_G(X)$ is an isomorphism whenever G is discrete and X is a finite G-CW-complex. (Since it is an isomorphism for orbits: $\mathbb{K}_G(G/H) \cong R(H) \cong K_G(G/H)$ for any finite $H \subseteq G$.) In a later, more technical paper, we will extend this result (still for discrete G) to arbitrary proper G-spaces with compact orbit space.

Phillips has also shown [16, Theorems 3.3 and 5.3] that $\varphi_G(-)$ is an isomorphism whenever the space of connected components of G is compact, and that the completion theorem holds for $K_G(-)$ whenever G is a Lie group with finitely many connected components. The key to showing this is a theorem of Abels [1], which says that any such group G contains a maximal compact subgroup K, such that any proper G-space X with paracompact orbit space maps to the quotient G/K. This means that $X \cong G \times_K Y$ for some K-space Y, and hence that $\mathbb{K}_G(X) \cong \mathbb{K}_K(Y) \cong K_K(Y) \cong K_G(X)$ if X/G is compact.

In order to explain from our point of view what goes wrong for nondiscrete groups, we now construct a Lie group G, which is somewhat simpler than that used by Phillips, together with examples to show that $\mathbb{K}_G^*$ satisfies neither excision nor exactness. In particular, $\mathbb{K}_G^*$ is not a cohomology theory, and so $\varphi_G(-)$ is not in general an isomorphism. We also show that the completion theorem fails for $K_G(-)$.

Set $T = S^1 \times S^1$, the 2-torus, and let $\alpha \in \mathrm{Aut}(T)$ be the automorphism $\alpha(x,y) = (x, xy)$. Write $G = T \overset{\alpha}{\rtimes} \mathbb{Z}$: the semidirect product determined by α. We also need to consider the subgroup

$$K = 1 \times S^1 = \{g^{-1} \cdot \alpha(g) \mid g \in G\}$$

(note that $G/K \cong (T/K) \times \mathbb{Z}$).

Lemma 5.1. *Let X be any proper connected (G/T)-CW-complex. Then for any G-vector bundle $E \to X$, K acts via the identity on E. In particular, for any closed G-invariant subcomplex $A \subseteq X$, $\mathbb{K}_G^*(X, A) \cong \mathbb{K}_{G/K}^*(X, A)$.*

Proof. For each $x \in X$, the fiber $E|_x$ is a T-representation. Since X is connected, these representations are all isomorphic to some given T-representation V. Also, V and $\alpha^* V$ are T-isomorphic, because multiplication with the generator $z \in \mathbb{Z}$ defines an α-equivariant linear isomorphism from $E|_x$ to $E|_{zx}$. For any irreducible T-representation W, with character $\chi_W \in \mathrm{Hom}(T, S^1)$, either the $\chi_W \circ \alpha^n$ are all distinct, or $\chi_W = \chi_W \circ \alpha$ and $K = \{g^{-1}\alpha(g) \mid g \in G\} \subseteq \mathrm{Ker}(\chi_W)$. Since $\dim(V) < \infty$, this shows that V must be a sum of irreducible (T/K)-representations, and thus that K acts on each fiber of E via the identity.

The last statement now follows immediately from the definitions. □

It is now easy to see that Lemma 3.7 fails for G. For example, set $(X, A) = (\mathbb{R}, \mathbb{Z})$ with the G-action induced by the translation action of $\mathbb{Z} = G/T$, and let V be a T-representation upon which K does not act trivially. Then $G \times_T V$ is a G-vector bundle over $A = \mathbb{Z} \cong G/T$, and it cannot be embedded into any G-vector bundle over $X = \mathbb{R}$ since K acts nontrivially.

More generally, let G be any Lie group and let (X, A) be any finite G-CW-pair. If $E_0 \to A$ is any G-vector bundle, then an obvious necessary condition for being able to embed it in a G-vector bundle over X is to be able to choose the fibers: to find for each $x \in X$ a G_x-representation V_x, such that $E_0|_x$ embeds into V_x for each $x \in A$, and such that the V_x in any connected component of X^H (for any $H \subseteq G$) are all isomorphic as H-representations. For discrete G, we can always choose the V_x to be appropriate multiples of the regular representation of G_x, and this was the first (and easiest) step towards proving Lemma 3.7. In contrast, for the group $G = T \overset{\alpha}{\rtimes} \mathbb{Z}$, the above example shows that this first step of the proof fails: we cannot even choose the representations V_x.

Given this example, it is not at all surprising that excision and exactness fail for $\mathbb{K}_G^*(-)$. By homotopy invariance,

$$\mathbb{K}_G^*(\mathbb{R}, \mathbb{Z}) \cong \mathbb{K}_G^*(\mathbb{R}, [0, \tfrac{1}{2}] + \mathbb{Z}) \quad \text{and} \quad \mathbb{K}_G^*(\mathbb{Z} \times I, \mathbb{Z} \times \partial I) \cong \mathbb{K}_G^*([\tfrac{1}{2}, 1] + \mathbb{Z}, \{0, \tfrac{1}{2}\} + \mathbb{Z}),$$

and hence these should be isomorphic if excision holds for $\mathbb{K}_G^*(-)$. However:

Example 5.2. For G as above,

$$\mathbb{K}_G^{-n}(\mathbb{R}, \mathbb{Z}) \cong \begin{cases} R(T/K) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad \mathbb{K}_G^{-n}(\mathbb{Z} \times I, \mathbb{Z} \times \partial I) \cong \begin{cases} R(T) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus, excision does not hold for $\mathbb{K}_G^*(-)$. Furthermore,

$$\mathbb{K}_G^{-n}(\mathbb{R}) \cong R(T/K) \quad (\text{for all } n) \quad \text{and} \quad \mathbb{K}_G^{-n}(\mathbb{Z}) \cong \begin{cases} R(T) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

and so there is no long exact sequence in $\mathbb{K}_G^*$ for the pair $(\mathbb{R}\,\mathbb{Z})$.

Proof. By Lemmas 3.5 and 5.1,

$$\begin{aligned} \mathbb{K}_G^{-n}(\mathbb{R},\mathbb{Z}) &\cong \mathbb{K}_{G/K}^{-n}(\mathbb{R},\mathbb{Z}) \cong \mathbb{K}_{(T/K)\times\mathbb{Z}}^{-n}(\mathbb{R},\mathbb{Z}) \\ &\cong \mathbb{K}_{T/K}^{-n}(\mathbb{R}/\mathbb{Z},\mathrm{pt}) \cong K_{T/K}^{-n-1}(\mathrm{pt}) \cong \begin{cases} R(T/K) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even;} \end{cases} \end{aligned}$$

and

$$\mathbb{K}_G^{-n}(\mathbb{R}) \cong \mathbb{K}_{G/K}^{-n}(\mathbb{R}) \cong \mathbb{K}_{(T/K)\times\mathbb{Z}}^{-n}(\mathbb{R}) \cong \mathbb{K}_{T/K}^{-n}(\mathbb{R}/\mathbb{Z}) \cong K_{T/K}^{-n}(\mathrm{pt}) \oplus K_{T/K}^{-n-1}(\mathrm{pt}) \cong R(T/K)$$

(for all n). Also, by Lemma 3.4,

$$\mathbb{K}_G^{-n}(\mathbb{Z} \times I, \mathbb{Z} \times \partial I) \cong \mathbb{K}_T^{-n}(I,\partial I) \cong K_T^{-n-1}(\mathrm{pt}) \cong \begin{cases} R(T) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even;} \end{cases}$$

and similarly

$$\mathbb{K}_G^{-n}(\mathbb{Z}) \cong \mathbb{K}_T^{-n}(\mathrm{pt}) \cong \begin{cases} R(T) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

All of these groups are $R(T/K)$-modules (via the isomorphism $G/(K \times \mathbb{Z}) \cong T/K$), and all natural maps between them are $R(T/K)$-linear. But there is no $R(T/K)$-linear exact sequence $\mathbb{K}^0_G(\mathbb{R}) \to \mathbb{K}^0_G(\mathbb{Z}) \to \mathbb{K}^{-1}_G(\mathbb{R},\mathbb{Z})$, since the middle term is infinitely generated and the others finitely generated. □

In fact, $\varphi_G(\mathbb{R})$ is an isomorphism in this case; i.e., $K^0_G(\mathbb{R}) \cong \mathbb{K}^0_G(\mathbb{R}) \cong R(T/K)$. To see this, consider the composite

$$\mathbb{K}^0_G(\mathbb{R}) \xrightarrow{\varphi_G(\mathbb{R})} K^0_G(\mathbb{R}) \xrightarrow{i^*} K^0_G(\mathbb{Z}),$$

where $i : \mathbb{Z} \to \mathbb{R}$ denotes the inclusion. Under the identifications $\mathbb{K}^0_G(\mathbb{R}) \cong R(T/K)$ (see Example 5.2) and $K^0_G(\mathbb{Z}) = K^0_G(G/T) \cong R(T)$, $i^* \circ \varphi_G(\mathbb{R})$ corresponds to the inclusion $R(T/K) \hookrightarrow R(T)$ defined by regarding T/K-representations as T-representations. We have seen, in the proof of Lemma 5.1, that $R(T/K) = R(T)^\alpha$: the subgroup of elements fixed by composition with $\alpha \in \mathrm{Aut}(T)$. The map i^* is injective since $K^0_G(\mathbb{R},\mathbb{Z}) \cong K^0_G(\mathbb{Z} \times I, \mathbb{Z} \times \partial I) = 0$. Hence it remains to show that the image of i^* is contained in $R(T)^\alpha$. This follows since the action of α on $K^0_G(\mathbb{Z}) = K^0_G(G/T) \cong R(T)$ corresponds to the map $t^* : K^0_G(\mathbb{Z}) \to K^0_G(\mathbb{Z})$ induced by the G-map $t : n \mapsto n+1$, and since the analogous G-map $t : \mathbb{R} \to \mathbb{R}$ is G-homotopic to the identity.

This can now be used to show that the completion theorem (stated in terms of K^*_G or $\mathbb{K}^*_G$) fails for this group G. Consider the space $X = \mathbb{Z}$. Since $\mathbb{R}^H = \mathbb{R}$ is contractible for any compact subgroup $H \subseteq G$, $\mathbb{R} \simeq E_{\mathscr{C}\mathscr{P}\mathscr{C}\mathscr{T}}(G)$ is a universal proper G-CW-complex. Then

$$\mathrm{Im}[K_G(\mathbb{R}) \xrightarrow{i^*} K_G(\mathbb{Z}) \cong R(T)] = R(T/K),$$

while

$$K^0(EG \times_G \mathbb{Z}) \cong K^0(BT) \cong R(T)\widehat{_{IR(T)}} \neq R(T)\widehat{_{IR(T/K)}} \cong K^0_G(\mathbb{Z})\widehat{_{IR(T/K)}}.$$

Here, the completions of $R(T)$ with respect to $IR(T)$ and $IR(T/K)$ are distinct since

$$R(T)\widehat{_{IR(T)}} \cong \mathbb{Z}[[x,y]] \supsetneqq \mathbb{Z}[[x]][t,t^{-1}] \cong R(T)\widehat{_{IR(T/K)}} \quad (\text{where } y = t-1).$$

In the example constructed above, the reason for the failure of representing $K_G(-)$ by G-vector bundles and of the completion theorem comes down to the fact that the isotropy subgroup in question is positive dimensional, and hence has infinitely many irreducible representations. For example, one of the key lemmas which makes possible our results for actions of a discrete group G is that any finite proper G-CW-complex has a G-vector bundle over it whose fibers are free as representations of the isotropy subgroups — and this makes sense only if the isotropy subgroups are finite. It is thus natural to ask whether these results hold for any Lie group G and any finite proper G-complex X all of whose isotropy subgroups are finite. This does, in fact, turn out to be the case — $\mathbb{K}_G(X) \cong K_G(X)$ and $K(EG \times_G X)$ is a completion of $K_G(X)$ — but proving it requires working out the details of obstruction theory for Lie group actions, which is more complicated than that used in Section 2.

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