Calculus of functors and model categories

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Motivation and Main Results

Homotopy Theory of Small Functors:
(joint work with B. Dwyer)

• Model category structure on the category of small functors $\mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$
• Quillen equivalence with the equivariant model structure on $\mathcal{S}^{\bullet}_{\downarrow\bullet} = \mathrm{Map}(\mathcal{S})$

Goodwillie's Calculus of Functors:

- Model category framework for Homotopy Functors
- The n-excisive part of a functor P_n as a localization
- Classification of homogeneous functors

Categorical Localization

Let $\mathcal{C}$ be any category.

A functor $L\colon \mathcal{C} \to \mathcal{C}$ is a **localization** of $\mathcal{C}$ if L is coaugmented and idempotent. In more details, the functor L is

- **coaugmented** if there exists a natural map $j_X\colon X \to LX$ for each $X \in \mathcal{C}$;

- **idempotent** if the natural maps

$$L(j_X), j_{LX}\colon LX \rightrightarrows LLX$$

are equal isomorphisms in $\mathcal{C}$.

Examples:

- Localization of modules over a ring:
$$M \mapsto M \otimes_R R[S^{-1}].$$
- Abelianization of groups: $G \mapsto G/[G,G]$.

Homotopical Localizations

Let $\mathcal{M}$ be a (simplicial) model category.
Definition 1. A functor $L\colon \mathcal{M} \to \mathcal{M}$ is a (homotopical) localization of $\mathcal{M}$ if L is a coaugmented, homotopy idempotent functor, which preserves weak equivalences.

In more details, a functor $L\colon \mathcal{M} \to \mathcal{M}$ is

- **coaugmented** if there exists a natural map $j_X\colon X \to LX$ for each $X \in \mathcal{M}$;

- **homotopy idempotent** if the natural maps $L(j_X), j_{LX}\colon LX \rightrightarrows LLX$ are simplicially homotopic weak equivalences.

Examples:

Let $\mathcal{M} = \mathcal{S}$.

• Postnikov section $X \mapsto P_n X$;

• Sullivan's localization with respect to a set of primes;

• Quillen-Sullivan's rationalization;

• Quillen's plus construction $X \mapsto X^+$;

• Bousfield: localization with respect to an arbitrary homology theory.

• Goodwillie's polynomial approximation of a homotopy functor?

Model Categories

Definition 2. A category $\mathcal{M}$ with three classes of morphisms $\mathcal{W}$, $\mathcal{C}of$ and $\mathcal{F}ib$ is a **model category** if

CM1: $\mathcal{M}$ is complete and cocomplete.

CM2: $\mathcal{W}$ satisfies the '2-out-of-3' property.

CM3: Morphisms of $\mathcal{W}$, $\mathcal{C}of$ and $\mathcal{F}ib$ are closed under retracts.

CM4: Any commutative square $\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$ has a lift if either $i \in \mathcal{W}$, or $p \in \mathcal{W}$.

CM5: Any $f : A \to X$ admits factorizations:

$$(i)\ \ A \xhookrightarrow{\sim} W \twoheadrightarrow X$$

$$(ii)\ \ A \hookrightarrow W' \xrightarrow{\sim} X$$

Quillen equivalence is a pair of adjoint functors 'preserving the structure'. For example,

$$|-| : s\mathcal{S}\text{ets} \rightleftarrows \mathcal{T}\text{op} : \text{Sing}(-)$$

is a Quillen equivalence.

Small Functors

Let $\mathcal{S}$ denote the category of simplicial sets. We will be interested mostly in this case, although many results apply to any closed monoidal, locally presentable category $\mathcal{V}$.

Definition 3. Let $\mathcal{C}$ be a (not necessarily small) simplicial category. A functor $\underset{\sim}{X}\colon \mathcal{C} \to \mathcal{S}$ is called **small** if $\underset{\sim}{X}$ is a small weighted colimit of representables.

The category of small functors $\mathcal{S}^{\mathcal{C}}$ is cocomplete, but not always **complete**.

Example: If $\mathcal{C}$ is a large discrete category (large set), then there is no final object in $\mathcal{S}^{\mathcal{C}}$.

Question: For which $\mathcal{C}$ the category of small functors is complete?

Answer for Set-valued small functors

Freyd:
- Introduced *petty* and *lucid* functors.
- The category of lucid functors is complete iff the domain category is *approximately complete*.

Rosický:
- If the domain category is approximately complete, then `lucid functors = small functors`

Partial answer for the enriched case

$\mathcal{V}$ – symmetric monoidal closed category, locally finitely presentable as a closed category.

Theorem 4 (S. Lack). *Let $\mathcal{K}$ be a complete $\mathcal{V}$-category, then the category of small functors $\mathcal{V}^{\mathcal{K}^{\mathrm{op}}}$ is complete.*

Example: $\mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$ and $\mathcal{S}^{\mathcal{S}}$ are complete and cocomplete.

Homotopy theory of small functors: Projective model structure

Let $\mathcal{K}$ be a complete simplicial category, e.g $\mathcal{K} = \mathcal{S}, \mathcal{S}^{\mathrm{op}}$.

Theorem 5 (—, W.G. Dwyer). *We say that a morphism $f\colon X \to Y$ in $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$ is a* weak equivalence *(resp., a* fibration*) if $f_k\colon X(k) \to Y(k)$ is a weak equivalence (resp., a fibration) of simplicial sets. The category of small functors $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$ is a model category with these definitions.*

$\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$ is class-cofibrantly generated with

$$I = \left\{ R^K \otimes \begin{array}{c} \partial\Delta^n \\ \downarrow \\ \Delta^n \end{array} \;\middle|\; K \in \mathcal{K}^{\mathrm{op}}, n \geq 0 \right\}$$

and

$$J = \left\{ R^K \otimes \begin{array}{c} \Lambda^n_k \\ \downarrow \\ \Delta^n \end{array} \;\middle|\; K \in \mathcal{K}^{\mathrm{op}}, n > k \geq 0 \right\}$$

Application: Localization of spaces with respect to a map with a functorial universal property.

Example: Equivariant homotopy theory

Bredon: $G \rightsquigarrow \mathcal{O}_G = \{G/H | H < G\}$

$$\mathcal{S}^G \underset{\text{fixed points}}{\rightleftarrows} \mathcal{S}^{\mathcal{O}_G^{\mathrm{op}}}$$

Dror Farjoun-Zabrodsky:

$$D \rightsquigarrow \mathcal{O}_D = \{\underset{\sim}{Q} \in \mathcal{S}^D | \operatorname{colim} \underset{\sim}{Q} = *\}$$

Example: $D = \begin{array}{c}\bullet\\ \downarrow \\ \bullet\end{array}$, then $\mathcal{O}_{\begin{array}{c}\bullet\\ \downarrow \\ \bullet\end{array}} = \left\{ \begin{array}{c} X \\ \downarrow \\ * \end{array} \;\middle|\; X \in \mathcal{S} \right\} = \mathcal{S}.$

Model structure on $\mathcal{S}^D$: a map $f : \underset{\sim}{X} \to \underset{\sim}{Y}$ is a **weak equivalence** (resp. **fibration**) if

$$map(\underset{\sim}{Q}, f) : map(\underset{\sim}{Q}, \underset{\sim}{X}) \to map(\underset{\sim}{Q}, \underset{\sim}{Y})$$

is a weak equivalence (resp. fibration).

Quillen equivalence:

$$\mathcal{S}^{\begin{array}{c}\bullet\\ \downarrow \\ \bullet\end{array}} \rightleftarrows \mathcal{S}^{\mathcal{S}^{\mathrm{op}}}$$

Homotopy theory of homotopy functors

Theorem 6 (Homotopy model structure). *There is a model structure on $\mathcal{S}^{\mathcal{S}}$ such that $f\colon X \to Y$ is a weak equivalence if $f_W\colon X(W) \to Y(W)$ is a weak equivalence for all* fibrant $W \in \mathcal{S}$. *Cofibrations are projective cofibrations.*

Fibrant objects = objectwise fibrant homotopy functors

Construction: Localize the projective model structure with respect to the following class of maps:

$$\mathcal{F} = \{f^*\colon R^B \to R^A \mid f\colon A \tilde{\hookrightarrow} B \text{ in } \mathcal{S}\}$$

Corollary 7. *Any small functor admits an approximation by a homotopy functor featuring (homotopy) universal property.*

Use the fibrant replacement.

Goodwillie's calculus

A homotopy functor F is

- **excisive** if it takes homotopy pushouts to homotopy pullbacks
- **reduced** if $F(*) \simeq *$
- **linear** if it is both excisive and reduced

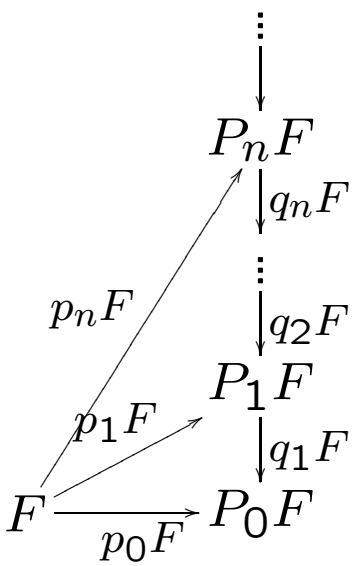
A cubical diagram is

- **strongly homotopy cocartesian** if all of its two-dimensional faces are homotopy pushouts
- **homotopy cartesian** if it is a 'homotopy pullback'

F is said to be $\mathbf{n}$**-excisive** if it takes strongly homotopy cocartesian (n+1)-cubical diagrams to homotopy cartesian diagrams.

Taylor tower

Every homotopy functor F may be approximated by the tower of functors:



such that P_nF is n-excisive and $F \to P_nF$ is universal, up to homotopy, with respect to maps into n-excisive functors.

F is said to be $\mathbf{n}$-**reduced** if $P_{n-1}F \simeq *$.

F is said to be $\mathbf{n}$-**homogeneous** if F is n-excicive and n-reduced.

Goodwillie's P_n as a localization

Theorem 8 (n-excisive model structure). *There is a model structure on $\mathcal{S}^{\mathcal{S}}$ such that $f\colon X \to Y$ is a weak equivalence if $P_n f$ is an objectwise weak equivalence, cofibrations = projective cofibrations.*

Fibrant objects = objectwise fibrant n-excisive functors

Idea of Proof: Localize with respect to the following class of maps:

$$\mathcal{F} = \{ \operatorname*{hocolim}_{U \in \mathcal{P}_0(\underline{n+1})} R^{K * U} \to R^K \mid K \in \mathcal{S} \}$$

$P_n \simeq$ fibrant replacement.

Same results apply to $\mathcal{S}_*^{\mathcal{S}_*}$.

Homotopy theory of spectum-valued functors

If $\mathcal{M}$ is a simplicial model category, then $\mathrm{Sp}(\mathcal{M})$ is a category with sequences $(X_0, X_1, \ldots)$ as objects with bonding morphisms $\Sigma X_n \to X_{n+1}$.

A functor from $\mathcal{S}_*$ to Sp is **small** if it is a left Kan extension of a functors from a small subcategory of $\mathcal{S}_*$.

Proposition 9. *A functor $F\colon \mathcal{S}_* \to \mathrm{Sp}$ is small iff F is levelwise small.*

Proposition 10. *There is an equivalence of categories $\mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*}) \cong \mathrm{Sp}^{\mathcal{S}_*}$.*

Corollary 11. *The category $\mathrm{Sp}^{\mathcal{S}_*}$ is complete.*

We obtain the projective model structure on $\mathrm{Sp}^{\mathcal{S}_*}$ by stabilization of the projective model structure on $\mathcal{S}_*^{\mathcal{S}_*}$. The homotopy and n-excisive model structures are obtained by consequtive localizations.

Colocalization and n-homogeneous structure

Define a new functor $M_nF\colon \mathrm{Sp}^{\mathcal{S}_*} \to \mathrm{Sp}^{\mathcal{S}_*}$, called the n-reduced part of F, by

$$M_nF = \mathrm{hofib}(F \to P_{n-1}F)$$

M_n is equipped with an augmentation $M_nF \to F$ and it is homotopy idempotent, i.e. it is a homotopical colocalization.

Denote also the n-homogeneous part of a functor by $D_nF = M_nP_nF$.

Theorem 12. *There exists a model structure on* $\mathrm{Sp}^{\mathcal{S}_*}$ *such that weak equivalences are maps of functors which induce weak equivalences of the* n*-homogeneous parts. Fibrations =* n*-excisive fibrations.*

Idea of Proof: Colocalize the n-excisive model structure with respect to M_n-equivalences.

Fibrant-cofibrant objects = projectively fibrant-cofibrant n-homogeneous homotopy functors

Classification of n-homogeneous functors

Theorem 13 (Goodwillie). *The homotopy category of n-homogeneous functors is equivalent to the homotopy category of spectra with Σ_n-action. Moreover, the equivalence is given by the functors:*

$$(- \wedge X^{\wedge n})_{h\Sigma_n} : \mathsf{Ho}(\mathsf{Sp}^{\Sigma_n}) \leftrightarrows$$
$$\mathsf{Ho}(n - \textit{homog. functors}) : \mathsf{cr}_n F(S^0, \ldots, S^0) \simeq \partial^n F(*),$$

where

$$\mathsf{cr}_n F(X_1 \ldots, X_n) = \mathsf{hofib}(F(\bigvee_{i=1}^{n} X_i) \to \mathsf{holim}_{T \in \mathcal{P}_0(\underline{n})} F(\bigvee_{\underline{n}-T} X_i))$$

Our reformulation of Goodwillie's theorem

This is actually a Quillen equivalence:

$$\lambda_n\colon\ \mathsf{Sp}^{\Sigma_n} \quad \leftrightarrows \quad \mathsf{Sp}^{\mathcal{S}_*}_{(n\text{-homog.})} \quad : \quad \rho_n,$$

where

$$\lambda_n E = \left(E \wedge \left(\widetilde{\bigwedge_{i=1}^{n} R^{S^0}} \right) \right)_{\Sigma_n}$$

and

$$\rho_n F = \hom\left(\widetilde{\bigwedge_{i=1}^{n} R^{S^0}}, F \right)$$

The small object argument for classes of maps

Theorem 14. *$\mathcal{C}$ – a cocomplete category; $I \subset \mathrm{Mor}\,\mathcal{C}$ – a class of maps satisfying:*

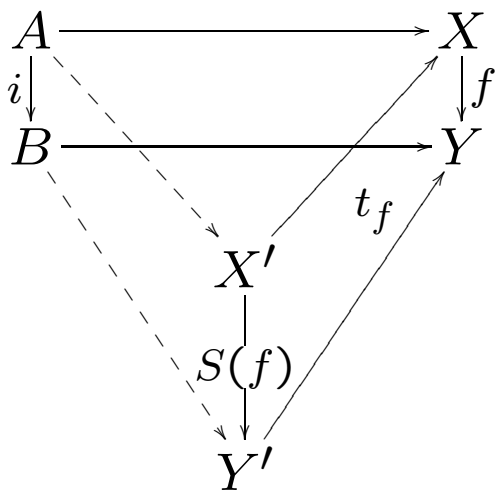
1. There exists a cardinal κ, such that each element $A \in \mathrm{dom}(I)$ is κ-small relative to I-cell;

2. There exists a (functorial) assignment $f \rightsquigarrow S(f) \in \mathrm{Map}\,\mathcal{C}$ equipped with a (natural) map

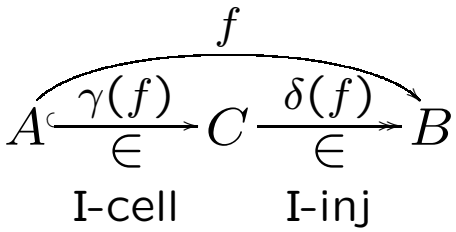
$$t_f \colon S(f) \to f,$$

such that $S(f) \in I$-cell for every $f \in \mathrm{Map}\,\mathcal{C}$ and any morphism of maps $i \to f$ with $i \in I$ factors through the (natural) map

$$t_f \colon S(f) \to f.$$



Then there is a (functorial) factorization (γ, δ) *on* $\mathcal{C}$ *such that, for all morphisms* f *in* $\mathcal{C}$*, the map* $\gamma(f)$ *is in* I*-cell and the map* $\delta(f)$ *is in* I*-inj.*



Non-functorial Bousfield-Friedlander localization technique

$\mathcal{M}$ – proper model category.
$\mathcal{Q}\colon \mathcal{M} \to \mathcal{M}$ – not necessarily functorial localization construction.
$f\colon A \to B$ is a $\mathcal{Q}$-equivalence if $\mathcal{Q}f\colon \mathcal{Q}A \to \mathcal{Q}B$ is a w.e. in $\mathcal{M}$.

$\mathcal{Q}$-fibrations are given by the RLP with respect to cofibrations, which are also $\mathcal{Q}$-equivalences.

Theorem 15. *Cofibrations, $\mathcal{Q}$-equivalences and $\mathcal{Q}$-fibrations form the $\mathcal{Q}$-local model structure on $\mathcal{M}$ if $\mathcal{Q}$-equivalences are preserved under pullbacks along $\mathcal{Q}$-fibrations.*