The Segal Burnside ring conjecture

Let BG denote the classifying space of a finite group G. Atiyah showed in 1960 that $KU^0(BG) \cong \hat{R}[G]$, where $KU^*(-)$ is complex periodic K-theory and $\hat{R}[G]$ is the complex representation ring of G completed at the aumentation ideal.

Similar results were proven for $KO^*(-)$, real periodic K-theory, for compact Lie groups by Atiyah and Segal.

Let $\pi^*(-)$ denote stable cohomotopy. This is a reduced cohomology theory on the stable category of spectra, and there is a homomorphism $\hat{A}[G] \to \pi^0(BG_+)$. The source of this map is the Burnside ring of finite G-sets, and the hat denotes completion at the augmentation ideal. Segal's original conjecture was that this map is an isomorphism, in analogy with the result sited above.

Segal showed that $\pi_G^*(S^0) \cong A[G]$, where $\pi_G^*(-)$ is the stable *equivariant* cohomotopy functor. There is a map

$$\hat{\pi}_G^*(S^0) \to \pi^*(BG_+)$$

thus it is possible to generalize the original conjecture and claim that this map should be an isomorphism of graded groups.

The conjecture in this form was proven in several steps. The first non-trivial case is $G = \mathbb{Z}/2$ which was proven by W.H. Lin using a complicated Ext-calculation. His proof was simplified and published in 1980 by Lin, Davis, Mahowald and Adams.

Gunawardena managed to prove the conjecture for odd primes, also using Ext-calculations. In a paper from 1981, Ravenel proves the conjecture for cyclic p-groups. His methods involved a modified Adams spectral sequence, and also involves a fair amount of homological algebra. May and McClure showed that the conjecture holds for all finite groups if and only if it holds for all p-groups. Carlsson finally proved in a paper from 1984, using homotopic theoretic methods, that the Segal conjecture holds for finite p-groups by reducing the problem to the known case of cyclic p-groups.

The proof for G cyclic of prime order involves analyzing the Ext-groups of the ring of Laurent polynomials $\mathbb{Z}/p[x, x^{-1}]$, as a module over the mod p-Steenrod algebra.

In the talk we will explore some of the machinery behind this analyzis. The methods will be at the heart of my next talk.