Products and push-forwards in parametrised cohomology theories

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Introduction

It is well known that generalised cohomology theories defined on topological spaces correspond to spectra. Under this correspondence, the cohomology groups $E^*(X)$ of a space X arise as maps in the stable homotopy category from the suspension spectrum $\Sigma^\infty X$ of X to E, the spectrum representing the cohomology theory. If one assumes that X is a CW-complex and E is an Ω-spectrum (that is, the maps $E(n) \to \Omega_m E(n+m)$ are weak homotopy equivalences), elements of the nth group can be realised as maps from X to $E(n)$. Maps from X to $E(n)$ are the same as sections in the trivial bundle over X with fibre $E(n)$. The idea to define twisted cohomology groups is to replace the trivial $E(n)$-bundle over X by a a non-trivial $E(n)$-bundle M over X and define the twisted groups as homotopy classes of sections of M. Assuming that $\mathcal{E}(n) \to B$ is the universal $E(n)$-bundle, there is a map $q : X \to B$ such that $M \cong q^*\mathcal{E}(n)$. Sections of M are nothing but maps $u : X \to \mathcal{E}(n)$ such that

$$
\begin{array}{ccc}
X & \xrightarrow{\quad u \quad} & \mathcal{E}(n) \\
& \searrow{\scriptstyle q} \quad \swarrow & \\
& B &
\end{array}
$$

commutes, i.e. maps from X to M in the category of spaces over B. This leads to the definition of a twisted or parametrised cohomology theory as a cohomology theory on the category of spaces over a fixed base space B. In this picture of twisted cohomology theories, the set of twists on a given space X corresponds to the set of maps $X \to B$; we will see later that a homotopy of twists defines an isomorphism of the corresponding twisted cohomology groups.

Since a cohomology theory should have suspension isomorphisms relating the groups in different degrees, the spaces $\mathcal{E}(m)$ should be interrelated. More precisely, there should be maps $\Sigma^m \mathcal{E}(n) \to \mathcal{E}(n+m)$, where Σ denotes fibre-wise suspension, i.e., $\mathcal{E}$ should be what is called a parametrised spectrum or ex-spectrum. Ex-spectra were first studied in [13, 14]. The category of ex-spectra $\mathcal{S}_B$ over a fixed base B is similar to the category of non-parametrised spectra $\mathcal{S}$. In particular, one can define the stable parametrised homotopy category $Ho\mathcal{S}_B$ in which the suspension functor is invertible. Moreover, just

as generalised cohomology theories on spaces correspond to spectra, cohomology theories on the category of spaces over a fixed base correspond to ex-spectra [32].

In [31], the authors discuss $Ho\mathcal{S}_B$ in the language of model categories. Using the concept of orthogonal spectra, they are able to define a symmetric monoidal structure which exists already before passage to the stable homotopy category.

An important point in the application of twisted cohomology theories is the existence of a product. In the examples, there is frequently the notion of the sum $\alpha + \beta$ of twists α, β. The products in the examples of multiplicative twisted cohomology theories and the sum of twists are related in that the product adds the twists, i.e. if u is an α-twisted class and v is a β-twisted class, $u \cup v$ is usually $\alpha + \beta$ twisted. The fact that multiplicative non-parametrised cohomology theories correspond to ring spectra suggests that multiplicative parametrised cohomology theories should correspond to parametrised ring spectra. However, the symmetric monoidal structure on $Ho\mathcal{S}_B$ constructed in [13, 14, 31] is not suitable for this purpose, because the product would not add twists. In addition, if one thinks of twists as maps to some space B, it is for a general B not clear what the sum of twists should be. If B is an H-space, one can use the product on B to define the sum $f + g$ of maps $f, g : X \rightarrow B$. To obtain our symmmetric monoidal structure on $Ho\mathcal{S}_B$, an H-space structure on B is not enough. We need better homotopical control of the product on B. In particular, the product should be not only homotopy associative, but the associativity homotopy should be unique up to higher homotopies. This is exactly what is captured by the notion of an action of an E_∞-operad on B. Such an action defines products on B in which all commutativity and associativity diagrams commute up to homotopy, and these homotopies are all unique up to higher homotopies. We define symmetric monoidal structures $\underline{p}$ on $Ho\mathcal{S}_B$ for base spaces B with an action of an E_∞-operad. In fact, we only need up to four-fold associativities. Therefore, a truncated action would be sufficient. We use the symmetric monoidal strucures $\underline{p}$ to define multiplicative parametrised cohomology theories on the category of spaces over such a B.

In the past few years, a twisted cohomology theory of particular interest was H^3-twisted K-theory [2, 3, 17–19, 34, 35]. There are various pictures of twisted K-theory, corresponding to different pictures of H^3.

One way to interpret elements of $H^3(X, \mathbb{Z})$ is as principal $PU(\mathbf{H})$-bundles over X, where $\mathbf{H}$ denotes some fixed separable Hilbert space. In [2, 3], the authors associate to a $PU(\mathbf{H})$-bundle a $Fred(\mathbf{H})$-bundle, using the conjugation action of $PU(\mathbf{H})$ on $Fred(\mathbf{H})$, and define twisted K-theory in terms of homotopy classes of sections of the $Fred(\mathbf{H})$-bundle. Another interpretation

of H^3 (which readily carries over to the equivariant setting) is used in [34, 35]. An H^3-class of a topological stack corresponds to a central S^1-extension of a groupoid representing the stack, and twisted K-theory is defined as the K-theory of the reduced C^*-algebra of this extension.

Recall that a a class in $H^3(X, \mathbb{Z})$ can be realised as a continuous map $X \rightarrow K(\mathbb{Z}, 3)$. Therefore, the category of "spaces with H^3-twists" may be identified with the category of spaces over $K(\mathbb{Z}, 3)$. In consequence, to define H^3 twisted K-theory as a multiplicative parametrised cohomology theory amounts to constructing a parametrised ring spetrum over $BPU(\mathbf{H}) \cong K(\mathbb{Z}, 3)$ representing twisted K-theory, and we do precisely that. The properties of twisted K-theory such as homotopy invariance, Mayer-Vietoris sequences, excision and so forth follow from the general statement that it is a multiplicative cohomology theory on spaces over $BPU(\mathbf{H})$. In particular, the Thom isomorphisms and push-forwards in twisted K-theory which were geometrically constructed in [12] follow from the corresponding construction for multiplicative twisted cohomology theories.

The text is organised as follows. In the first chapter, we summarise the construction of $Ho\mathcal{S}_B$ and some of its properties from [31]. We define the symmetric monoidal structure $\underline{p}$ in Proposition 32. If a group G acts on a spectrum F, an operad acts on both G and F and the actions are compatible, we construct a parametrised ring spectrum with fibre F over BG in Lemma 35.

In the second chapter, we define twisted cohomology theories and products in twisted cohomology theories. We prove a Thom isomorphism theorem for generalised twisted cohomology theories and define push-forward homomorphisms for proper oriented maps. In addition, we prove functoriality of push-forwards and a projection formula.

In the last chapter, we construct parametrised ring spectra representing twisted K-theory and twisted $Spin^c$-cobordism.

Throughout the text, we use Quillens notion of a model category as defined in [36]. A concise introduction to model categories is [15], a detailed account with an eye to examples is [20]. All model categories in our text are cofibrantly generated and therefore admit cofibrant and fibrant replacement functors which we denote by Q and R. For a topologically enriched category $\mathcal{C}$ and $X, Y \in Ob(\mathcal{C})$, we denote both the morphism set and the morphism space by $\mathcal{C}(X, Y)$.

Chapter 1

Basic definitions and constructions

1.1   Ex-spaces

Definition 1 *An ex-space over a topological space B is a space X with continuous maps $q : X \to B$ and $s : B \to X$ such that $q \circ s = id_B$.*

The point-set topology to set up convenient categories of ex-spaces is quite tedious. We refrain ourselves from a lengthy discussion, but rather refer the interested reader to [5–9,22,31] and give a short account of the results we need. Recall that a topological space X is called a *k*-space if a subset $A \subset X$ is closed if $f^{-1}(A)$ is closed for all compact Hausdorff spaces K and all continuous maps $f : K \to X$. X is called weak Hausdorff if $f(K)$ is closed for all compact Hausdorff spaces K and all continuous maps $f : K \to X$. Let $\mathcal{U}$ denote the category of weak Hausdorff *k*-spaces and $\mathcal{K}$ the category of *k*-spaces.

Definition 2 *For $B \in \mathcal{U}$, define $\mathcal{K}_B$ to be the category of ex-spaces X over B with $X \in Ob(\mathcal{K})$. Define $\mathcal{K}/B$ to be the category of k-spaces over B. The functor $_+ : \mathcal{K}/B \to \mathcal{K}_B$ is given by adjoing a disjoint basepoint, i.e. a disjoint copy of B to spaces X over B.*

We topologize the morphism sets with the subspace topology, where we use the compact-open topology for $\mathcal{K}$. To improve legibility, we will omit the maps $q : X \to B$ and $s : B \to X$ from our notation most of the time and only include them when necessary. For $K \in \mathcal{K}$ and $(X, q) \in \mathcal{K}/B$ we define

$$Map_B(K, X) := \{f : K \to X \mid \exists\, b \in B \ s.t. \ f(K) \subset q^{-1}(b)\} \subset Map(K, X),$$

again using the compact-open topology on $Map(K, X)$ and the obvious map to B. Together with $X \times_B K := (X \times K, q \circ pr_X)$ this defines a tensor

and cotensor structur of the topological category $\mathcal{K}/B$ over $\mathcal{K}$, i.e. there are natural homeomorphisms

$$\mathcal{K}/B(X \times_B K, Y) \cong \mathcal{K}(K, \mathcal{K}/B(X, Y)) \cong \mathcal{K}/B(X, Map_B(K, Y))$$

Similarly, the pointed category $\mathcal{K}_B$ is tensored and cotensored as topological category over $\mathcal{K}_*$.

Definition 3 *For ex-spaces X and Y over B, define the smash product $X \wedge_B Y$ by the following pushout in $\mathcal{K}/B$.*

$$
\begin{array}{ccc}
X \vee_B Y & \longrightarrow & X \times_B Y \\
\downarrow & & \downarrow \\
*_B & \longrightarrow & X \wedge_B Y.
\end{array}
$$

There are internal mapping spaces $Map_B(.,.)$ in $\mathcal{K}/B$ and $F_B(.,.)$ in $\mathcal{K}_B$, giving $\mathcal{K}/B$ the structure of a closed cartesian category and $\mathcal{K}_B$ the structure of a closed symmetric monoidal category [5–7].
A continuous map $f : A \to B$ defines functors

$$f_! : \mathcal{K}_A \overset{\longrightarrow}{\longleftarrow} \mathcal{K}_B : f^*$$

by setting

$$
\begin{array}{ccc}
A & \overset{f}{\longrightarrow} & B \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\
X & \longrightarrow & f_!X
\end{array}
\qquad\qquad
\begin{array}{ccc}
f^*Y & \longrightarrow & Y \\
{\scriptstyle r'}\downarrow & & \downarrow{\scriptstyle q'} \\
A & \overset{f}{\longrightarrow} & B
\end{array}
$$

for $(X, q, s) \in \mathcal{K}_A$, $(Y, q', s') \in \mathcal{K}_B$. The first diagram is a pushout and we define the missing map $r : f_!X \to B$ to be the pushout of $id_B : B \to B$ and $f \circ q : X \to B$; the second diagram is a pullback and $t' : A \to f^*Y$ is defined to be the pullback of $id_A : A \to A$ and $s' \circ f : A \to Y$. These functors are adjoint as functors of topologically enriched categories, i.e. they are continuous and the adjunction isomorphisms

$$\mathcal{K}_B(f_!X, Y) \cong \mathcal{K}_A(X, f^*Y)$$

are homeomorphisms [31, 2.1.2].
Furthermore, there are external products $\overline{\times} : \mathcal{K}/A \times \mathcal{K}/B \to \mathcal{K}/(A \times B)$ and $\overline{\wedge} : \mathcal{K}_A \times \mathcal{K}_B \to \mathcal{K}_{A \times B}$ and mapping spaces $\overline{Map}$ and $\overline{F}$ with natural isomorphisms

$$\mathcal{K}/(A \times B)(X \overline{\times} Y, Z) \cong \mathcal{K}/A(X, \overline{Map}(Y, Z))$$

and

$$\mathcal{K}_{A \times B}(X \overline{\wedge} Y, Z) \cong \mathcal{K}_A(X, \overline{F}(Y, Z)).$$

The base change functors can be used to derive the external products from the internal ones; for example, $X \overline{\wedge} Y := \pi_A^* X \wedge_{A \times B} \pi_B^* Y$. A pointed space $K \in \mathcal{K}_*$ is just an ex-space over a point. Using the projection $\pi : B \to *$, the trivial ex-space with fibre K is

$$K_B := \pi^* K = K \times B \in \mathcal{K}_B.$$

Notice that there are natural isomorphisms

$$X \overline{\wedge} K \cong X \wedge_B K \cong X \wedge K_B,$$

relating the tensor structure of $\mathcal{K}_B$ over $\mathcal{K}$, the external and the internal smash product; see [31] for a more thorough discussion of the functors $\overline{\wedge}, \wedge, f^*, f_!, \wedge_B, Map_B, F_B, \overline{F}$ and their relations.

Proposition 4 *[31, 2.2.1] For a map $f : A \to B$, the functor $f^* : \mathcal{K}_B \to \mathcal{K}_A$ is closed symmetric monoidal.*

In [31], the authors develop model structures on $\mathcal{K}/B$ and $\mathcal{K}_B$. We will now give a short summary of their results, again omitting the proofs.

Definition 5 $f \in \mathcal{K}/B(X, Y)$ *is a fibrewise cofibration if it has the left lifting property with respect to the maps*

$$\pi_0 : Map_B(I, Z) \to Z \ \forall Z \in \mathcal{K}/B,$$

where π_0 is given by evaluation at zero.

Denote the upper (lower) hemisphere of a n-disc by S_+^{n-1} (S_-^{n-1}) with the inclusion $i_\pm : S_\pm^{n-1} \to D^n$. Similarly, we write

$$i : S^{n-1} \to D^n \qquad \text{and} \qquad j_- : S^{n-1} \to S_-^n$$

for the inclusions.

Definition 6

$$
\begin{aligned}
I_B &:= \{i : (S^{n-1}, d \circ i) \to (D^n, d) \mid i \text{ is a fibrewise cofibration}\} \\
J_B &:= \{i_+ : (S_+^n, d \circ i_+) \to (D^{n+1}, d) \mid i : (S^n, d \circ i) \to (D^{n+1}, d) \text{ and} \\
&\qquad j_- : (S^{n-1}, d \circ i_- \circ j_-) \to (S_-^n, d \circ i_-) \text{ are fibrewise cofibrations}\}
\end{aligned}
$$

Theorem 7 *[31, 6.2.5] $\mathcal{K}/B$ is a cofibrantly generated model category, with weak equivalences given by weak homotopy equivalences of total spaces and I_B (J_B) as set of generating (acyclic) cofibrations.*

Theorem 8 *[31, 6.2.6] $\mathcal{K}_B$ is a cofibrantly generated model category, with weak equivalences given by weak homotopy equivalences of total spaces and $(I_B)_+$ $((J_B)_+)$ as set of generating (acyclic) cofibrations.*

Proposition 9 *[31, 7.3.4, 7.3.5] For a continuous map $f : A \to B$, the adjunction $(f_!, f^*)$ is a Quillen adjunction between $\mathcal{K}_A$ and $\mathcal{K}_B$. It is a Quillen equivalence if f is a weak homotopy equivalence.*

The internal smash product $\wedge : \mathcal{K}_B \times \mathcal{K}_B \to \mathcal{K}_B$ is not a Quillen bifunctor. Nonetheless, the authors of [31] construct a "derived" internal smash product on the homotopy category. However, the external smash product

$$\overline{\wedge} : \mathcal{K}_A \times \mathcal{K}_B \to \mathcal{K}_{A \times B}$$

is a Quillen bifunctor [31, 7.3] and thus can be derived to obtain

$$L\overline{\wedge} : Ho\mathcal{K}_A \times Ho\mathcal{K}_B \to Ho\mathcal{K}_{A \times B}.$$

For an ex-space $E \in \mathcal{K}_A$, there is a natural map

$$i_{0!}QE \cong QE \coprod_{A \times \{0\}} A \times I \tilde{\to} RE \times I \cong \pi_A^* RE$$

given by the extension of $QE \to RQE \to RE$. This is a weak equivalence, hence it induces a natural isomorphism of total derived functors $Li_{0!} \cong R\pi_A^*$. Similarly, $Li_{1!} \cong R\pi_A^*$ and therefore we get a natural isomorphism $\rho : Li_{0!} \tilde{\to} Li_{1!}$.

Proposition 10 *A homotopy $h : A \times I \to B$ between $f, g : A \to B$ gives rise to a natural isomorphism $\phi : Lf_! \overset{\sim}{\to} Lg_!$.*
If two homotopies h_0, h_1 are homotopic (as homotopies) via H, the isomorphisms ϕ_0 and ϕ_1 coincide.

Proof: The natural isomorphisms

$$h_! i_{0!} \cong f_!, \qquad h_! i_{1!} \cong g_!$$

between left Quillen functors yield isomorphisms between the derived functors [20]. We combine these and the natural isomorphism $\rho : Li_{0!} \tilde{\to} Li_{1!}$ to obtain the desired isomorphism

$$Lf_! \cong Lh_!\, Li_{0!} \overset{Lh_!(\rho)}{\longrightarrow} Lh_!\, Li_{1!} \cong Lg_!.$$

Let us now discuss the case in which there are two homotopic homotopies h_0, h_1 from f to g. For the inclusions $j_0, j_1 : A \times I \to A \times I \times I$, and $E \in \mathcal{K}_A$, we have again natural weak equivalences in $\mathcal{K}_{A \times I \times I}$

$$j_{0!}Qi_{0!}QE \overset{\alpha_0}{\to} \pi_{A \times I}^* Ri_{0!}QE \overset{\alpha_1}{\leftarrow} j_{1!}Qi_{0!}QE$$

$$j_{0!}Qi_{1!}QE \xrightarrow{\beta_0} \pi^*_{A\times I}Ri_{1!}QE \xleftarrow{\beta_1} j_{1!}Qi_{1!}QE$$

$$j_{0!}Q\pi^*_A RE \xrightarrow{\gamma_0} \pi^*_{A\times I}RQ\pi^*_A RE \xleftarrow{\gamma_1} j_{1!}Q\pi^*_A RE.$$

Note that $\pi^*_{A\times I} \cong \overline{\wedge}S^0_I$, hence it preserves cofibrant objects. For $E \in \mathcal{K}_A$, commutativity of the diagram

$$
\begin{array}{ccccc}
& h_{0!}Qi_{0!}QE & \xrightarrow{Lh_{0!}(\rho)} & h_{0!}Qi_{1!}QE & \\
& \downarrow{\scriptstyle H_!(\alpha_0)} & & \downarrow{\scriptstyle H_!(\beta_0)} & \\
f_!QE & H_!\pi^*_{A\times I}Ri_{0!}QE & & H_!\pi^*_{A\times I}Ri_{1!}QE & g_!QE \\
& \downarrow{\scriptstyle H_!(\alpha_1)^{-1}} & & \downarrow{\scriptstyle H_!(\beta_1)^{-1}} & \\
& h_{1!}Qi_{0!}QE & \xrightarrow{Lh_{1!}(\rho)} & h_{1!}Qi_{1!}QE &
\end{array}
$$

in $Ho\mathcal{K}_B$ gives the result, so let us first show that the middle square commutes. A lift of the square to $\mathcal{K}_B$ is $H_!$ applied to the following diagram in $\mathcal{K}_{A\times I\times I}$.

$$
\begin{array}{ccccc}
j_{0!}Qi_{0!}QE & \longrightarrow & j_{0!}Q\pi^*_A RE & \longleftarrow & j_{0!}Qi_{1!}QE \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle \beta_0} \\
\pi^*_{A\times I}Ri_{0!}QE & \longrightarrow & \pi^*_{A\times I}RQ\pi^*_A RE & \longleftarrow & \pi^*_{A\times I}Ri_{1!}QE \\
\uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \gamma_1} & & \uparrow{\scriptstyle \beta_1} \\
j_{1!}Qi_{0!}QE & \longrightarrow & j_{1!}Q\pi^*_A RE & \longleftarrow & j_{1!}Qi_{1!}QE.
\end{array}
$$

The top and bottom horizontal maps stem from the natural maps

$$i_{0!}QE \to \pi^*_A RE \qquad\qquad i_{1!}QE \to \pi^*_A RE.$$

Since $i_{0!}QE$, $i_{1!}QE$ are cofibrant, these maps can be lifted to maps

$$i_{0!}QE \to Q\pi^*_A RE \qquad\qquad i_{1!}QE \to Q\pi^*_A RE$$

which yield the middle horizontal arrows. The above diagram commutes in $\mathcal{K}_{A\times I\times I}$ and all maps are weak equivalences. Since all objects in the diagram are cofibrant and $H_!$, being a left Quillen functor, preserves all weak equivalences between cofibrant objects, the middle square commutes in $Ho\mathcal{K}_B$.
To show commutativity of the left triangle, we choose again a lift to $\mathcal{K}_B$, namely

$$
\begin{array}{ccc}
& h_{0!}Qi_{0!}QE & \\
{\scriptstyle h_{0!}(q)}\swarrow & & \downarrow{\scriptstyle H_!(\alpha_0)} \\
f_!QE & & H_!\pi^*_{A\times I}Ri_{0!}QE \\
{\scriptstyle h_{1!}(q)}\nwarrow & & \uparrow{\scriptstyle H_!(\alpha_1)} \\
& h_{1!}Qi_{0!}QE, &
\end{array}
$$

with q being the natural map $Qi_{0!}QE \to i_{0!}QE$. Observe that we have factorisations

$$\alpha_0 : j_{0!}Qi_{0!}QE \to \pi^*_{A\times I}i_{0!}QE \to \pi^*_{A\times I}Ri_{0!}QE$$

$$\alpha_1 : j_{1!}Qi_{0!}QE \to \pi^*_{A\times I}i_{0!}QE \to \pi^*_{A\times I}Ri_{0!}QE.$$

We use these to see that $H_!(\alpha_0)$ and $H_!(\alpha_1)$ factor through $H_!\pi^*_{A\times I}i_{0!}QE$ and we get a commutative diagram

$$\begin{array}{ccc}
 & h_{0!}Qi_{0!}QE & \\
\swarrow^{h_{0!}(q)} & \downarrow & \searrow^{H_!(\alpha_0)} \\
f_!QE \quad H_!\pi^*_{A\times I}i_{0!}QE & \longrightarrow & H_!\pi^*_{A\times I}Ri_{0!}QE \\
\nwarrow_{h_{1!}(q)} & \uparrow & \nearrow_{H_!(\alpha_1)} \\
 & h_{1!}Qi_{0!}QE &
\end{array} \quad .$$

Furthermore, there are maps

$$\xi_s : j_{s!}i_{0!}QE \to \pi^*_{A\times I}i_{0!}QE \quad \forall\, s \in I$$

given by extension of the identitiy $i_{0!}QE \to i_{0!}QE$. The triangles

$$\begin{array}{ccc}
 & h_{0!}Qi_{0!}QE & \\
\nearrow^{h_{0!}(q)} & & \downarrow \\
f_!QE \xrightarrow{\quad H_!(\xi_0)\quad} & H_!\pi^*_{A\times I}i_{0!}QE &
\end{array}$$

and

$$\begin{array}{ccc}
f_!QE \xrightarrow{\quad H_!(\xi_1)\quad} & H_!\pi^*_{A\times I}i_{0!}QE & \\
 & \searrow_{h_{1!}(q)} & \uparrow \\
 & h_{1!}Qi_{0!}QE &
\end{array}$$

commute. Since $H \circ j_s \circ i_0 = f \ \ \forall\, s \in I$, $H_!(\xi_0)$ and $H_!(\xi_1)$ are homotopic over B. Thus, the left triangle commutes in $Ho\mathcal{K}_B$.

The argument for the right triangle is similar. $\qquad\qquad\qquad\square$

In addition, the correspondence between homotopies and natural equivalences is functorial in the following sense.

Proposition 11 *Let h_0 be a homotopy between $f_0, f_1 : A \to B$ and h_1 be one connecting f_1 and f_2, and denote by h the homotopy from f_0 to f_2 given by composing h_0 and h_1. Then the corresponding isomorphisms $\phi_0 : Lf_{0!}\tilde{\to}Lf_{1!}$, $\phi_1 : Lf_{1!}\tilde{\to}Lf_{2!}$ and $\phi : Lf_{0!}\tilde{\to}Lf_{2!}$ satisfy $\phi_1 \circ \phi_0 = \phi : Lf_{0!}\tilde{\to}Lf_{2!}$.*

Proof: Consider the double interval $J := [0, 2]$ and the maps

$$\pi : A \times J \to A, \; j_0 : A \times I \to A \times J, \; j_1 : A \times I \to A \times J, \; i_{kl} := j_k \circ i_l$$

and h as a map $A \times J \to B$. There are identities

$$f_{k+l} = h_k \circ i_l = h \circ i_{kl} \; \forall k, l \in \{0, 1\}.$$

Consider the diagram

$$
\begin{array}{ccc}
 & Qj_{0!}i_{0!}QE & \\
\nearrow^{\alpha_2} \downarrow \searrow & \\
j_{0!}Qi_{0!}QE \longrightarrow j_{0!}i_{0!}QE \overset{\alpha_0}{\longrightarrow} Q\pi^*RE & \\
\downarrow \quad {\scriptstyle\alpha_1}\downarrow \quad \downarrow & \\
j_{0!}Q\pi_A^*RE \longrightarrow j_{0!}\pi_A^*RE \longrightarrow \pi^*RE &
\end{array}
$$

in which all maps except α_i are either natural "extensions by zero" or of the form $QF \to F$, and α_i are obtained by lifting the respective maps to the cofibrant replacements. Since maps of the form $QF \to F$ are weak equivalences, the diagram commutes in $Ho\mathcal{K}_{A \times J}$. Similarly, let α_3 be defined to be the lift in

$$
\begin{array}{ccc}
 & Q\pi^*RE & \\
\nearrow^{\alpha_3} & \downarrow & \\
j_{0!}i_{1!}QE \longrightarrow j_{0!}\pi_A^*RE \longrightarrow \pi^*RE &
\end{array}
$$

and observe that

$$
\begin{array}{ccc}
j_{0!}Qi_{1!}QE \longrightarrow j_{0!}Q\pi_A^*RE \overset{\alpha_1}{\longrightarrow} Q\pi^*RE \\
\downarrow \quad\quad\quad {\scriptstyle\alpha_3}\nearrow \\
j_{0!}i_{1!}QE
\end{array}
$$

commutes in $Ho\mathcal{K}_{A \times J}$. Putting everything together, applying $h_!$ and using the natural identifications

$$h_! \circ j_{k!} \circ i_{l!} \cong f_{k+l!},$$

we obtain

$$
\begin{array}{c}
h_!Qi_{00!}QE \xrightarrow{\hspace{6cm}} h_!Q\pi^*RE \\[-0.5em]
\end{array}
$$

with the diagram containing:
$h_!Qi_{00!}QE$ at top left, $h_!Q\pi^*RE$ at top right, connected by a horizontal arrow.
$h_!(\alpha_2)$ labels the left vertical arrow from $h_{0!}Qi_{0!}QE$ to $h_!Qi_{00!}QE$.
$h_{0!}Q\pi_A^*RE$ in the middle.
$h_!(\alpha_1)$ labels arrow from $h_{0!}Q\pi_A^*RE$ to $h_!Q\pi^*RE$.
$h_!(\alpha_3)$ labels the right vertical arrow from $f_{1!}QE$ to $h_!Q\pi^*RE$.
$h_{0!}Qi_{0!}QE$ and $h_{0!}Qi_{1!}QE$ in the row below.
$f_{0!}QE$ and $f_{1!}QE$ at the bottom.

One can repeat the construction with h_1 instead of h_0 and the second half of J. This completes the previous diagram to

$$
\begin{array}{c}
h_!Qi_{00!}QE \xrightarrow{\hspace{3cm}} h_!Q\pi^*RE \xleftarrow{\hspace{3cm}} h_!Qi_{11!}QE
\end{array}
$$

with:
$h_{0!}Q\pi_A^*RE$ and $h_{1!}Q\pi_A^*RE$ in the second row.
$h_{0!}Qi_{0!}QE$, $h_{0!}Qi_{1!}QE$, $h_{1!}Qi_{0!}QE$, $h_{1!}Qi_{1!}QE$ in the third row.
$f_{0!}QE$, $f_{1!}QE$, $f_{2!}QE$ at the bottom.

The natural transformation corresponding to h is obtained by the left and right vertical arrows and the top horizontal ones whereas the ϕ_i correspond to the left and right 'hats'. $\qquad\square$

1.2   Ex-spectra

Definition 12 *Let $\mathcal{J}$ denote the category of finite dimensional real inner product spaces and linear isometric isomorphisms. A $\mathcal{J}$-space over $B \in \mathcal{U}$ is a continuous functor $E : \mathcal{J} \to \mathcal{K}_B$; $\mathcal{J}\mathcal{K}_B$ is the category of $\mathcal{J}$-spaces over B.*

The external smash product of $X, Y \in \mathcal{J}\mathcal{K}_B$ is given by

$$
X \overline{\wedge}_B Y := \wedge \circ (X \times Y) : \mathcal{J} \times \mathcal{J} \to \mathcal{K}_B.
$$

Define the smash product $X \wedge_B Y$ to be the topological left Kan extension of $X \overline{\wedge}_B Y$ along $\oplus : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$.

Theorem 13 *[31, 11.1.3, 11.1.6] $(\mathcal{JK}_B, \wedge_B)$ is a closed symmetric monoidal category. It is tensored and cotensored over $\mathcal{K}_B$.*

The tensor and cotensor structure are defined levelwise. Moreover, for a map $f : A \to B$ of base spaces, levelwise application of the adjunction $(f_!, f^*)$ defines an adjunction

$$f_! : \mathcal{JK}_A \underset{\leftarrow}{\overset{\rightarrow}{}} \mathcal{JK}_B : f^*.$$

For a real inner product space V, let S^V denote the one-point compactification. Define the sphere S_B over B to be the $\mathcal{J}$-space sending V to S_B^V. This is a commutative monoid in $\mathcal{JK}_B$, so we can define $\mathcal{S}_B$, the ex-spectra over B, to be the S_B-modules. The smash product $X \wedge Y$ of ex-spectra X, Y is the coequalizer (in $\mathcal{JK}_B$) of

$$X \wedge_B S_B \wedge_B Y \overset{\longrightarrow}{\longrightarrow} X \wedge_B Y.$$

Proposition 14 *[31, 11.2.5] $(\mathcal{S}_B, \wedge)$ is a closed symmetric monoidal category with unit S_B.*

Again, levelwise application of the functors $f_!, f^*$ gives an adjunction $f_! : \mathcal{S}_A \underset{\leftarrow}{\overset{\rightarrow}{}} \mathcal{S}_B : f^*$ and the functor $f^* : \mathcal{S}_B \to \mathcal{S}_A$ is closed symmetric monoidal.

Definition 15 *[31]For base spaces A, B, the external smash product is*

$$\overline{\wedge} := \wedge \circ (\pi_A^* \times \pi_B^*) : \mathcal{S}_A \times \mathcal{S}_B \to \mathcal{S}_{A \times B}.$$

There is an obvious pair of adjoint functors

$$\Sigma^\infty : \mathcal{K}_B \to \mathcal{S}_B \qquad \text{and} \qquad \Omega^\infty : \mathcal{S}_B \to \mathcal{K}_B,$$

given by $\Sigma^\infty(X)(V) = X \wedge_B S_B(V)$ and $\Omega^\infty(E) := E(0)$. More generally, the functor Ω_V^∞ given by evaluation at V has a left adjoint Σ_V^∞ [31].

Proposition 16 *[31, 11.2.5] $\mathcal{S}_B$ is tensored and cotensored over $\mathcal{K}_B$. For $X \in \mathcal{S}_B$ and $K \in \mathcal{K}_B$, there are natural isomorphims*

$$X \wedge_B K \cong X \wedge_B \Sigma^\infty K.$$

1.3 Model structures on ex-spectra

In this section, we give a short summary of the model structures on ex-spectra developed in [31]. First, a level model structure is defined which is then utilised to obtain a stable model structure, where the adjectives "level" and "stable" refer to the respective classes of weak equivalences.

Definition 17 *[31] A map $f : X \to Y$ of spectra over B is called a level weak equivalence if $f(V) : X(V) \to Y(V)$ is a weak homotopy equivalence for all $V \in \mathcal{J}$. It is called a level fibration if each $f(V)$ is a fibration in the model structure on $\mathcal{K}_B$ defined in Theorem 8.*

Fix a skeleton $skel(\mathcal{J})$ of $\mathcal{J}$.

Definition 18 *[31]*

$$
\begin{aligned}
FI_B &:= \{\Sigma_V^\infty i \,|\, i \in I_B,\ V \in skel(\mathcal{J})\} \\
FJ_B &:= \{\Sigma_V^\infty j \,|\, j \in J_B,\ V \in skel(\mathcal{J})\}
\end{aligned}
$$

Theorem 19 *[31, 12.1.7] $\mathcal{S}_B$ is a cofibrantly generated topological model category with level weak equivalences as weak equivalences, level fibrations as fibrations and $FI_B(FJ_B)$ as generating (acyclic) cofibrations .*

Recall that the homotopy groups $\pi_q(X)$ of an (unparametrised) spectrum X are defined as the colimits of the groups $\pi_q(\Omega^V(X(V)))$.

Definition 20 *[31] A map $f : X \to Y$ of parametrised spectra is called a stable weak equivalence if, after level fibrant approximation R^l, it induces an isomorphism $i_b^* R^l f : i_b^* R^l X \to i_b^* R^l Y$ on homotopy groups of (unparametrised) spectra for all points $b \in B$.*

A map of ex-spectra is called s-cofibration if it is a cofibration in the level model structure.

Theorem 21 *[31, 12.3.10] $\mathcal{S}_B$ is a cofibrantly generated model category with the stable weak equivalences as weak equivalences and the s-cofibrations as cofibrations.*

As in the unparametrised case, any level weak equivalence is a stable weak equivalence. The stable model structure from the above theorem is indeed stable in the sense that the suspension functor $\Sigma : \mathcal{S}_B \to \mathcal{S}_B, \Sigma(E) = E \wedge_B S^1$ is invertible in the homotopy category.

Definition 22 *[31] An Ω-spectrum over B is a level fibrant spectrum over B such that each of its adjoint structure maps is a weak equivalence.*

It is shown in [31, 12.3.10,12.3.14] that the Ω-spectra over B are the fibrant objects in the stable model structure. Moreover, the sets of level and stable weak equivalences between Ω-spectra coincide.

Proposition 23 *[31, 12.6.2] The adjunction $\Sigma_V^\infty : \mathcal{K}_B \overset{\longrightarrow}{\longleftarrow} \mathcal{S}_B : \Omega_V^\infty$ is a Quillen adjunction for all $V \in \mathcal{I}$.*

Proposition 24 *[31, 12.6.7] For a continuous map $f : A \to B$, the adjunction $(f_!, f^*)$ is a Quillen adjunction. It is a Quillen equivalence if f is a weak homotopy equivalence.*

The external smash product $\overline{\wedge} : \mathcal{S}_A \times \mathcal{S}_B \to \mathcal{S}_{A \times B}$ is homotopically well-behaved in the stable model structure and compatible with the base change functors; more precisely, we have the following proposition.

Proposition 25 *[31, 12.6.5,13.7.2] $\overline{\wedge} : \mathcal{S}_A \times \mathcal{S}_B \to \mathcal{S}_{A \times B}$ is a Quillen bifunctor. For maps $f : A \to B$ and $g : A' \to B'$, there are natural equivalences*

$$L(\overline{\wedge} \circ (f_! \times g_!)) \cong L((f \times g)_! \circ \overline{\wedge}).$$

For a spectrum E over A, there is a natural map $i_{0!}QE \to i_{0!}RQE \to \pi_A^* RE$ of spectra over $A \times I$. The first map is a stable weak equivalence since $i_{0!}$ is a Quillen left adjoint, and $RQE \to RE$ is a stable weak equivalence between fibrant spectra, hence it is a level weak equivalence which implies that the second map is a (level and therefore stable) weak equivalence as well. Moreover, since the base change functors are defined levelwise, the same proofs as for Proposition 10 and 11 give the corresponding statements for $Ho\mathcal{S}_B$.

Proposition 26 *A homotopy $h : A \times I \to B$ between $f, g : A \to B$ gives rise to a natural isomorphism $\phi : Lf_! \overset{\sim}{\to} Lg_!$ of functors $Ho\mathcal{S}_A \to Ho\mathcal{S}_B$. If two homotopies h_0, h_1 are homotopic (as homotopies) via H, the isomorphims ϕ_0 and ϕ_1 coincide.*

Proposition 27 *Let h_0 be a homotopy between $f_0, f_1 : A \to B$ and h_1 be one connecting f_1 and f_2, and denote by h the homotopy from f_0 to f_2 given by composing h_0 and h_1. Then the corresponding isomorphisms $\phi_0 : Lf_{0!} \overset{\sim}{\to} Lf_{1!}$, $\phi_1 : Lf_{1!} \overset{\sim}{\to} Lf_{2!}$ and $\phi : Lf_{0!} \overset{\sim}{\to} Lf_{2!}$ satisfy $\phi_1 \circ \phi_0 = \phi : Lf_{0!} \overset{\sim}{\to} Lf_{2!}$.*

1.4   Operads

Definition 28 *Let Δ denote the category with the sets $(n) = \{0, ..., n\}, n \geq 0$ as objects and the non-decreasing functions as morphisms. The category of simplicial spaces $\mathbf{S}\mathcal{U}$ is defined as the category of functors $\Delta^{op} \to \mathcal{U}$.*

Using the functor $\triangle : \Delta \to \mathcal{U}$ mapping (n) to the topological n-simplex

$$\triangle^n := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \Sigma t_i = 1, \ t_i \geq 0\}$$

we define the geometric realisation $|\cdot| : \mathbf{S}\mathcal{U} \to \mathcal{U}$ as

$$|X_\bullet| := \int^{(n) \in \Delta} X_n \times \triangle^n,$$

where the coend is defined as the initial object of $X_\bullet \times \triangle^\bullet / \mathcal{U}$, the category whose objects are spaces $Y \in \mathcal{U}$ together with maps $X_n \times \triangle^n \to Y$ for all n such that

$$
\begin{array}{ccc}
X_k \times \triangle^l & \longrightarrow & X_k \times \triangle^k \\
\downarrow & & \downarrow \\
X_l \times \triangle^l & \longrightarrow & Y
\end{array}
$$

commutes for all morphisms in $\Delta((l),(k))$.

Proposition 29 *[28, 11.5] Geometric realisation preserves finite products, i.e. there are natural homeomorphisms $\xi : |X_\bullet \times Y_\bullet| \cong |X_\bullet| \times |Y_\bullet|$ for all simplicial spaces $X_\bullet, Y_\bullet \in \mathbf{S}\mathcal{U}$.*

Definition 30 *[27, 28] An operad (in $\mathcal{U}$) is a sequence of spaces $\mathcal{P}(j) \in \mathcal{U}$, $j \in \mathbb{N}$ with right Σ_j-actions on $\mathcal{P}(j)$, continuous Σ-equivariant maps*

$$
\gamma : \mathcal{P}(k) \times \mathcal{P}(j_1) \times ... \times \mathcal{P}(j_k) \to \mathcal{P}(j_1 + j_2 + ...j_k)
$$

satisfying associativity and a unit $1 \in \mathcal{P}(1)$. A morphism of operads $\theta : \mathcal{P} \to \mathcal{P}'$ is a sequence of continuous Σ_j-equivariant maps $\theta(j) : \mathcal{P}(j) \to \mathcal{P}'(j)$ such that

$$
\theta(j_1 + ... + j_k) \circ \gamma = \gamma' \circ (\theta(k) \times \theta(j_1) \times ... \times \theta(j_k)).
$$

An E_∞-operad is an operad $\mathcal{P}$ such that all spaces $\mathcal{P}(j)$ are contractible.

An operad is called pointed if it is indexed on $\mathbb{N}_0$ and $\mathcal{P}(0)$ consists of a single point.

Example: Let $(\mathcal{C}, \wedge)$ be a symmetric monoidal category enriched over $\mathcal{U}$. For any $X \in \mathcal{C}$, the endomorphism operad End_X is defined as

$$
End_X(j) := \mathcal{C}(X^{\wedge j}, X) \qquad \gamma(f, g_1, ..., g_k) := f \circ (g_1 \wedge ... \wedge g_k)
$$

with the identity map as unit.

Definition 31 *An action of an operad $\mathcal{P}$ on $X \in \mathcal{C}$ is a morphism of operads $\theta : \mathcal{P} \to End_X$. (X, θ) is called an algebra over $\mathcal{P}$. We denote the category of algebras over $\mathcal{P}$ in $\mathcal{C}$ by $\mathcal{P}[\mathcal{C}]$.*

Proposition 32 *If a pointed E_∞-operad $\mathcal{P}$ acts on a space B, any $p \in \mathcal{P}(2)$ endows $Ho\mathcal{S}_B$ with the structure of a symmetric monoidal category.*

Proof: For any $q \in \mathcal{P}(j)$, the $\mathcal{P}$-algebra structure yields a map $\theta(q) : B^j \to B$ and any path in $\mathcal{P}(j)$ from q to q' yields a homotopy from $\theta(q)$ to $\theta(q')$. By Proposition 26, this homotopy yields a natural transformation

$$
L\theta(q)_! \tilde{\to} L\theta(q')_!
$$

and since $\mathcal{P}(j)$ is contractible by assumption, any two such paths are homotopic and thus define the same natural transformation.

We define the bifunctor associated to $p \in \mathcal{P}(2)$ to be

$$\underline{p} := L(\theta(p)_! \circ \overline{\wedge}) : Ho\mathcal{S}_B \times Ho\mathcal{S}_B \to Ho\mathcal{S}_B.$$

and the unit by $\mathbf{1}_B := i_{*!}S$ with

$$i_* := \theta(\mathcal{P}(0)) : * \to B$$

and S the sphere spectrum. Observe that S and hence also $\mathbf{1}_B$ is cofibrant. The idea to obtain unit, symmetry and associativity isomorphisms for $\underline{p}$ is very simple: the things one wants to relate are obtained as push-forwards along maps corresponding to different points in the operad. These points in $\mathcal{P}(j)$ can be connected by paths in $\mathcal{P}(j)$ and we use the natural transformations corresponding to these paths to construct the isomorphisms.

Throughout the proof, we suppress the associativity, unit and commutativity isomorphisms for $\overline{\wedge}$ from the notation.

To get the (left) unit isomorphism, observe that the map

$$p_e : B \to B, \quad b \mapsto \theta(p)(*, b)$$

is given by $\theta(\gamma(p; *, id))$. Since $\mathcal{P}(1)$ is connected, p_e is homotopic to the identity [29]. Combining Propositions 25 and 26, we obtain

$$\underline{p}(\mathbf{1}_B, X) \cong \theta(p)_!((i_{*!}S)\overline{\wedge}QX) \cong \theta(p)_!(i_* \times id)_!S\overline{\wedge}QX \cong p_{e!}QX \cong X.$$

Similarly, we make use of the fact that the maps $p_{12} := \theta(\gamma(p; p, 1)) : B^3 \to B$ and $p_{21} := \theta(\gamma(p; 1, p)) : B^3 \to B$ are homotopic.

$$\begin{aligned}
\underline{p}(\underline{p}(X, Y), Z) &\cong \theta(p)_!(\theta(p)_!(QX\overline{\wedge}QY)\overline{\wedge}QZ) \\
&\cong \theta(p)_!(\theta(p) \times id)_!((QX\overline{\wedge}QY)\overline{\wedge}QZ) \\
&\cong p_{12!}((QX\overline{\wedge}QY)\overline{\wedge}QZ) \cong p_{21!}(QX\overline{\wedge}(QY\overline{\wedge}QZ)) \\
&\cong \underline{p}(X, \underline{p}(Y, Z))
\end{aligned}$$

which yields the associativity transformation. For the non-trivial element $\tau \in \Sigma_2$, the maps $\theta(p)$ and $\theta(p\tau)$ are homotopic, hence Proposition 26 gives the commutativity isomorphism T.

What remains to be checked are the usual coherence diagrams which can be found for example in [24].

For spectra $X, Y, W, Z \in \mathcal{S}_B$ we need to show that

$$\begin{array}{ccccc}
\underline{p}(X, \underline{p}(Y, \underline{p}(Z, W))) & \longrightarrow & \underline{p}(\underline{p}(X, Y), \underline{p}(Z, W)) & \longrightarrow & \underline{p}(\underline{p}(\underline{p}(X, Y), Z), W) \\
\downarrow & & & & \uparrow \\
\underline{p}(X, \underline{p}(\underline{p}(Y, Z), W)) & & \longrightarrow & & \underline{p}(\underline{p}(X, \underline{p}(Y, Z)), W)
\end{array}$$

commutes. Using Propositions 25 and 26 and writing

$$E := QX \overline{\wedge} QY \overline{\wedge} QZ \overline{\wedge} QW,$$

this reduces to the commutativity of

$$
\begin{array}{ccccc}
p_{1!}\, E & \longrightarrow & p_{2!}\, E & \longrightarrow & p_{3!}\, E \\
\downarrow & & & & \uparrow \\
p_{4!}\, E & \longrightarrow & & & p_{5!}\, E
\end{array}
$$

where the p_i denote the maps corresponding to points in $\mathcal{P}(4)$ (for example, $p_1 = \theta(\gamma(p; 1, \gamma(p; 1, p)))$) and the arrows correspond to natural isomorphisms induced from homotopies (i.e., paths in $\mathcal{P}(4)$) between the maps. $\mathcal{P}(4)$ is contractible, so all paths connecting two points are homotopic. This observation, together with Proposition 27, gives the result. By the same argument,

$$
\begin{array}{ccc}
\underline{p}(X,Y) & \longrightarrow & \underline{p}(Y,X) \\
 & \searrow{\scriptstyle id} & \downarrow \\
 & \underline{p}(X,Y) &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\underline{p}(\mathbf{1}_B, \underline{p}(X,Y)) & \longrightarrow & \underline{p}(X,Y) \\
\downarrow & \nearrow & \\
\underline{p}(\underline{p}(\mathbf{1}_B, X), Y) &
\end{array}
$$

and

$$
\begin{array}{ccccc}
\underline{p}(X, \underline{p}(Y,Z)) & \longrightarrow & \underline{p}(\underline{p}(X,Y), Z) & \longrightarrow & \underline{p}(Z, \underline{p}(X,Y)) \\
\downarrow & & & & \downarrow \\
\underline{p}(X, \underline{p}(Z,Y)) & \longrightarrow & \underline{p}(\underline{p}(X,Z), Y) & \longrightarrow & \underline{p}(\underline{p}(Z,X), Y)
\end{array}
$$

commute. $\qquad\qquad\square$

Note that $\theta(p), \theta(q)$ are homotopic for all $p, q \in \mathcal{P}(2)$. Therefore, the symmetric monoidal structures $\underline{p}, \underline{q}$ are isomorphic.

Proposition 33 *[28, 12.2] If $(X_\bullet, \theta) \in \mathcal{P}[\mathbf{S}\mathcal{U}]$, $(|X_\bullet|, \tilde{\theta})$ is an algebra over $\mathcal{P}$, where for $p \in \mathcal{P}$, the action is given by $\tilde{\theta}(p) := \xi \circ |\theta(p)| \circ \xi^{-1} : |X_\bullet|^j \to |X_\bullet|$.*

Let $\mathcal{B}(., ., .)$ denote the two-sided bar construction [27, 28]. For a topological group G, we use the models $BG := |\mathcal{B}(*, G, *)|$ and $EG := |\mathcal{B}(*, G, G)|$ for a classifying space and a contractible G-space.

Lemma 34 *[28] If (G, θ) is a group in $\mathcal{P}[\mathcal{U}]$, there are actions of $\mathcal{P}$ on BG and EG. Furthermore, the maps $EG \to BG$ and $EG \times G \to EG$ are maps of algebras over $\mathcal{P}$.*

Proof: First, let us show that $\mathcal{B}(*, G, *)$ and $\mathcal{B}(*, G, G)$ are algebras over $\mathcal{P}$. For any $p \in \mathcal{P}(j)$, the map $\theta_j(p) : G^j \to G$ gives rise to maps

$\theta_j(p)^q : \mathcal{B}_q(*, G, *)^j \cong (G^q)^j \cong (G^j)^q \to G^q \cong \mathcal{B}_q(*, G, *)$ and
$\theta_j(p)^{q+1} : \mathcal{B}_q(*, G, G)^j \cong (G^{q+1})^j \cong (G^j)^{q+1} \to G^{q+1} \cong \mathcal{B}_q(*, G, G)$. Compatibility of θ with the group structure of G implies that these are maps of simplicial spaces. Since

$$\theta_j : \mathcal{P}(j) \to \mathcal{U}(G^j, G) \qquad \text{and} \qquad (.)^q : \mathcal{U}(G^j, G) \to \mathcal{U}((G^j)^q, G^q)$$

are continuous, $\mathcal{B}(*, G, *)$ and $\mathcal{B}(*, G, G)$ are indeed in $\mathcal{P}[\mathcal{SU}]$. Proposition 33 gives $\mathcal{P}$-algebra structures on θ^B on BG and θ^E on EG. The commutativity of the diagrams

$$
\begin{array}{ccc}
\mathcal{B}_q(*, G, G)^j & \longrightarrow & \mathcal{B}_q(*, G, *)^j \\
\downarrow {\scriptstyle \theta_j^E(p)} & & \downarrow {\scriptstyle \theta_j^B(p)} \\
\mathcal{B}_q(*, G, G) & \longrightarrow & \mathcal{B}_q(*, G, *)
\end{array}
\qquad
\begin{array}{ccc}
(\mathcal{B}_q(*, G, G) \times G)^j & \longrightarrow & \mathcal{B}_q(*, G, G)^j \\
\downarrow {\scriptstyle \theta_j^E(p) \times \theta_j(p)} & & \downarrow {\scriptstyle \theta_j^E(p)} \\
\mathcal{B}_q(*, G, G) \times G & \longrightarrow & \mathcal{B}_q(*, G, G)
\end{array}
$$

implies that $EG \to BG$ and $EG \times G \to EG$ are maps of $\mathcal{P}$-algebras. $\qquad\square$

If we have an action $m : G_+ \wedge F \to F$ of G on a spectrum $F \in \mathcal{S}$, we can define an ex-spectrum BGF over BG by setting $BGF(V) := EG \times_G F(V)$ with the evident projection and section. The $O(V)$ action is the one induced by the action on $F(V)$ and since $(EG \times_G F(V)) \wedge S(W) \cong EG \times_G (F(V) \wedge S(W))$, the structure maps of F give rise to structure maps
$\sigma_{V,W} : BGF(V) \wedge S(W) \to BGF(V \oplus W)$.

Lemma 35 *Let $(F, \phi) \in \mathcal{P}[\mathcal{S}]$, (G, θ) a group in $\mathcal{P}[\mathcal{U}]$ with an action $m : G_+ \wedge F \to F$ in $\mathcal{P}[\mathcal{S}]$ and $\mathcal{P}$ a pointed E_∞ operad. Then for each $p \in \mathcal{P}(2)$, BGF is a monoid in $(Ho\mathcal{S}_{BG}, \underline{p})$, i.e. there is a map $\mu(p) : \underline{p}(BGF, BGF) \to BGF$ and a unit $\eta : 1_{BG} \to BGF$ satisfying the usual conditions.*

Proof: To construct $\mu(p)$, we have to define $O(V_1) \times O(V_2)$ -equivariant maps

$$\mu(p)_{V_1, V_2} : BGF(V_1) \overline{\wedge} BGF(V_2) \to BGF(V_1 \oplus V_2)$$

for all $V_1, V_2 \in \mathcal{I}$. Recall the construction of the $\mathcal{P}$-actions θ^B on BG and θ^E on EG from Lemma 34. The $\mathcal{P}$-structure of F gives

$$\phi(p)_{V_1, V_2} : F(V_1) \wedge F(V_2) \to F(V_1 \oplus V_2).$$

Define

$$
\begin{aligned}
\tilde{\mu}(p)_{V_1, V_2} \ : \ & BGF(V_1) \overline{\wedge} BGF(V_2) \to \theta^B(p)^* BGF(V_1 \oplus V_2) \\
& [[e_1, f_1], [e_2, f_2]] \mapsto [\theta^E(p)(e_1, e_2), \phi(p)_{V_1, V_2}(f_1, f_2)].
\end{aligned}
$$

These maps are by Proposition 34 well defined and since they are compatible with the structure maps as well as $O(V) \times O(W)$ equivariant, we obtain

$$\tilde{\mu}(p) : BGF \overline{\wedge} BGF \to \theta^B(p)^* BGF.$$

Letting $\mu(p)'$ denote the adjoint of $\tilde{\mu}(p)$, $\mu(p)$ is given by (the homotopy class of)

$$\underline{p}(BGF, BGF) = \theta^B(p)_!(QBGF \overline{\wedge} QBGF) \to \theta^B(p)_!(BGF \overline{\wedge} BGF) \overset{\mu(p)'}{\to} BGF.$$

Notice that the same procedure yields maps

$$\mu(q)' : \theta^B(q)_! BGF^{\overline{\wedge}^j} \to BGF$$

for each $q \in \mathcal{P}(j)$. Similarly, $\eta : 1_{BG} = i_{b!}S \to BGF$ is the adjoint of the unit $1_F : S \to F \cong i_b^* BGF$. What remains to be shown is the compatibility of $\mu(p)$ and η and the associativity and unit isomorphisms of $\underline{p}$, i.e. commutativity of the diagrams

$$
\begin{array}{ccc}
\underline{p}(BGF, \underline{p}(BGF, BGF)) & \overset{\sim}{\longrightarrow} & \underline{p}(\underline{p}(BGF, BGF), BGF) \\
\downarrow{\scriptstyle \underline{p}(id, \mu(p))} & & \downarrow{\scriptstyle p(\mu(p), id)} \\
\underline{p}(BGF, BGF) \overset{\mu(p)}{\longrightarrow} BGF & \longleftarrow & \underline{p}(BGF, BGF) \\
& {\scriptstyle \mu(p)} &
\end{array}
$$

and

$$
\begin{array}{ccc}
\underline{p}(1_{BG}, BGF) & \overset{\underline{p}(\eta, id)}{\longrightarrow} & \underline{p}(BGF, BGF) \\
& \searrow{\scriptstyle \sim} & \downarrow{\scriptstyle \mu(p)} \\
& & BGF
\end{array}
$$

Recall that those isomorphisms were constructed using homotopies given by paths in the operad. Using the construction above, we define maps of spectra over these homotopies which we use to prove the result.

The diagram for the unit transformation is the following, where h is a path in $\mathcal{P}(1)$ connecting $p_* := \gamma(p; *, id)$ and 1 and $E := BGF$.

$$
\begin{array}{ccc}
\theta^B(p)_!(Q1_{BG}\overline{\wedge}QE) & \xrightarrow{\ \theta^B(p)_!(Q\eta\overline{\wedge}id)\ } & \theta^B(p)_!(QE\overline{\wedge}QE) \\[4pt]
\downarrow{\scriptstyle\sim} & & \downarrow \\[4pt]
\theta^B(p)_!(1_{BG}\overline{\wedge}QE) & \longrightarrow & \theta^B(p)_!(1_{BG}\overline{\wedge}E) \\[4pt]
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\[4pt]
\theta^B(p)_!(i_*\times id)_!(S\overline{\wedge}QE) & \longrightarrow & \theta^B(p)_!(i_*\times id)_!(S\overline{\wedge}E) \qquad \theta^B(p)_!(E\overline{\wedge}E) \\[4pt]
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \qquad \mathbf{I} \\[4pt]
\theta^B(p_*)_!QE & \longrightarrow & \theta^B(p_*)_!E \\[4pt]
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \qquad \mathbf{II}\quad \mu(p_*) \\[4pt]
h_!i_{0!}QE & \longrightarrow & h_!i_{0!}E \\[4pt]
\downarrow{\scriptstyle\sim} & & \downarrow \\[4pt]
h_!\pi^*RQE & \longrightarrow & h_!\pi^*RE \xleftarrow{\ \sim\ } h_!\pi^*E \xrightarrow{\ \mu'(h_t)\ } E \\[4pt]
\uparrow{\scriptstyle\sim} & & \uparrow \qquad \mathbf{III} \\[4pt]
h_!i_{1!}QE & \longrightarrow & h_!i_{1!}E \\[4pt]
\uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \qquad id \\[4pt]
QE & \xrightarrow{\ \sim\ } & E
\end{array}
$$

(with $\mu(p)$ the diagonal arrow from $\theta^B(p)_!(E\overline{\wedge}E)$ to E)

The commutativity **I, II, III** follows from the definition of the maps $\mu(q)'$ for $q \in \mathcal{P}(j)$ and the definition of an action of an operad. Similarly, the compatibility with the associativity transformations is shown by the following diagram. The top horizontal line gives the associativity iso-morphism and h is the homotopy corresponding to a path in $\mathcal{P}(3)$ connecting $\gamma(p; 1, p)$ and $\gamma(p; p, 1)$.

$$\theta^B(p)_!(QE\overline{\wedge}\theta^B(p)_!(QE\overline{\wedge}QE))$$

$$h_!Qi_{0!}(QE^{\overline{\wedge}^3}) \xrightarrow{\sim} h_!Q\pi^*R(QE^{\overline{\wedge}^3}) \xleftarrow{\sim} h_!Qi_{1!}(QE^{\overline{\wedge}^3})$$

$$\theta^B(p)_!(\theta^B(p)_!(QE\overline{\wedge}QE)\overline{\wedge}QE)$$

$$h_!i_{0!}(QE^{\overline{\wedge}^3}) \xrightarrow{\sim} h_!\pi^*R(QE^{\overline{\wedge}^3}) \xleftarrow{\sim} h_!i_{1!}(QE^{\overline{\wedge}^3})$$

$$\theta^B(p)_!(QE\overline{\wedge}E) \qquad h_!i_{0!}(E^{\overline{\wedge}^3}) \qquad h_!\pi^*(QE^{\overline{\wedge}^3}) \qquad h_!i_{1!}(E^{\overline{\wedge}^3}) \qquad \theta^B(p)_!(E\overline{\wedge}QE)$$

$$\theta^B(p)_!(E\overline{\wedge}E) \qquad h_!\pi^*(E^{\overline{\wedge}^3}) \qquad \theta^B(p)_!(E\overline{\wedge}E)$$

$$F$$

$$\mu(p) \qquad \mu'(h_t) \qquad \mu(p)$$

$\square$

Chapter 2

Twisted cohomology theories

2.1 Definition and basic properties

Recall that generalised cohomology theories on $\mathcal{U}_*$ correspond to spectra, where the theory corresponding to a spectrum E can be defined as

$$\tilde{E}^n(X) := [(L\Sigma^\infty_{\mathbb{R}^n})X, E] = [\Sigma^\infty_{\mathbb{R}^n}QX, E] \cong [X, (RE)(\mathbb{R}^n)].$$

By Browns representability theorem, any cohomology theory on $\mathcal{U}_*$ arises in this way. The situation for twisted cohomology theories is very similar, where we replace $\mathcal{U}_*$ by the category of ex-spaces $\mathcal{K}_B$ and $\mathcal{S}$ by the category of ex-spaces $\mathcal{S}_B$. Just as in the unparametrised case, the morphism sets in $Ho\mathcal{S}_B$ can be equipped with an abelian group structure as follows. S^2 is a homotopy commutative H-cogroup with comultiplication given by the 'pinch map' $S^2 \to S^2 \vee S^2$. Hence $\Sigma^2 E \cong E \wedge S^2$ is an H-cogroup as well for all $E \in \mathcal{S}_B$. This comultiplication defines a commutative product on $[\Sigma^2 E, F]$ for any F and since $Ho\mathcal{S}_B$ is stable,

$$[E, G] \cong [\Sigma E, \Sigma G] \cong [\Sigma^2 E, \Sigma^2 G]$$

is an abelian group for any $E, G \in \mathcal{S}_B$.

We will now define reduced twisted cohomology theories and then use these to define the unreduced versions.

Recall that in pointed model categories the cofiber of a map $f : X \to Y$ is defined to be the coequalizer $g : Y \to Z$ of f and the zero map. If f is a cofibration of cofibrant objects,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called cofiber sequence.

Definition 36 *A (reduced) generalised cohomology theory on $\mathcal{K}_B$ consists of contravariant functors $\tilde{H}^* : \mathcal{K}_B \to Ab$ indexed on $\mathbb{Z}$ and natural isomorphisms $\sigma^n : \tilde{H}^{n+1} \circ \Sigma \to \tilde{H}^n$ such that*
- *$\tilde{H}^*$ factors through $\gamma : \mathcal{K}_B \to Ho\mathcal{K}_B$;*
- *If $X \to Y \to Z$ is a cofiber sequence, $\tilde{H}^n(Z) \to \tilde{H}^n(Y) \to \tilde{H}^n(X)$ is exact for all n;*
- *$\tilde{H}^*(\coprod_{i \in I} X_i) \cong \prod_{i \in I} \tilde{H}^*(X_i)$.*

Lemma 37 *The generalised cohomology theory on $\mathcal{K}_B$ correponding to an ex-spectrum $E \in \mathcal{S}_B$ defined by*

$$\tilde{E}^n(X) := [(L\Sigma^\infty_{\mathbb{R}^n})(X), E] \qquad f^* = \tilde{E}^n(f) := (L\Sigma^\infty_{\mathbb{R}^n})(f)^* \quad \forall n \in \mathbb{N}_0$$

and

$$\tilde{E}^{-n}(X) := \tilde{E}^0(\Sigma^n X)$$

is indeed a cohomology theory.

Proof: Left derived functors preserve cofiber sequences, hence

$$\tilde{E}^n(Z) \to \tilde{E}^n(Y) \to \tilde{E}^n(X)$$

is exact for all cofiber sequences $X \to Y \to Z$. For the suspension isomorphism, recall that the adjoints of the structure maps $E(V) \to \Omega_W E(V \oplus W)$ are weak equivalences of ex-spaces for all fibrant ex-spectra E, hence

$$[(L\Sigma^\infty_{\mathbb{R}^{n+1}})\Sigma X, E] \cong [\Sigma X, (RE)(\mathbb{R}^{n+1})] \cong [X, \Omega(RE)(\mathbb{R}^{n+1})] \cong [X, (RE)(\mathbb{R}^n)].$$

$\square$

Lemma 38 *[32] For every generalised cohomology theory $\tilde{H}$, there is an ex-spectrum E such that $\tilde{H}^n(X) = [(L\Sigma^\infty_{\mathbb{R}^n})X, E]$*

Proof: In [10], the author proves a general representability theorem for functors satisfying the wedge and the Mayer-Vietoris axiom on what he calls homotopy categories. In [31, 7.5], it is shown that $Ho\mathcal{K}_B$ is a homotopy category in the sense of [10], thus the functors H^n are representable by $E(\mathbb{R}^n) \in \mathcal{K}_B$. The transformations σ^n yield the maps $\Sigma E(\mathbb{R}^n) \to E(\mathbb{R}^{n+1})$. $\square$

To obtain the unreduced theories, we proceed exactly as in the unparametrised case.

Definition 39 *Let $K^2_{/B}$ be the category of pairs in $K_{/B}$. We define the cone $C : K^2_{/B} \to \mathcal{K}_B$ by*

$$C(X, A) := QX \coprod_{QA \times \{0\}} QA \times I \coprod_{QA \times \{1\}} B \qquad and \qquad C(X, \emptyset) := QX_+.$$

The unreduced cohomology groups are defined by

$$E^n(X, A) := \tilde{E}^n(C(X, A)), \qquad E^n(X) := E^n(X, \emptyset).$$

Note that there are natural maps

$$C(X, A) \to C(X, A)/_B QX_+ \cong \Sigma QA_+.$$

Combining these with the suspension isomorphism, we obtain natural maps

$$\delta^n : E^{n-1}(A) \cong \tilde{E}^n(\Sigma QA_+) \to \tilde{E}^n(C(X, A)) = E^n(X, A).$$

Lemma 40 *[31, 5.6.2] The sequence*

$$.. \to E^n(X) \to E^n(A) \overset{\delta^{n+1}}{\to} E^{n+1}(X, A) \to E^{n+1}(X) \to ...$$

is exact.

The same proof as in the unparametrised case shows that E^* satisfies excision.

Lemma 41 *If $X = \mathring{A} \cup \mathring{D}$, the inclusion $i : (A, A \cap D) \to (X, D)$ induces an isomorphism $i^* : E^n(X, D) \to E^n(A, A \cap D)$.*

Proof: First, let us assume that X is a CW-complex and A, D are subcomplexes. The natural maps

$$C(X, D) \to X \cup D \times I \cup B \qquad C(A, A \cap D) \to A \cup (A \cap D) \times I \cup B$$

are weak equivalences by the 5-lemma and [31, 5.6.2]. We use a representation of $(D, A \cap D)$ as NDR-pair to construct a homotopy inverse of the inclusion

$$\iota : A \cup (A \cap D) \times I \cup B \to X \cup D \times I \cup B.$$

Recall that a representation as NDR-pair is given by maps

$$u : D \to I \qquad h : D \times I \to D$$

such that

$$u^{-1}(0) = A \cap D, \qquad h_1(u^{-1}[0, 1)) = A \cap D, \qquad h_0 = id, \qquad h_{t|A \cap D} = id.$$

Now, a homotopy inverse of ι is

$$
\begin{array}{ccccc}
(A \cup D) \cup & D \times I & \cup B & \to & A \cup \qquad (A \cap D) \times I \qquad \cup B \\
a & & & \mapsto & a \\
& d & & \mapsto & (h_1(d), u(d)) \\
& (d, t) & & \mapsto & (h_1(d), max(t, u(d))) \\
& & b & \mapsto & b.
\end{array}
$$

This shows tht ι is a weak equivalence of ex-spaces and therefore induces an isomorphism. To give the proof for general X, one uses CW-substitutes. The crucial observation is that if $X = \overset{\circ}{A} \cup \overset{\circ}{D}$ and one has CW-substitutes A' of A and D' of D which are obtained as extensions of a CW-substitute $(A \cap D)'$ of $A \cap D$, then $A' \cup D'$ is a CW-substitute of X. $\qquad\square$

Thus, E^* does indeed define a cohomology theory on pairs.

Definition 42 *A triad* $(X; A, D)$ *over* B *is called proper with respect to* E *if the inclusions*

$$i_A : (A, A \cap D) \to (A \cup D, D) \quad and \quad i_D : (D, A \cap D) \to (A \cup D, A)$$

induce isomorphisms

$$i_A^* : E^*(A, A \cap D) \to E^*(A \cup D, D) \quad and \quad i_D^* : E^*(D, A \cap D) \to E^*(A \cup D, A).$$

The Mayer-Vietoris sequence, the exact sequence of a triple and the exact sequence of a proper triad follow formally from Lemma 40 and 41.

Lemma 43 *If* $A \subset D \subset X$*, the sequence*

$$... \to E^{n-1}(A, D) \overset{\Delta}{\to} E^n(X, A) \to E^n(X, D) \to E^n(A, D) \to ...$$

where Δ *is the composition*

$$E^{n-1}(A, D) \to E^{n-1}(A) \overset{\delta^n}{\to} E^n(X, A)$$

is exact.

Lemma 44 *If* $X = \overset{\circ}{A} \cup \overset{\circ}{D}$*, the sequence*

$$... \to E^{n-1}(A \cup D) \overset{\Delta}{\to} E^n(X) \to E^n(A) \oplus E^n(D) \to E^n(A \cup D) \to ...$$

where Δ *is the composition*

$$E^{n-1}(A \cup D) \overset{\delta^n}{\to} E^n(A, A \cup D) \cong E^n(X, D) \to E^n(X)$$

is exact.

Lemma 45 *If* $(X; A, D)$ *is a proper triad, the sequence*

$$... \to E^{n-1}(A, A \cap D) \overset{\Delta}{\to} E^n(X, A \cup D) \to E^n(X, D) \to E^n(A, A \cap D) \to ...$$

where Δ *is the composition*

$$E^{n-1}(A, A \cap D) \cong E^{n-1}(A \cup D, D) \to E^{n-1}(A \cup D) \overset{\delta^n}{\to} E^n(X, A \cup D)$$

is exact.

There is an additional feature of parametrised cohomology theories which does not show up in the classical ones, namely the following.

Definition 46 *If $q, r : X \to B$ are homotopic via $h : X \times I \to B$, the isomorphism induced by h is*

$$\psi(h) := i_1^*(i_0^*)^{-1} : E^n(X, A, q) \to E^n((X, A) \times I, h) \to E^n(X, A, r).$$

In particular, maps $h : X \times S^1 \to B$ yield isomorphisms

$$\psi(h) : E^n((X, A), h_{|X \times \{1\}}) \to E^n((X, A), h_{|X \times \{1\}}).$$

Lemma 47 *If two homotopies h_0, h_1 are homotopic through H, they induce the same isomorphism. If h is the composition of the homotopies h_0 and h_1, $\psi(h) = \psi(h_1)\psi(h_0)$.*

Proof: Consider the diagram

$$
\begin{array}{ccccc}
((X, A), q) & \xrightarrow{i_0} & ((X, A) \times I, h_0) & \xleftarrow{i_1} & ((X, A), r) \\
& & \downarrow{\scriptstyle j_0} & & \\
& & ((X, A) \times I^2, H) & & \\
& & \uparrow{\scriptstyle j_1} & & \\
((X, A), q) & \xrightarrow{i_0} & ((X, A) \times I, h_1) & \xleftarrow{i_1} & ((X, A), r).
\end{array}
$$

Observe that

$$j_0 \circ i_0, j_1 \circ i_0 : ((X, A), q) \to ((X, A) \times I^2, H)$$

as well as

$$j_0 \circ i_1, j_1 \circ i_1 : ((X, A), r) \to ((X, A) \times I^2, H)$$

are homotopic over B and therefore

$$(j_0 \circ i_0)^* = (j_1 \circ i_0)^* \qquad \text{and} \qquad (j_0 \circ i_1)^* = (j_1 \circ i_1)^*.$$

Thus,

$$\psi(h_0) = i_1^*(i_0^*)^{-1} = (j_0 \circ i_1)^*((j_0 \circ i_0)^*)^{-1} = (j_1 \circ i_1)^*((j_1 \circ i_0)^*)^{-1} = \psi(h_1).$$

The proof for the second statement is similar. $\square$

2.2   Product structures

Recall the monoidal structure $\underline{p}$ on $Ho\mathcal{S}_B$ for base spaces with an action by an E_∞-operad from Proposition 32. Note that the same construction can be carried through with $\mathcal{K}_B$ instead of $\mathcal{S}_B$. This defines a monoidal structure on $Ho\mathcal{K}_B$, which we again denote by $\underline{p}$. Observe that there are natural isomorphisms

$$L\Sigma^\infty_{V \oplus W} \circ \underline{p} \cong \underline{p} \circ (L\Sigma^\infty_V \times L\Sigma^\infty_W),$$

justifying the use of the same notation for the monoidal structures on spaces and spectra.

Definition 48 *A parametrised ring spectrum over B is a monoid in $(Ho\mathcal{S}_B, \underline{p})$.*

We use parametrised ring spectra to define products in parametrised cohomology theories, mimicking the use of ring spectra in the construction of multiplicative theories as in [1, 33]. There are three different products, the wedge product for the reduced theories and the external and internal (or cup) product for the unreduced ones. All of these are very similar to their counterparts in the unparametrised case. Since the latter are discussed extensively in the literature and the proofs carry over to our setting, we will make it short and give little more than the definitions. The formal reason why the proofs carry over is the following. In [30], the definition when a triangulation and a closed symmetric monoidal structure on a category are compatible is given. Moreover, he shows how to prove compatibility if the category in question is the stable homotopy category of a topological model category and the symmetric monoidal structure is obtained from a Quillen bifunctor. The parametrised stable homotopy category $Ho\mathcal{S}_B$ is triangulated [31, 13.1.5] and the symmetric monoidal structure $\underline{p}$ on $Ho\mathcal{S}_B$ is obtained from a Quillen bifunctor, hence one can follow [30] to proof compatibility.

Definition 49 *Let (E, μ) be a parametrised ring spectrum. The product*

$$\wedge_p : \tilde{E}^n(X) \times \tilde{E}^m(Y) \to \tilde{E}^{n+m}(\underline{p}(X, Y))$$

in $\tilde{E}$ is given by

$$
\begin{aligned}
\wedge_p : [L\Sigma^\infty_{\mathbb{R}^n} X, E] \times [L\Sigma^\infty_{\mathbb{R}^m} Y, E] \;\;&\xrightarrow{\underline{p}}\;\; [\underline{p}(L\Sigma^\infty_{\mathbb{R}^n} X, L\Sigma^\infty_{\mathbb{R}^m} Y), \underline{p}(E, E)] \\
&\cong\;\; [L\Sigma^\infty_{\mathbb{R}^{n+m}} \underline{p}(X, Y), \underline{p}(E, E)] \\
&\xrightarrow{\mu_*}\;\; [L\Sigma^\infty_{\mathbb{R}^{n+m}} \underline{p}(X, Y), E].
\end{aligned}
$$

The wedge product has the same properties as its non-parametrised analogue, i.e. it is associative, additive in each variable, commutes with suspension and

so on.

Since $\theta(p)_!$ is a left Quillen functor, the natural weak eqivalences of cofibrant objects

$$C(X,A)\overline{\wedge}C(Y,D)\xrightarrow{\sim}C(X\overline{\times}Y, X\overline{\times}D \cup A\overline{\times}Y)$$

yield weak equivalences

$$\underline{p}(C(X,A), C(Y,D))\xrightarrow{\sim}\theta(p)_!C(X\overline{\times}Y, X\overline{\times}D \cup A\overline{\times}Y).$$

We abuse notation and write $(X,A)\overline{\times}(Y,D) := (X\overline{\times}Y, A\overline{\times}Y \cup X\overline{\times}D)$. Using the above equivalences, we obtain the external product

$$\times_p : E^n(X,A) \times E^m(Y,D) \to E^{n+m}\theta(p)_!((X,A)\overline{\times}(Y,D))$$

and again, this product shares the main features of its non-parametrised analogue. Moreover, the product is compatible with the isomorphisms induced by homotopies of twists in the following sense. If $r, r' : Y \to B$ are homotopic via $h : Y \times I \to B$,

$$\theta(p) \circ (q \times h) : X \times Y \times I \to B$$

is a homotopy from $\theta(p) \circ (q \times r)$ to $\theta(p) \circ (q \times r')$ and we have the following commutative diagram:

$$
\begin{array}{ccc}
E^n((X,A),q) \times E^m((Y,D),r) & \xrightarrow{\;\;\times_p\;\;} & E^{n+m}(\theta(p)_!((X,A)\overline{\times}(Y,D), q \times r)) \\
\Big\downarrow{\scriptstyle 1\times\psi(h)} & & \Big\downarrow{\scriptstyle \psi(p\circ(q\times h))} \\
E^n((X,A),q) \times E^m((Y,D),r') & \xrightarrow{\;\;\times_p\;\;} & E^{n+m}(\theta(p)_!((X,A)\overline{\times}(Y,D), q \times r')).
\end{array}
$$

Finally, we turn to the parametrised analogue of internal products. Note that the $\mathcal{P}$-action equips $[X,B]$ with the structure of an abelian group, where the sum is given by

$$[q] + [r] = [q +_p r] := [p \circ (q \times r) \circ \Delta]$$

and the neutral element is the constant map $pr_0 : X \to * \xrightarrow{i_*} B$ onto the homotopy unit. To define internal products, we need parametrised analogues of the diagonal

$$\Delta : (X, A \cup D) \to (X,A) \times (X,D)$$

and the sum of twists given by the $\mathcal{P}$-action is exactly what is needed to make this map a morphism over B. If $q, r : X \to B$ are two twists, we equip $(X, A \cup D)$ with the twist $q +_p r$, obtaining

$$\Delta : ((X, A \cup D), q +_p r) \to \theta(p)_!((X,A) \times (X,D), q \times r)$$

which yields the internal product

$$\cup_p := \Delta^* \circ \times_p : E^n((X, A), q) \times E^m((X, D), r) \to E^{n+m}((X, A \cup D), q +_p r).$$

The $\mathcal{P}$-action can be used to define associativity and unit homotopies h_{ass} and h_1, connecting $q +_p (r +_p s)$ to $(q +_p r) +_p s$ and $q +_p pr_o$ to q. These are unique up to higher homotopies, so induce unique isomorphisms

$$\psi(h_{ass}) : E^*((X, A), q +_p (r +_p s)) \to E^*((X, A), (q +_p r) +_p s)$$

and

$$\psi(h_1) : E^*((X, A), q +_p pr_0) \to E^*((X, A), q).$$

We will henceforth make frequent use of these isomorphisms without further mention. For example, combining the unit isomorphism and the cup product makes the untwisted groups $E^*((X, A), pr_0)$ a ring. Similarly, the twisted groups $E^*((X, A), q)$ are $E^*((X, A), pr_0)$-modules for any twist $q : X \to B$.

2.3   Orientation

Let $\pi : X \to M$ be a vector bundle. We denote the restriction of X to $U \subset M$ by X_U and the complement of the zero section by X'.

Definition 50 *Let $\pi : X \to M$ be a vector bundle over $M \xrightarrow{q} B$ and $\|.\|$ a fibrewise norm on X. The parametrised Thom space is*

$$Th_B(X, q) := X/_B(X \setminus D),$$

where $D \subset X$ is the open unit disc bundle.

We will frequently omit the twist from our notation. Since all bundles are assumed to be bundles over B, we use the same letter for the twist on M and for the corresponding twist on X. Of course, $Th_B(X)$ is weakly equivalent to the cone $C(X, X')$ which was defined in Definition 39. From now on, we assume that the base space B is connected, i.e. that any constant map is homotopic to the constant map pr_0 onto the homotopy unit of B. This implies that

$$E^{i+n}(X_m, X'_m, q_{|m}) \cong \tilde{E}^{i+n}(S^n, pr_{q(m)}) \cong \tilde{E}^i(S^0, pr_0)$$

for any n-dimensional vector bundle $X \to M$ and any $m \in M$. In particular, $E^*((X_m, X'_m, q_{|m}))$ is a free $\tilde{E}^*(S^0, pr_0)$-module with one generator of degree n. Note that the above isomorphism depends on the choice of a path between $q(m)$ and the homotopy unit.

Definition 51 *A vector bundle $X^n \to M$ over $q : M \to B$ is called (E, q)-oriented by $u \in E^n(X, X')$ if $i_m^* u$ is a generator of $E^*(X_m, X_m')$ as $\tilde{E}^*(S^0, pr_0)$-module.*

Theorem 52 *If u is an (E, q)-orientation of $X^n \to M$, and M is a finite CW-complex, the homomorphism*

$$Th(u) : E^i(M, r) \xrightarrow{\pi^*} E^i(X, r) \xrightarrow{\cup u} E^{i+n}((X, X'), r +_p q) \cong \tilde{E}^{i+n}(Th_B(X, r +_p q))$$

is an isomorphism for all $r : M \to B$.

Proof: We examine the spectral sequences of the fibrations

$$M \xrightarrow{id} M \qquad \text{and} \qquad X \to M$$

and observe that $Th(u)$ induces a morphism of spectral sequences which is an isomorphism on the first pages. Let us first recall the construction of the spectral sequence. Denote the preimage of the s-skeleton M^s by X^s. The exact sequence of the triad $(X^s; X^{s-1}, X'^s)$ is

$$\begin{aligned} \ldots \quad &\to \quad E^{n-1}(X^{s-1}, X'^{s-1}) \to E^n(X^s, X'^s \cup X^{s-1}) \\ &\to \quad E^n(X^s, X'^s) \to E^n(X^{s-1}, X'^{s-1}) \to E^{n+1}(X^s, X'^s \cup X^{s-1}) \to \ldots \end{aligned}$$

and thus, the groups

$$A^{s,t} := E^{s+t}(X^s, X'^s), \quad C^{s,t} := E^{s+t}(X^s, X'^s \cup X^{s-1})$$

form an exact couple. The standard machinery constructs a spectral sequence with E_1-page equal to $C^{s,t}$ and converging to $E^*(X, X')$.
For each s-cell $j : D^s \to M$, we denote the bundle $j^* X$ by X_{D^s} and regard D^s as a disc over B with the twist given by the pull-back of the twist of M. The map j yields a map of ex-spaces

$$C(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \to C(X^s, X^{s-1} \cup X'^s).$$

Taking the coproduct over all cells, we obtain

$$\coprod_{D^s \subset M^s} C(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \to C(X^s, X^{s-1} \cup X'^s).$$

To show that this map is a weak equivalence we use a representation of (X^s, X^{s-1}) as an NDR-pair, i.e. maps

$$u : X^s \to I \qquad h : X^s \times I \to X^s$$

such that

$$u^{-1}(1) = X^{s-1}, \quad h_{t|X^{s-1}} = id, \quad h_0 = id, \quad h_1(u^{-1}(0,1]) = X^{s-1}.$$

We use this representation to define

$$
\begin{array}{ccccc}
X^s \cup & (X^{s-1} \cup X'^s) \times I & \cup B & \to & \coprod_{D^s} X_{D^s} \cup & (X_{S^{s-1}} \cup X'_{D^s}) \times I & \cup B \\
x & & & \mapsto & & (h_1(x), u(x)) & \\
& (x,t) & & \mapsto & & (h_1(x), max(t, u(x))) & \\
& & b & \mapsto & & & b.
\end{array}
$$

Since for $x \in X^s \setminus X^{s-1}$ there is a unique cell such that $x \in X_{D^s}$ and $X^{s-1} \times I$ is mapped to B, this is welldefined. Note that for general twists, the map is not (and cannot be chosen to be) a morphism of ex-spaces. Nonetheless, it is a homotopy inverse of the map induced by

$$\coprod_{D^s \subset M^s} (X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \to (X^s, X^{s-1} \cup X'^s)$$

which is a morphism of ex-spaces. Therefore, we get an isomorphism

$$E^*(X^s, X^{s-1} \cup X'^s) \cong \bigoplus_{D^s \subset M^s} E^*(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}).$$

To analyse the groups $E^*(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s})$, denote the upper half disc by $D_+^{s-1} \subset S^{s-1}$. For any cell $D^s \subset M^s$, the exact sequence of the triple $(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}, X_{D_+^{s-1}} \cup X'_{D^s})$ is

$$
\begin{aligned}
\dots \to\ & E^{n-1}(X_{S^{s-1}} \cup X'_{D^s}, X_{D_+^{s-1}} \cup X'_{D^s})) \to E^n(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \\
\to\ & E^n(X_{D^s}, X_{D_+^{s-1}} \cup X'_{D^s}) \to E^n(X_{S^{s-1}} \cup X'_{D^s}, X_{D_+^{s-1}} \cup X'_{D^s}) \to \dots
\end{aligned}
$$

Since the inclusion

$$X_{D_+^{s-1}} \cup X'_{D^s} \to X_{D^s}$$

is a weak equivalence, we obtain isomorphisms

$$E^{i-1}(X_{S^{s-1}} \cup X'_{D^s}, X_{D_+^{s-1}} \cup X'_{D^s})) \cong E^i(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}),$$

while excision yields

$$E^{i-1}(X_{S^{s-1}} \cup X'_{D^s}, X_{D_+^{s-1}} \cup X'_{D^s})) \cong E^{i-1}(X_{D_-^{s-1}}, X_{\partial D_-^{s-1}} \cup X'_{D_-^{s-1}}).$$

Combining these isomorphisms, we get

$$E^{i+s}(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \cong E^{i+s-1}(X_{D_-^{s-1}}, X_{\partial D_-^{s-1}} \cup X'_{D_-^{s-1}}) \cong \dots \cong E^i(X_{v_s}, X'_{v_s})$$

with v_s being the south pole of D^s. Summarising the above discussion,

$$C^{s,t} \cong \bigoplus_{D^s \subset M^s} E^*(X_{D^s}, X_{S^{s-1}} \cup X'_{D^s}) \cong \bigoplus_{D^s \subset M^s} E^t(X_{v_s}, X'_{v_s}).$$

Similarly, the spectral sequence of the fibration $M \to M$ is constructed from the exact couple

$$A_M^{s,t} := E^{s+t}(M^s), \qquad C_M^{s,t} := E^{s+t}(M^s, M^{s-1}),$$

and we have isomorphisms

$$C_M^{s,t} \cong \bigoplus_{D^s \subset M^s} E^t(v_s).$$

The restriction of $Th(u)$ to the respective groups defines a morphism of exact couples. Now, for any $r : M \to B$, we have

$$
\begin{array}{ccc}
C_M^{s,t} & \xrightarrow{\quad Th(u) \quad} & C^{s,t} \\
\Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} \\
\bigoplus_{D^s} E^t(v_s) \xrightarrow{\pi^*} \bigoplus_{D^s} E^t(X_{v_s}) \xrightarrow{\cup i^*_{v_s} u} & \bigoplus_{D^s} E^{t+n}(X_{v_s}, X'_{v_s}). &
\end{array}
$$

The bottom horizontal line is an isomorphism by assumption. The vertical isomorphisms were constructed using only excision and the exact sequence of a triple, thus the diagram commutes. $\qquad\square$

Notice that the only part of the proof where we used the finiteness assumption on M was the convergence of the spectral sequence. If the spectral sequence converges for other reasons (for example, if the coefficients of the cohomology theory are bounded), the theorem is true for arbitrary M.

The notion of orientation is stable in the sense that if $u \in E^n(X, X')$ is a q-orientation, then $\sigma^m(u) \in E^{n+m}(X \oplus \mathbb{R}^m, (X \oplus \mathbb{R}^m)')$ is a q-orientation as well. Sums and pull-backs of oriented bundles are again oriented in the following way. If $u \in E^n((X, X'), q)$ is an (E, q)-orientation of a bundle $X \to M$, f^*u is an (E, f^*q)-orientation of f^*X for all $f : N \to M$. If $Y \to M$ is another bundle over M and $v \in E^m((Y, Y'), r)$ is its orientation, $\Delta^*(u \times_p v)$ is an $(E, q +_p r)$-orientation of $X \oplus Y \cong \Delta^*(X \times_p Y)$ with the diagonal $\Delta : M \to M \times M$.

2.4 Integration

We will now discuss parametrised push-forward homomorphisms for oriented maps. Let us first recall the construction of push-forwards in the non-parametrised setting. Starting with a smooth proper map $f : (M, f^*q) \to (N, q)$,

we choose a closed embedding

$$\tilde{f} : M \xrightarrow{\quad f \times \iota \quad} N \times \mathbb{R}^m$$
$$\downarrow{\pi_N}$$
$$f \searrow \quad N$$

for some large m. We will frequently assume that the image of $\tilde{f}$ is contained in $N \times D^m$. Note that any two such embeddings are stably isotopic. This implies that the normal bundle $\nu(\tilde{f})$ is, up to stable isomorphism, independent of the chosen map ι. The stable isomorphism class of $\nu(\tilde{f})$ is called stable normal bundle. f is called oriented if the stable normal bundle of f is oriented; since an orientation class of a bundle defines orientation classes on the stabilisations of the bundle, this notion does make sense. Now, let $\phi : \nu(\tilde{f}) \xrightarrow{\sim} U \subset N \times \mathbb{R}^m$ be a tubular neighbourhood. Using ϕ, one constructs a collapsing map

$$\hat{f} : N_+ \wedge S^m \to Th(\nu(\tilde{f}))$$

and defines

$$f_! : E^*(M) \xrightarrow{Th(u)} \tilde{E}^*(Th(\nu(f))) \xrightarrow{\hat{f}^*} \tilde{E}^*(N_+ \wedge S^m) \cong E^*(N).$$

We will now mimic the construction of $f_!$ in the parametrised setting.

Definition 53 *A smooth proper map $f : M \to N$ is called q-oriented for a twist $q : N \to B$ if (a representative of) the stable normal bundle of f is $f^*(q)$-oriented.*

Note that we demand that the twist of $\nu(\tilde{f})$ is pulled back from N. The main difficulty lies in the fact that for general twists q, ϕ can't be chosen as a map over B. However, the choice of a fibrewise norm $\|.\|$ on $\nu(\tilde{f})$ enables us to deform $q : N \times \mathbb{R}^n \to B$ so that at least the restriction of ϕ to the unit disc bundle is indeed a map over B. The idea is to make q constant along the fibres of the embedded normal bundle. More precisely, define $\tilde{q}^{\|.\|} : N \times \mathbb{R}^m \to B$ by

$$\tilde{q}^{\|.\|}(n,x) := \begin{cases} q(f(m)) & if & (n,x) = \phi(m,v), \|v\| \le 1 \\ q(\pi_N \phi(m, (\|v\| - 1)v)) & if & (n,x) = \phi(m,v), 1 \le \|v\| \le 2 \\ q(n) & else. \end{cases}$$

If we compose ϕ with a homeomorphism from $\nu(\tilde{f})$ to the open unit disc bundle $D \subset \nu(\tilde{f})$, we obtain a tubular neighbourhood over B. Note that we have a homotopy $h : N \times \mathbb{R}^m \times I \to N \times \mathbb{R}^m$ such that h_1 is the identitiy and $q \circ h_0 = \tilde{q}^{\|.\|}$. We define $\tilde{r}^{\|.\|} := r \circ h_0$ for any twist $r : N \to B$. Observe that

$$\widetilde{r +_p r'}^{\|.\|} = \tilde{r}^{\|.\|} +_p \tilde{r'}^{\|.\|}.$$

Write $Y := N \times \mathbb{R}^m$, $Y^{c_{\|\cdot\|}} := N \times \mathbb{R}^m \setminus \phi(D)$ and define the parametrised Pontrjagin-Thom map

$$\hat{f}^{\|\cdot\|} : (Y/_B Y^{c_{\|\cdot\|}}, \tilde{q}^{\|\cdot\|}) \to (Th_B(\nu(\tilde{f}), f^*q)), \quad y \mapsto \begin{cases} \tilde{q}^{\|\cdot\|}(y) & y \notin \phi(D) \\ \phi^{-1}(y) & y \in \phi(D) \end{cases}$$

By the construction of $\tilde{q}^{\|\cdot\|}$, this is a map of ex-spaces. Now, we define

$$
\begin{aligned}
f_! : E^*(M, f^*r) & \overset{Th(u)}{\to} & \tilde{E}^* Th_B(\nu(\tilde{f}), f^*r +_p f^*q) \overset{\hat{f}^{\|\cdot\|*}}{\to} \tilde{E}^*(Y/_B Y^{c_{\|\cdot\|}}, \widetilde{r +_p q}^{\|\cdot\|}) \\
& \cong & E^*((Y, Y^{c_{\|\cdot\|}}), \widetilde{r +_p q}^{\|\cdot\|}) \overset{\psi(h)}{\to} E^*((Y, Y^{c_{\|\cdot\|}}), r +_p q) \\
& \to & E^*(N \times \mathbb{R}^m, N \times (\mathbb{R}^m \setminus D^m)), r +_p q) \\
& \cong & \tilde{E}^*(\Sigma^m N_+, r +_p q) \cong E^*(N, r +_p q).
\end{aligned}
$$

Proposition 54 *$f_!$ is independent of the choice of the fibrewise norm, the embedding and the tubular neighbourhood.*

Proof: Let us first show that it is independent of the fibrewise norm. If $\|.\|_0, \|.\|_1$ are two fibrewise norms on $\nu(\tilde{f})$, we can find a path G in the space of norms connecting $\|.\|_0$ and $\|.\|_1$. We use G to show that the morphism

$$
\begin{aligned}
E^*(N \times \mathbb{R}^m, N \times (\mathbb{R}^m \setminus D^m)) & \to & E^*(Y, Y^{c_{\|\cdot\|_i}}) \overset{\psi(h_{\|\cdot\|_i})}{\to} E^*(Y, Y^{c_{\|\cdot\|_i}}) \\
& \to \tilde{E}^*(Y/_B Y^{c_{\|\cdot\|_i}}) \overset{(\hat{f}^{\|\cdot\|_i})^*}{\to} & \tilde{E}^*(Th_B^{\|\cdot\|_i}(\nu(\tilde{f}))) \overset{\sim}{\leftarrow} E^*(\nu(\tilde{f}), \nu(\tilde{f})')
\end{aligned}
$$

is the same for $\|.\|_0$ and $\|.\|_1$. Set

$$D^G := \{(v, t) \mid \|v\|^{G(t)} < 1\} \subset \nu(\hat{f}) \times I$$

and

$$(Y \times I)^{c_G} := N \times \mathbb{R}^m \times I \setminus (\phi \times id_I)(D^G).$$

$\|.\|_0$ corresponds to the upper path in the following diagram, whereas the bottom path corresponds to $\|.\|_1$. The middle squares commute and the left and right triangle commute up to homotopy, thus the two morphisms agree in $Ho\mathcal{K}_B$.

$$
\begin{array}{ccc}
& C(Y, Y^{c_{\|\cdot\|_0}}) & \xrightarrow{\ i_0\ } C((Y, Y^{c_{\|\cdot\|_0}}) \times I) \longleftarrow \\[2mm]
& \downarrow{\scriptstyle\sim} & \downarrow{\scriptstyle\sim} \\[2mm]
C(N \times \mathbb{R}^m, N \times (\mathbb{R}^m \setminus D^m)) & C(Y \times I, (Y \times I)^{c_G}) \longrightarrow C(Y \times I, (Y \times I)^{c_G}) \times I) \longleftarrow \\[2mm]
& \uparrow{\scriptstyle\sim} & \uparrow{\scriptstyle\sim} \\[2mm]
& C(Y, Y^{c_{\|\cdot\|_1}}) & \xrightarrow{\ i_0\ } C(Y, Y^{c_{\|\cdot\|_1}}) \times I \longleftarrow
\end{array}
$$

$$
\begin{array}{ccccc}
\longleftarrow{\scriptstyle i_1} & C(Y, Y^{c_{\|\cdot\|_0}}) & \xrightarrow{\ \sim\ } & Y/_B Y^{c_{\|\cdot\|_0}} & \xrightarrow{\ \hat{f}^{\|\cdot\|_0}\ } Th_B^{\|\cdot\|_0}(\nu(\hat{f})) \\[2mm]
& \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & \downarrow{\scriptstyle\sim} \\[2mm]
\longleftarrow & C(Y \times I, (Y \times I)^{c_G}) & \longrightarrow & Y \times I/_B(Y \times I)^{c_G} & \longrightarrow \nu(\tilde{f}) \times I/_B(\nu(\tilde{f}) \times I \setminus D^G) \qquad C(\nu(\tilde{f}), \nu(\tilde{f})') \\[2mm]
& \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & \uparrow{\scriptstyle\sim} \\[2mm]
\longleftarrow{\scriptstyle i_1} & C(Y, Y^{c_{\|\cdot\|_1}}) & \xrightarrow{\ \sim\ } & Y/_B Y^{c_{\|\cdot\|_1}} & \xrightarrow{\ \hat{f}^{\|\cdot\|_1}\ } Th_B^{\|\cdot\|_1}(\nu(\hat{f}))
\end{array}
$$

The proof that different embeddings and tubular neighbourhoods yield the

same $f_!$ is similar. If

$$\iota_0 : M \to D^{m_0}, \qquad \iota_1 : M \to D^{m_1}$$

are two embeddings and

$$\phi_0 : \nu(\tilde{f}^0) \to U_0 \subset N \times D^{m_0}, \qquad \phi_1 : \nu(\tilde{f}^1) \to U_1 \subset N \times D^{m_1}$$

tubular neighbourhoods, they are (stably) isotopic, i.e. after stabilising, there are $\iota : M \times I \to D^m \times I, \Phi : \nu(\tilde{f}) \times I \to U \subset N \times D^m \times I$ from (ι_0, ϕ_0) to (ι_1, ϕ_1). As in the proof for the different norms, these isotopies are used to show that the different embeddings give the same $f_!$. $\square$

Since the push-forward is indpendent of the chosen norm, we will henceforth only include the norm in the notation if it is relevant. Otherwise, we write $\hat{f}, Y^c, \tilde{q}, h$ instead of $\hat{f}^{\|\cdot\|}, Y^{c_{\|\cdot\|}}, \tilde{q}^{\|\cdot\|}, h^{\|\cdot\|}$.

Theorem 55 *If f is q-oriented,*

$$f_!(f^*(x) \cup y) = x \cup f_! y$$

for all $x \in E^*(N, r)$, $y \in E^*(M, f^*s)$.

Proof The proof is very similar to the one in the unparametrised case, see e.g [16]. Let us first show that

$$\hat{f}^*(\pi_M^* f^* x \cup y') = \psi(h)^{-1} \pi_N^* x \cup \hat{f}^* y' \ \forall x \in \tilde{E}^*(N, r), y' \in \tilde{E}^*(Th_B(\nu(\tilde{f}), f^*s)),$$

where h is the homotopy needed to make the tubular neighbourhood a map over B. To see this, observe that the following diagram commutes.

Starting with $x \in \tilde{E}^*(N, r), y' \in \tilde{E}^*(Th_B(\nu(\tilde{f}), f^*s))$, following the arrows on the left it is mapped to $\hat{f}^*(\pi_M^* f^* x \cup y')$ whereas it is mapped to $\psi(h)^{-1} \pi_N^* x \cup \hat{f}^* y'$ by the arrows on the right, proving the equality

$$\hat{f}^*(\pi_M^* f^* x \cup y') = \psi(h)^{-1} \pi_N^* x \cup \hat{f}^* y'.$$

Recall that

$$f_!(x') := \sigma^{-n} i^* \psi(h) \hat{f}^*(\pi_M^* x \cup u)$$

with $i : (N \times \mathbb{R}^m, N \times (\mathbb{R}^m - D^m)) \to (Y, Y^{c_g})$. Using the above equality,

$$
\begin{aligned}
f_!(f^* x \cup y) &= \sigma^{-n} i^* \psi(h) \hat{f}^*(\pi_M^*(f^* x \cup y) \cup u) \\
&= \sigma^{-n} i^* \psi(h) \hat{f}^*(\pi_M^* f^* x \cup (\pi_M^* y \cup u)) \\
&= \sigma^{-n}(\pi_N^* x \cup i^* \psi(h) \hat{f}^*(\pi_M^* y \cup u)) \\
&= x \cup f_! y.
\end{aligned}
$$

$\square$

Let us now discuss the functoriality of the push-forward, so take

$$f : M \to N, \ g : N \to L$$

smooth and assume that f is $g^* q$-oriented by $u \in E^*((\nu(\tilde{f}), \nu(\tilde{f})'), f^* g^* q)$ and g is r-oriented by $v \in E^*((\nu(\tilde{g}), \nu(\tilde{g})'), g^* r)$. We choose tubular neighbourhoods

$$\phi^M : \nu(\tilde{f}) \to N \times \mathbb{R}^m \qquad \phi^N : \nu(\tilde{g}) \to L \times \mathbb{R}^n$$

and fibrewise norms on $\nu(\tilde{f}), \nu(\tilde{g})$. To see that there is an induced orientation of $g \circ f$, observe that the normal bundle of

$$\widetilde{gf} := (\tilde{g} \times 1) \circ \tilde{f} : M \to L \times \mathbb{R}^{n+m}$$

is

$$\nu(\widetilde{gf}) \cong \nu(\tilde{f}) \oplus \tilde{f}^* \nu(\tilde{g} \times 1) \cong \nu(\tilde{f}) \oplus f^* \nu(\tilde{g}).$$

We denote the induced orientation by

$$uv := \Delta^*(u \times_p f^* v) \in E^*((\nu(\widetilde{gf}), \nu(\widetilde{gf})'), f^* g^*(q +_p r)).$$

Moreover, the tubular neighbourhoods of $\tilde{f}$ and $\tilde{g}$ can be used to construct a tubular neighbourhood of $\widetilde{gf}$ as follows. For any $w \in \nu(\tilde{f})_m$, there is a natural path $\tilde{\gamma}_w$ joining $\tilde{f}(m)$ and $\phi^M(w)$, namely

$$\tilde{\gamma}_w : I \to N \times \mathbb{R}^m, \ t \mapsto \phi^M(tw).$$

Choose a connection and thus the notion of parallel transport on $\nu(\tilde{g} \times 1) \cong \nu(\tilde{g}) \times \mathbb{R}^n$ which is compatible with the chosen norm on $\nu(\tilde{g})$. We denote the isomorphism given by parallel transport along $\tilde{\gamma}_w$ by

$$\gamma_w : \nu(\tilde{g} \times 1)_{|\tilde{f}(m)} \to \nu(\tilde{g} \times 1)_{|\phi^m(w)}.$$

A tubular neighbourhood is

$$\phi := (\phi^N \times 1) \circ \phi^1 : \nu(\widetilde{gf}) \to L \times \mathbb{R}^{n+m}$$

with

$$
\begin{array}{rcl}
\phi^1 : \nu(\widetilde{gf}) \cong \nu(\tilde{f}) + \tilde{f}^*\nu(\tilde{g} \times 1) & \to & \nu(\tilde{g}) \times \mathbb{R}^m \cong \nu(\tilde{g} \times 1) \\
(w_1, w_2) & \mapsto & \gamma_{w_1}(w_2).
\end{array}
$$

Theorem 56 $(g \circ f)_! = g_! \circ f_!$ *for the induced orientation of* gf.

Proof: We use the tubular neighbourhood ϕ defined above and the norm on

$$\nu(\tilde{gf}) = \nu(\tilde{f}) + f^*\nu(\tilde{g}) \qquad \text{given by} \qquad \|w_1 + w_2\| := max\{\|w_1\|, \|w_2\|\}.$$

The advantage of this norm is that the various adjustments of the twists needed to make the tubular neighbourhoods maps over B and the homotopies between the original twists and the deformed ones are compatible in the following sense. The homotopy

$$h^M : N \times \mathbb{R}^m \times I \to N \times \mathbb{R}^m$$

making ϕ^M a map over B is just the restriction of the homotopy h^1 which turns ϕ^1 into a map over B. We abuse notation and denote all homotopies by h. Since $(g \circ f)_!$ depends only on the orientation, it is sufficient to prove the equality for these choices. We write D for the unit disc bundle of a Riemannian vector bundle and

$$
\begin{array}{ccc}
Y := N \times \mathbb{R}^m & Y^c := N \times \mathbb{R}^m \setminus \phi^M(D) & Z := \nu(\tilde{g} \times 1) \\
Z^c := \nu(\tilde{g} \times 1) \setminus \phi^1(D) & W := L \times \mathbb{R}^{n+m} & W^c := L \times \mathbb{R}^{n+m} \setminus \phi(D) \\
X := L \times \mathbb{R}^n & X^c := L \times \mathbb{R}^n \setminus \phi^N(D) & \\
TF := Th_B(\nu(\tilde{f})) & TG := Th_B(\nu(\tilde{g})) & TGF := Th_B(\nu(\widetilde{gf})).
\end{array}
$$

$\hat{gf} : W/_B W^c \overset{G}{\to} Z/_B Z^c \overset{F}{\to} TGF$ factors over $Z/_B Z^c$, using the factorisation of $\phi = (\phi^N \times 1) \circ \phi^1$. In the following commutative diagramm, the left hand vertical morphism is $f_!$, the bottom horizontal one is $g_!$ and the diagonal is

$(g \circ f)!:$

$E^*(M)$

$\xrightarrow{Th(uv)}$

$\downarrow Th(u)$

$\tilde{E}^*(TF) \xrightarrow{\wedge f^*v} \tilde{E}^*\underline{p}(TF, Th_B f^*\nu(\tilde{g})) \xrightarrow{\Delta^*} \tilde{E}^*(TGF)$

$\downarrow \hat{f}^* \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow F^* \qquad \searrow \hat{g}\hat{f}^*$

$\tilde{E}^*(Y/_B Y^c) \xrightarrow{\wedge_p v} \tilde{E}^*(\underline{p}(Y/_B Y^c, TG)) \xrightarrow{\Delta^*} \tilde{E}^*(Z/_B Z^c) \xrightarrow{G^*} \tilde{E}^*(W/_B W^c)$

$\downarrow \psi(h) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \psi(h) \qquad\qquad \searrow \psi(h)$

$\tilde{E}^*(Y/_B Y^c) \qquad\qquad\qquad \tilde{E}^*(Z/_B Z^c) \qquad\qquad\qquad \tilde{E}^*(W/_B W^c)$

$\tilde{E}^*(Th_B(Y)) \xrightarrow{\wedge_p v} \tilde{E}^*\underline{p}(Th_B(Y), TG) \longrightarrow \tilde{E}^*(Th_B(\nu(\tilde{g})+\mathbb{R}^m)) \qquad \tilde{E}^*(\Sigma^{n+m}L_+)$

$\downarrow \sim \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \sim$

$\tilde{E}^*(\Sigma^m N_+) \qquad\qquad\qquad \tilde{E}^*(\Sigma^m TG) \qquad\qquad \tilde{E}^*(\Sigma^n L_+) \qquad\qquad \searrow \sigma^{-m}$

$\downarrow \sigma^{-m} \qquad\qquad\qquad\qquad\qquad \downarrow \sigma^{-m} \qquad\qquad \uparrow \qquad\qquad\qquad \downarrow \sigma^{-m-n}$

$E^*(N) \xrightarrow{\qquad Th(v) \qquad} \tilde{E}^*(TG) \xrightarrow{\psi(h)\circ\hat{g}^*} \tilde{E}^*(X/_B X^c) \qquad\qquad E^*(L)$

$\square$

Chapter 3

Examples

3.1 Twisted K-theory

We will now define BPU-twisted (complex) K-theory. In [11], the authors developed an orthogonal (strict) ring spectrum representing K-theory. Their construction uses the picture of KK-theory from [4], where unbounded operators are used to represent KK-classes. We modify the construction of the K-spectrum in the sense that we weaken the product (obtaining an E_∞-spectrum), but make way for a PU-action on the spectrum which is compatible with the product structure. First, we recall the results from [11].

Let A be a C^*-algebra and M a countably generated $\mathbb{Z}_2$-graded Hilbert A-module. We denote by $B(M)$ the C^*-algebra of bounded A-linear operators that admit an adjoint with respect to the A-valued scalar product.

Definition 57 $\mathbf{F}_M := \{F \in B(M) \mid F^* = F,\ F^2 - 1 \text{ compact and } ||F|| \le 1\}$
We equip $\mathbf{F}_M$ with the weakest topology such that the maps

$$\mathbf{F}_M \to B(M),\ F \mapsto F^2 \qquad and \qquad \mathbf{F}_M \to M,\ F \mapsto F(\psi)$$

are continuous (with respect to the norm topology on $B(M)$ and M) for all $\psi \in M$.

Let $\mathbf{U}_M$ be the subspace of unitary elements in $\mathbf{F}_M$ and denote by $\mathbf{F}_M/\mathbf{U}_M$ the quotient (in topological spaces).

Definition 58 $\mathbf{F}_M^+$ *is the set $\mathbf{F}_M/\mathbf{U}_M$ with the topology generated by the open sets in $\mathbf{F}_M \setminus \mathbf{U}_M$, and for all $\varepsilon > 0$ the sets*

$$\{F \in \mathbf{F}_M \mid 0 < ||F^2 - 1|| < \varepsilon\} \cup *.$$

Proposition 59 *The conjugation action of the unitary group $U(M) \subset B(M)$ (with the norm topology) on $\mathbf{F}_M$ is continuous and extends to a continuous action on $\mathbf{F}_M^+$.*

Proof: Let us first show that $U(M)$ acts continuously on $\mathbf{F}_M$. Observe that for $U, U_0 \in U(M)$ we have $\|U^{-1} - U_0^{-1}\| = \|U_0(U_0 - U)U_0^{-1}\| = \|U - U_0\|$. Now, for $\psi \in M$ and $F, F_0 \in \mathbf{F}_M$ we have

$$
\begin{aligned}
\|UFU^{-1}\psi - U_0 F_0 U_0^{-1}\psi\| &\leq \|UFU^{-1}\psi - UFU_0^{-1}\psi\| \\
&\quad + \|UFU_0^{-1}\psi - UF_0 U_0^{-1}\psi\| \\
&\quad + \|UF_0 U_0^{-1}\psi - U_0 F_0 U_0^{-1}\psi\| \\
&\leq 2\|U - U_0\|\|\psi\| + \|(F - F_0)U_0\psi\|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|(UFU^{-1})^2 - (U_0 F_0 U_0^{-1})^2\| &\leq \|UF^2 U^{-1} - UF^2 U_0^{-1}\| \\
&\quad + \|UF^2 U_0^{-1} - UF_0^2 U_0^{-1}\| \\
&\quad + \|UF_0^2 U_0^{-1} - U_0 F_0^2 U_0^{-1}\| \\
&\leq 2\|U - U_0\| + \|F^2 - F_0^2\|
\end{aligned}
$$

so the action of $U(M)$ on $\mathbf{F}_M$ is indeed continuous. Since $\mathbf{U}_M$ is invariant under this action and $\|F^2 - 1\| = \|UF^2 U^{-1} - 1\|$, it extends to an action on $\mathbf{F}_M^+$. $\qquad\square$

Recall that an A-linear operator D on M is called regular if it is closed densely defined, dom D^* is dense and $(1 + D^*D)$ has dense image. The unbounded picture of KK-theory as developed in [4] has the advantage that it allows for an easier description of the product.

Definition 60 *[11]*

$$
\mathbf{R}_M := \{D \text{ regular on } M \mid D^* = D, \; D \text{ odd}, (1 + D^2)^{-1} \text{ compact}\}.
$$

Let $J : \mathbf{R}_M \to \mathbf{F}_M$ be the map $D \mapsto D(1 + D^2)^{-\frac{1}{2}}$. J maps $\mathbf{R}_M$ indeed to $\mathbf{F}_M$, is injective, and $im(J) \cap \mathbf{U}_M = \emptyset$, hence we may regard $\mathbf{R}_M$ as a subset of $\mathbf{F}_M^+$ [4, 11].

Definition 61 *Let $\mathbf{R}_M^+$ be the pointed set $\mathbf{R}_M \cup *$, equipped with the subspace topology induced by $\mathbf{R}_M^+ \subset \mathbf{F}_M^+$.*

Since $\mathbf{R}_M^+ \subset \mathbf{F}_M^+$ is invariant under the conjugation action of the even part $U^0(M)$ of $U(M)$, the $U(M)$-action on $\mathbf{F}_M$ induces a continuous action of $U^0(M)$ on $\mathbf{R}_M^+$.

If M and M' are Hilbert modules over A and A', we define

$$
\mu : \mathbf{R}_M \times \mathbf{R}_{M'} \to \mathbf{R}_{M \otimes M'}, \quad (D, D') \mapsto D \otimes 1 + 1 \otimes D'.
$$

Proposition 62 *[11, 3.6] $\mu : \mathbf{R}_M \times \mathbf{R}_{M'} \to \mathbf{R}_{M \otimes M'}$ is continuous and has a continuous pointed extension $\mu^+ : \mathbf{R}_M^+ \wedge \mathbf{R}_{M'}^+ \to \mathbf{R}_{M \otimes M'}^+$.*

For a finite dimensional real inner product space (V, g), we denote the Clifford algebra of $(V \otimes \mathbb{C}, g \otimes \mathbb{C})$ by $\mathbb{C}l(V)$. Note that $\mathbb{C}l(V)$ is functorial in (V, g) and that there are natural isomorphisms $\mathbb{C}l(V) \otimes \mathbb{C}l(W) \cong \mathbb{C}l(V \oplus W)$. Consider the $\mathbb{Z}_2$-graded vector space $C_0^\infty(V, \mathbb{C}l(V))$ and define $E_V := L^2(V, \mathbb{C}l(V))$ to be its L^2-completion.

Definition 63 *[11]*

- *The Dirac operator $\partial_V \in \mathbf{R}_{E_V}$ is the closure of the operator on $C_0^\infty(V, \mathbb{C}l(V))$ given on the functions of pure degree by*

$$\partial_V \sigma(x) := (-1)^{|\sigma|} \sum_{i \in I} \left(\frac{\partial \sigma}{\partial v_i}(x) \right) v_i, \qquad x \in V$$

  where $(v_i)_{i \in I}$ is any orthonormal basis of V.

- *The Clifford operator $L_V \in \mathbf{R}_{E_V}$ is defined by*

$$L_V(\sigma)(v) = v\sigma(v), \quad v \in V, \quad \sigma \in C_0^\infty(V, \mathbb{C}l(V))$$

- *The Bott-Dirac operator $D_V \in \mathbf{R}_{E_V}$ is $D_V := \partial_V + L_V$.*

Proposition 64 *[11, 3.9] There are canonical isomorphisms $E_V \otimes E_W \cong E_{V \oplus W}$. Using these isomorphisms, we have identities*

$$\partial_{V \oplus W} = \mu(\partial_V, \partial_W) \qquad and \qquad D_{V \oplus W} = \mu(D_V, D_W).$$

Recall that an orthogonal spectrum is defined as a module over the orthogonal sphere spectrum, therefore the data for an orthogonal spectrum K is given by

- a pointed space $K(V)$ with a left $O(V)$-action

- $O(V) \times O(W)$-equivariant maps $\sigma_{V,W} : K(V) \wedge S^W \to K(V \oplus W)$

for all finite dimensional real inner product spaces V, W. Moreover, the appropriate associativity and unit diagrams have to commute.

To obtain the spaces of the spectrum representing complex K-theory, we regard $H'(V) := \mathbb{C}l(V) \otimes E_V$ as a right $\mathbb{C}l(V)$-module, where $\mathbb{C}l(V)$ acts by right multiplication on $\mathbb{C}l(V)$ and trivially on E_V, and define $K'(V) := \mathbf{R}_{H'(V)}^+$.

To construct the $O(V)$-action, let $Pin^c(V) \subset \mathbb{C}l(V)^*$ be the subgroup generated by elements of V of norm one. The Pin^c-representation

$$\rho_V : Pin^c(V) \to O(V)$$

is defined as follows. The subspace $V \subset \mathbb{C}l(V)$ is invariant under the twisted conjugation action

$$Pin^c(V) \times \mathbb{C}l(V) \to \mathbb{C}l(V), \qquad (g, v) \mapsto (-1)^{|g|} g v g^{-1}$$

and ρ_V is simply the restriction of this action to V. $Pin^c(V)$ is a central S^1-extension of $O(V)$, i.e.

$$1 \to S^1 \to Pin^c(V) \xrightarrow{\rho_V} O(V) \to 1.$$

We define an action of $Pin^c(V)$ on E_V by

$$g\sigma(v) := \rho_V(g)\sigma(\rho_V(g)^{-1}v)$$

for $g \in Pin^c(V), \sigma \in C_0^\infty(V, \mathbb{C}l(V))$ and $v \in V$. Combining this with the $Pin^c(V)$-action on $\mathbb{C}l(V)$ given by left multiplication, we obtain a unitary representation $\varrho'_V : Pin^c(V) \to \mathbf{U}_{H'(V)}$. The twisted conjugation action on $\mathbf{F}_{H'(V)}$ given by $p : F \mapsto (-1)^{deg(p)}\varrho'_V(p)F\varrho'_V(p)^{-1}$ factors through an $O(V)$-action. Moreover, this $O(V)$-action induces one on $\mathbf{F}^+_{H'(V)}$ [11]. The subspace $\mathbf{R}^+_{H'(V)}$ is $O(V)$-invariant, thus we get an action on $K'(V)$.

For $v \in V$, denote the operator on $\mathbb{C}l(V)$ given by left Clifford multiplication by l_v and observe that the map

$$\tilde{\eta}_V : V \to \mathbf{R}_{H'(V)}, \quad v \mapsto \mu(l_v, D_V)$$

has a continuous pointed extension $\eta_V : S^V \to \mathbf{R}^+_{H'(V)} = K'(V)$ since

$$\|J(\tilde{\eta}_V(v))^2 - 1\| = \|\frac{(l_v \otimes 1 + 1 \otimes D_V)^2}{1 + (l_v \otimes 1 + 1 \otimes D_V)^2} - 1\| \xrightarrow{v \to \infty} 0.$$

We define the structure maps of K' by

$$K'(V) \wedge S^W \xrightarrow{1 \wedge \eta_W} K'(V) \wedge K'(W) \xrightarrow{\mu^+} \mathbf{R}^+_{H'(V) \otimes H'(W)} \cong \mathbf{R}^+_{H'(V \oplus W)} = K'(V \oplus W).$$

These maps are $O(V) \times O(W)$-equivariant, associative and unital, hence K' is indeed an orthogonal spectrum.

Proposition 65   *[11] K' represents complex K-theory.*
Moreover,

$$\mu : K' \wedge K' \to K' \text{ given by } K'(V) \wedge K'(W) \xrightarrow{\mu^+} \mathbf{R}^+_{H'(V \oplus W)} = K'(V \oplus W)$$

induces the product in K-theory and μ, η make K' a ring spectrum.

With this definition of the K-spectrum at hand, we will now describe the modification that enables us to define BPU-twisted K-theory. The spaces of the spectrum K' are given as spaces of operators on Hilbert modules indexed on $\mathcal{I}$. The idea is to tensor all these modules by a fixed Hilbert space insomuch that the projective unitary group of this space acts on the spectrum, so let us fix the separable $\mathbb{Z}_2$-graded Hilbert space $\mathbf{H} := L^2(\mathbb{R}, \mathbb{C}l(\mathbb{R}))$ and define

$$PU := U^0(\mathbf{H})/S^1 \qquad H(V) := H'(V) \otimes \mathbf{H} \qquad K(V) := \mathbf{R}^+_{H(V)}.$$

To get the $O(V)$-action, we extend the unitary $\mathbb{C}l(V)$-representation $\varrho'(V) : Pin^c(V) \to \mathbf{U}_{H'(V)}$ trivially, i.e.

$$\varrho(V) := \varrho'(V) \otimes 1 : Pin^c(V) \to \mathbf{U}_{H(V)} = \mathbf{U}_{H'(V) \otimes \mathbf{H}}.$$

As before, the twisted conjugation action factors through an $O(V)$-action, which yields an $O(V)$-action on $K(V)$. Similarly, we define the structure maps by

$$K(V) \wedge S^W \xrightarrow{1 \wedge \eta_W} K(V) \wedge K'(W) \xrightarrow{\mu^+} \mathbf{R}_{H(V) \otimes H'(W)} \cong \mathbf{R}_{H(V \oplus W)} = K(V \oplus W).$$

The $O(V) \times O(W)$-equivariance as well as the unitality and associativity follow as before, so we have an orthogonal spectrum, again. The PU-action defined by

$$PU \times K(V) \qquad ([u], D) \mapsto (1 \otimes u)D(1 \otimes u)^{-1}$$

is $O(V)$-equivariant as well as compatible with the structure maps and thus yields a PU-action on K.

Moreover,

$$\phi_V := \mu(., D_\mathbb{R}) : K'(V) = \mathbf{R}^+_{H'(V)} \to \mathbf{R}^+_{H(V)} = K(V)$$

is a weak equivalence for all $V \neq \{0\}$ [11, 3.12]. Since these maps are $O(V)$-equivariant and compatible with the structure maps, they assemble to a stable equivalence $\phi : K' \to K$.

Definition 66 *Let $\mathcal{P}$ be the E_∞-operad given by $\mathcal{P}(j) := \mathrm{Isom}^0(\mathbf{H}^{\otimes j}, \mathbf{H})$ with the obvious Σ_j-action and structure maps, where Isom^0 denotes the space of even isometric isomorphisms with the norm topology.*

Recall that $\mathcal{P}[\mathcal{U}]$ is the set of algebras over $\mathcal{P}$, i.e. spaces $X \in \mathcal{U}$ with morphisms $\theta_j : \mathcal{P}(j) \times X^j \to X$ which are compatible with the operadic structure of $\mathcal{P}$.

Lemma 67 *PU is a group object in $\mathcal{P}[\mathcal{U}]$.*

Proof: Define the action of $\mathcal{P}$ by

$$\theta : \mathcal{P}(j) \times PU^j \to PU, \quad (p, [u_1], .., [u_j]) \mapsto [p(u_1 \otimes ... \otimes u_j)p^{-1}]$$

and check that it commutes with the product $\mu : PU \times PU \to PU$ and the unit $e : * \to PU$. $\qquad\square$

Since the category of spectra $\mathcal{S}$ is enriched, tensored and cotensored over $\mathcal{U}$, the category $\mathcal{P}[\mathcal{S}]$ of $\mathcal{P}$-algebras in $\mathcal{S}$ is defined.

Lemma 68 *K is in $\mathcal{P}[\mathcal{S}]$.*

Proof: Each $p \in \mathcal{P}(j)$ defines a map $\mathbf{R}_{M \otimes \mathbf{H} ... \otimes \mathbf{H}} \to \mathbf{R}_{M \otimes \mathbf{H}}$ for all M, given by $D \mapsto (1 \otimes p)D(1 \otimes p)^{-1}$. Using this, the action of $\mathcal{P}$ is defined by the maps

$$\phi_{V_1,..,V_j} : \mathcal{P}(j)_+ \wedge K(V_1) \wedge ... \wedge K(V_j) \overset{1 \times \mu^+}{\to} \mathcal{P}(j) \times \mathbf{R}_{H'(V_1 \oplus ... \oplus V_j) \otimes \mathbf{H} ... \otimes \mathbf{H}}$$
$$\to \quad \mathbf{R}_{H'(V_1 \oplus ... \oplus V_j) \otimes \mathbf{H}}$$
$$= \quad K(V_1 \oplus ... \oplus V_j).$$

$\qquad\square$

Lemma 69 *The PU-action on K is compatible with the $\mathcal{P}$-actions on PU and K, i.e. $m : PU_+ \wedge K \to K$ is a map in $\mathcal{P}[\mathcal{S}]$.*

Proof: We have to show commutativity of

$$\mathcal{P}(j)_+ \wedge PU_+ \wedge K(V_1) \wedge ... \wedge PU_+ \wedge K(V_j) \longrightarrow \mathcal{P}(j)_+ \wedge K(V_1) \wedge ... \wedge K(V_j)$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$
$$PU_+ \wedge K(V_1 \oplus ... \oplus V_j) \longrightarrow K(V_1 \oplus ... \oplus V_j).$$

On inspection of the definitions, this reduces to the observation that for all $D_i \in K(V_i)$ and $[u_i] \in PU$
$$\mu^+((1 \otimes u_1)D_1(1 \otimes u_1)^{-1}, ..., (1 \otimes u_j)D_j(1 \otimes u_j)^{-1}) =$$
$$(1 \otimes u_1... \otimes u_j)\mu^+(D_1, .., D_j)(1 \otimes u_1... \otimes u_j)^{-1}.\square$$

Proposition 70 *Any $p \in \mathcal{P}(2)$ endows $Ho\mathcal{S}_{BPU}$ with the structure of a symmetric monoidal category.*

Proof: First, recall that BPU is a $K(\mathbb{Z}, 3)$-space. We will make repeated use of this fact throughout the proof.

In Proposition 32, we have shown how to construct a symmetric monoidal structure on $Ho\mathcal{S}_B$ from an action of a pointed E_∞-operad on B. $\mathcal{P}$ is an E_∞-operad and it acts on BPU, but it is not pointed and therefore we can't

apply Proposition 32 directly. However, the only parts of the proof where the pointedness of the operad were used were the construction of the unit isomorphism and the coherence diagramm involving it, so we will just reproof that.

We define the bifunctor associated to $p \in \mathcal{P}$ to be

$$\underline{p} := L(\theta(p)_! \circ \overline{\wedge}) : Ho\mathcal{S}_{BPU} \times Ho\mathcal{S}_{BPU} \to Ho\mathcal{S}_{BPU}$$

and the unit to be $\mathbf{1}_{BPU} := i_{*!}S$.

To get the unit isomorphism, we need to find a homotopy from

$$p_e : BPU \xrightarrow{\ i_* \times id\ } BPU \times BPU \xrightarrow{\ \theta(p)\ } BPU$$

to the identity. Note that

$$\theta(p) : BPU \times BPU \to BPU$$

induces the sum on H^3, i.e.

$$\theta(p)_* = + : H^3(X, \mathbb{Z}) \times H^3(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}) \quad \forall X \in \mathcal{U}$$

and thus p_e is indeed homotopic to the identity. Choosing such a homotopy defines by Proposition 26 a natural transformation $\phi : Lp_{e!} \xrightarrow{\sim} Lid_!$. This yields the required natural isomorphism

$$\underline{p}(\mathbf{1}_{BPU}, X) \cong Lp_{e!}X \cong QX \cong X.$$

Moreover, the identity component of $\mathcal{U}(BPU, BPU)$ is simply connected (since $H^2(BPU, \mathbb{Z}) = 0$), so all homotopies connecting p_e and id_{BPU} are homotopic and the isomorphism does not depend on the choice of the homotopy. It remains to check commutativity of

$$\underline{p}(\mathbf{1}_{BPU}, \underline{p}(X, Y)) \longrightarrow \underline{p}(X, Y) .$$
$$\downarrow \qquad \nearrow$$
$$\underline{p}(\underline{p}(\mathbf{1}_{BPU}, X), Y)$$

Recall that the associativity transformation was defined using a path in the operad connecting $\gamma(p; 1, p)$ and $\gamma(p; p, 1)$. We denote the natural isomorphisms induced by the various homotopies by ϕ_i. The commutativity of the

above diagram is the same as the commutativity of

$$\theta(p)_!(i_* \times id)_!\theta(p)_!(QX\overline{\wedge}QY) \xrightarrow{\sim} p_{e!}\theta(p)_!(QX\overline{\wedge}QY)$$

$$\downarrow{\sim}$$

$$\theta(p)_!(1 \times \theta(p))_!(i_* \times id)_!(QX\overline{\wedge}QY) \qquad\qquad \phi_0$$

$$\downarrow{\phi_1}$$

$$\theta(p)_!(\theta(p) \times 1)_!(i_* \times id)_!(QX\overline{\wedge}QY) \qquad \theta(p)_!(QX\overline{\wedge}QY)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \phi_2 \nearrow$$

$$\downarrow{\sim}$$

$$\theta(p)_!(p_e \times id)_!(QX\overline{\wedge}QY)$$

for all $X, Y \in \mathcal{S}_{BPU}$. Note that $H^2(BPU \times BPU, \mathbb{Z}) = 0$ and hence all homotopies between maps
$f, g : BPU \times BPU \to BPU$ are homotopic, so the application of Propositions 26, 27 finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proposition 71 *The gauge group $\mathcal{G}(EPU)$ of bundle automorphisms of the principal PU-bundle $EPU \to BPU$ is connected.*

Proof: A gauge transformation $g \in \mathcal{G}(EPU)$ defines a principal PU-bundle $P_g \to BPU \times S^1$ by identifying $EPU \times \{0\}$ and $EPU \times \{1\}$ via g in $EPU \times I$. Now, $H^3(BPU \times S^1, \mathbb{Z}) \cong H^3(BPU, \mathbb{Z})$ and therefore P_g is isomorphic to π^*EPU, where

$$\pi : BPU \times S^1 \to BPU$$

is the projection. This implies that one can extend P_g to a PU-bundle on $BPU \times D^2$. The choice of such an extension yields a path connecting g and id_{EPU} in $\mathcal{G}(EPU)$.

$$\square$$

Lemma 72 $E := BPUK := EPU \times_{PU} K$ *is a monoid in $(Ho\mathcal{S}_{BPU}, \underline{p})$ for all $p \in \mathcal{P}(2)$.*

Proof: The non-pointedness of $\mathcal{P}$ obstructs the direct application of Lemma 35, thus we will redo those parts of the proof where the basepoints of the operad were used, namely the construction of the unit $\eta : \mathbf{1}_{BPU} \to E$. Note that the map $\tilde{\eta} : S \xrightarrow{\eta'} K' \xrightarrow{\phi} K$ is a homotopy unit for each $p \in \mathcal{P}(2)$ since

$$K'(V) \wedge K'(W) \xrightarrow{\mu^+} K'(V \oplus W)$$

$$\downarrow{\phi \wedge \phi} \qquad\qquad\qquad\qquad\qquad \downarrow{\phi}$$

$$K(V) \wedge K(W) \xrightarrow{\mu^+} R^+_{H'(V \oplus W) \otimes \mathbf{H} \otimes \mathbf{H}} \longrightarrow K(V \oplus W)$$

is homotopy commutative. Define $\eta : \mathbf{1}_{BPU} = i_{*!}S \to E$ to be the adjoint of $\tilde{\eta}$. The last thing to check is the commutativity of

$$\underline{p}(\mathbf{1}_{BPU}, E) \longrightarrow \underline{p}(E, E)$$

$$E.$$

By the definition of the unit isomorphism, all we have to find are maps ψ_1, ψ_2 that make the following diagram homotopy commutative. In the proof of Lemma 35, these were constructed from the action of $\mathcal{P}$. Since p_e need not be of the form $\theta(q)$ for some $q \in \mathcal{P}(1)$, we cannot copy that construction.
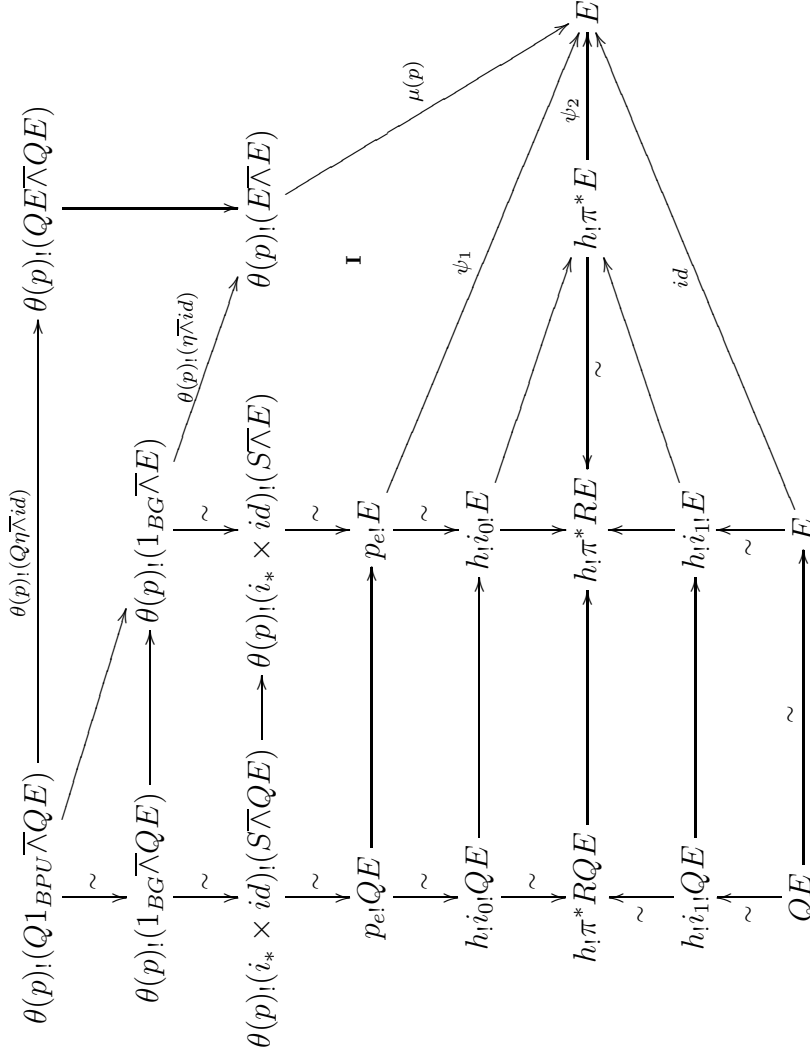
To obtain ψ_1, let us first choose an isomorphism $\tilde{\psi_1}' : EPU \to p_e^* EPU$ and use this to define $\tilde{\psi_1} : E \to p_e^* E$.

Letting $\psi_1 : p_{e!}E \to E$ be the adjoint, the quadrangle **I** in the diagram commutes in $Ho\mathcal{S}_{BPU}$ since $\tilde{\eta}$ is a homotopy unit. Moreover, since the gauge group

of EPU is connected, one can extend the isomorphism $\tilde{\psi_1}'$ to
$\tilde{\psi_2}' : \pi^* EPU \to h^* EPU$ in such a way that $i_0^*(\tilde{\psi_2}') = \tilde{\psi_1}'$ and $i_1^*(\tilde{\psi_2}') = id_{EPU}$.
Now, $\tilde{\psi_2}'$ yields $\tilde{\psi_2} : \pi^* E \to h^* E$. Defining ψ_2 to be the adjoint of $\tilde{\psi_2}$ completes the argument. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2   Twisted $Spin^c$-cobordism

To define twisted $Spin^c$-cobordism, we build upon the construction of $MSpin^c$
in [23]. Recall from the previous section the representation

$$\varrho(V) : Pin^c(V) \to \mathbf{U}_{H(V)}.$$

Restricting it to the subgroup of even elements $Spin^c(V) \subset Pin^c(V)$, we
obtain a representation $Spin^c(V) \to \mathbf{U}_{H(V)}^0$ which we call again $\varrho(V)$. Since
$\mathbf{U}_{H(V)}^0$ is contractible, $\mathbf{U}_{H(V)}^0 \times_{Spin^c(V)} V$ is a model for the universal $Spin^c(V)$-
bundle. We define the spaces of the spectrum as

$$MSpin^c(V) := Th(\mathbf{U}_{H(V)}^0 \times_{Spin^c(V)} V) = \mathbf{U}_{H(V)+}^0 \wedge_{Spin^c(V)} S^V.$$

The $O(V)$-action is obtained by combining the $O(V)$-action induced by the
conjugation action of $Spin^c(V)$ on $\mathbf{U}_{H(V)}^0$ and the standard action on S^V. To
define the structure maps

$$\sigma_{V,W} : MSpin^c(V) \wedge S^W \to MSpin^c(V \oplus W),$$

we use the canonical isomorphism $H(V \oplus W) \cong H(V) \otimes H'(W)$ and the
homeomorphism $f : S^V \wedge S^W \to S^{V \oplus W}$ and define

$$\sigma_{V,W}[U, v, w] := [U \otimes 1, f(v, w)].$$

Note that $PU = PU^0(\mathbf{H})$ acts on $MSpin^c$ via

$$PU \times MSpin^c(V) \to MSpin^c(V), \qquad ([u], [U, v]) \mapsto [(1 \otimes u)U(1 \otimes u)^{-1}, v].$$

Lemma 73 *$MSpin^c$ is in $\mathcal{P}(\mathcal{S})$ and the PU-action on $MSpin^c$ is compatible
with the $\mathcal{P}$-algebra structures on PU and $MSpin^c$.*

Finally, there is an obvious map

$$\alpha : MSpin^c \to K, \ \alpha(U, v) := U\tilde{\eta}(v)U^{-1} \text{ for } U \in \mathbf{U}_{H(V)}^0, v \in V$$

which is a PU-equivariant map of $\mathcal{P}$-spectra.

Proposition 74 *[23] $MSpin^c$ represents $Spin^c$-cobordism and α induces the classical orientation homomorphism $MSpin^{c*} \to K^*$.*

Analogous to the K-spectrum over BPU, we define an ex-spectrum over BPU with fibre $MSpin^c$ by $\mathcal{M}_{Spin^c}(V) := EPU \times_{PU} MSpin^c$. A similar proof as for the parametrised K-spectrum shows that $\mathcal{M}_{Spin^c}$ is a parametrised ring spectrum over BPU.

Definition 75 *The parametrised $MSpin^c$-orientation is*

$$\beta := 1 \times \alpha : EPU \times_{PU} MSpin^c \to EPU \times_{PU} K.$$

Bibliography

[1] J.F. Adams, Stable Homotopy and Generalised Homology, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago (1974)

[2] M. Atiyah and G.Segal, Twisted K-theory, arXiv:math.KT/0407054 (2004)

[3] M. Atiyah and G.Segal, Twisted K-theory and cohomology, arXiv:math.KT/0510674 (2005)

[4] S. Baaj and P. Julg, Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-module hilbertiens, C.R. Acad. Sci. Paris Sér. I Math.296:21 (1983)

[5] P.I. Booth, The exponential law of maps I, Proc. London Math. Soc. (3) 20 (1970)

[6] P.I. Booth, The section problem and the lifting problem Math. Z. 121 (1971)

[7] P.I. Booth, The exponential law of maps II,Math. Z. 121 (1971)

[8] P.I. Booth and R. Brown, Spaces of partial maps, fibred mapping spaces and the compact-open topology., General Topology and Appl. 8 (1978)

[9] P.I. Booth and R. Brown, On the application of fibred mapping spaces to exponential laws for bundles, ex-spaces and other categories of maps., General Topology and Appl. 8 (1978)

[10] E.H. Brown, Abstract Homotopy Theory, Trans. Amer. Math. Soc. 119 (1965)

[11] U.Bunke, M.Joachim and S.Stolz, Classifying spaces and spectra representing the K-theory of a graded C^*-algebra, High-dimensional manifold topology, 80–102 (2003)

[12] A. L. Carey and B.-L. Wang Thom isomorphism and Push-forward map in twisted K-theory, arXiv:math.KT/0507414 (2005)

[13] M. Clapp, Duality and transfer for parametrized spectra, Arch. Math. (Basel) 37, no. 5 (1981)

[14] M. Clapp and D. Puppe, The homotopy category of parametrized spectra, Manuscripta Math. 45, no. 3 (1984)

[15] W.G Dwyer and J. Spalinski, Homotopy theories and model categories Handbook of algebraic topology, North-Holland, Amsterdam (1995)

[16] E.Dyer, Cohomology Theories Mathematical Lecture Notes Series, W.A. Benjamin Inc, New York, (1969)

[17] D. S. Freed, M. J. Hopkins and C. Teleman, Twisted equivariant K-theory with complex coefficients, arXiv:math.AT/0206257 (2002)

[18] D. S. Freed, M. J. Hopkins and C. Teleman, Twisted K-theory and loop group representations, arXiv:math.AT/0312155 (2003)

[19] D. S. Freed, M. J. Hopkins and C. Teleman, Loop Groups and Twisted K-Theory II, arXiv:math.AT/0511232 (2005)

[20] M. Hovey, Model Categories. Amer. Math. Soc. Surveys and Monographs 63 (1999)

[21] M. Intermont, M.W. Johnson, Model structures on the category of ex-spaces Topology Appl. 119 (2002)

[22] I.M. James, Fibrewise topology, Cambridge Tracts in mathematics 91, Cambridge University Press, Cambridge (1989)

[23] M. Joachim, Higher coherences for equivariant K-Theory Structured Ring Spectra, London Math. Soc. Lecture Notes Ser. 315, Cambridge University Press, Cambridge (2004)

[24] G.M.Kelly, On MacLane's Conditions for Coherence of Natural Associativities, Commutativities, etc. Journal of Algebra 1 (1964)

[25] M.A. Mandell and J.P. May, Equivariant orthogonal spectra and S-modules, Mem. Amer. Math. Soc. 159, no. 755 (2002)

[26] M.A. Mandell, J.P. May, S. Schwede and B. Shipley, Model categories of diagram spectra, Proc. London Math. Soc (3) 82 (2001)

[27] M. Markl, S. Shnider and J. Stasheff, Operads in Algebra, Topology and Physics Mathematical Surveys and Monographs, 96, AMS, Providence (2002)

[28] J.P. May The Geometry of Iterated Loop Spaces, Lecture Notes in Mathematics, Vol. 271, Springer-Verlag, Berlin-New York (1972)

[29] J.P. May E_∞ spaces, group completions, and permutative categories New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972), London Math. Soc. Lecture Note Ser., No. 11, Cambridge Univ. Press, London (1974)

[30] J.P. May, The additivity of traces in triangulated categories Adv. Math. 163, no. 1, (2001)

[31] J.P. May and J. Sigurdsson, Parametrized Homotopy Theory, arXiv:math.AT/0411656 (2004)

[32] R. Schön, On representable half-exact functors over B, Quaestiones Math. 13, no. 3-4, (1990)

[33] R.M. Switzer, Algebraic Topology - Homotopy and Homology, Die Grundlehren der mathematischen Wissenschaften 212, Springer-Verlag New York-Heidelberg (1975)

[34] J.-L. Tu, P. Xu and C. Laurent-Gengoux Twisted K-theory of differentiable stacks, arXiv:math.KT/0306138 (2003)

[35] J.-L. Tu and P. Xu The ring structure for equivariant twisted K-theory, arXiv:math.KT/0604160 (2006)

[36] D.G Quillen, Homotopical algebra, lecture notes in Mathematics 43, Springer-Verlag New York-Heidelberg (1967)