SPACES WITH NOETHERIAN COHOMOLOGY

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Abstract. Is the cohomology of the classifying space of a p-compact group, with Noetherian twisted coefficients, a Noetherian module? This note provides, over the ring of p-adic integers, such a generalization to p-compact groups of the Evens-Venkov Theorem. We consider the cohomology of a space with coefficients in a module, and we compare Noetherianity over the field with p elements, with Noetherianity over the p-adic integers, in the case when the fundamental group is a finite p-group.

Introduction

The main theorem of Dwyer and Wilkerson in [13] states that the mod p cohomology of the classifying space of a p-compact group is a finitely generated algebra. This generalizes to p-compact groups the Evens-Venkov Theorem [14] on the cohomology of a finite group G. There are however two main differences between these two results. Evens' statements allow a general base ring — any Noetherian ring is allowed, and they include the case of general twisted coefficients (contrary to the early work by Golod, [16], or Venkov, [25]) as follows: if M is Noetherian over a ring R, then so is $H^*(G;M)$ over $H^*(G;R)$. Beautiful finite generation statements on cohomology have since been proved in numerous situations. For statements as general as Evens' however, proofs have been surprisingly elusive.

This note is concerned with these generalizations for p-compact groups and p-local finite groups, as defined by Broto, Levi, and Oliver, [9]. We ask more generally when Noetherianity of the mod p cohomology algebra $H^*(Y;\mathbb{F}_p)$ of a space Y implies that the cohomology with coefficients in a $R[\pi_1(Y)]$-module M, $H^*(Y;M)$, is a Noetherian module over the algebra $H^*(Y;R)$. Because the classifying space BX of a p-compact group is p-complete by definition, we work over p-complete rings (for example $H^*((BS^3)^\wedge_p;\mathbb{Z})$ is not Noetherian).

Theorem 2.4. *Let Y be a connected space with finite fundamental group. Then, the graded $\mathbb{Z}^\wedge_p$-algebra $H^*(Y;\mathbb{Z}^\wedge_p)$ is Noetherian if, and only if, the graded $\mathbb{F}_p$-algebra $H^*(Y;\mathbb{F}_p)$ is Noetherian and the torsion in $H^*(Y;\mathbb{Z}^\wedge_p)$ is bounded.*

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Theorem 3.6. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group. Let M be a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module, which is finitely generated over $\mathbb{Z}_p^\wedge$. If the graded $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ is Noetherian, then $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{Z}_p^\wedge)$.*

This applies to p-compact group and to p-local finite groups to show that their p-adic cohomology algebra is Noetherian, see Theorem 4.2 and 4.5. Note that our proof makes no use of the recent classification of p-compact groups by Andersen, Grodal, Møller, and Viruel, [3], [4], [22]. Even in the case of a compact Lie group G, our theorem provides a general finiteness theorem for the cohomology of BG with twisted coefficients. One of the few explicit computations available in the literature is the case of $O(n)$, due to Čadek [11], (see also Greenblatt [17]).

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1. The cohomology as a graded module

Before considering the mod p or p-adic cohomology as an algebra, we first make explicit the relationship between two standard milder finiteness assumptions. When the graded vector space $H^*(Y;\mathbb{F}_p)$ is of finite type, i.e. $H^n(Y;\mathbb{F}_p)$ is a finite dimensional vector space in each degree n, is $H^n(Y;\mathbb{Z}_p^\wedge)$ a finitely generated $\mathbb{Z}_p^\wedge$-module in each degree n as well? This is clearly a necessary condition for the cohomology algebra to be finitely generated. We show that it holds when $\pi_1(Y)$ is finite.

The main tool to relate the mod p and the p-adic cohomology is the universal coefficient exact sequence — see for example [24, Theorem 5.5.10] for spaces and [1, Proposition 6.6] for spectra:

$$0 \to H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p \xrightarrow{\rho} H^*(Y;\mathbb{F}_p) \xrightarrow{\partial} \mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p) \to 0 \tag{1}$$

which applies, since $\mathbb{Z}_p^\wedge$ is a PID and $\mathbb{Z}/p$ is a finitely generated $\mathbb{Z}_p^\wedge$-module.

Remark 1.1. The morphism ρ in (1) is a ring homomorphism which makes the middle term $H^*(Y;\mathbb{F}_p)$ an $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$-module. Evens observed in [15, p. 272] that ∂ is also a homomorphism of $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$-modules, where $\mathrm{Tor}(H^*(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p)$ has the natural module structure he introduced in [15, Lemma 2].

Lemma 1.2. *Let G be a finite p-group, K a field of characteristic p and V a KG-module. If V^G is a finite dimensional K-vector space, then so is V.*

Proof. Let $n = \dim_K V^G$ and let $F = (KG)^n$ be a free KG-module of rank n. Note that $\dim_K F^G = n$, so there is an isomorphism of KG-modules $\alpha\colon V^G \to F^G$. Since F is an injective KG-module, α extends to a homomorphism $\alpha'\colon V \to F$ of KG-modules, which we now prove is injective. Clearly $(\mathrm{Ker}\,\alpha')^G = \mathrm{Ker}\,\alpha' \cap V^G = \mathrm{Ker}\,\alpha = 0$. Since G is a finite p-group, it follows that $\mathrm{Ker}\,\alpha' = 0$. Hence V embeds in F, so V is finite dimensional. ∎

Proposition 1.3. *Let Y be a connected space with finite fundamental group. The group $H^n(Y;\mathbb{F}_p)$ is finite for every positive integer n if and only if the $\mathbb{Z}_p^\wedge$-module $H^n(Y;\mathbb{Z}_p^\wedge)$ is finitely generated for every n. Under this condition, the $\mathbb{Z}_p^\wedge$-module $H^n(Y;M)$ is finitely generated for any n and every $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module M which is finitely generated over $\mathbb{Z}_p^\wedge$.*

Proof. If $H^n(Y;\mathbb{Z}_p^\wedge)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for any n, the universal coefficient exact sequence (1) implies that $H^n(Y;\mathbb{F}_p)$ is finite for any n.

Conversely, assume that $H^n(Y;\mathbb{F}_p)$ is finite for every n. Since the fundamental group of Y is finite, the space Y is p-good by [8, Proposition VII.5.1] and therefore $H^n(Y_p^\wedge;\mathbb{F}_p) \cong H^n(Y;\mathbb{F}_p)$. Likewise, since cohomology with p-adic coefficients is represented by Eilenberg-MacLane spaces $K(\mathbb{Z}_p^\wedge, n)$, which are p-complete, $H^n(Y_p^\wedge;\mathbb{Z}_p^\wedge) \cong H^n(Y;\mathbb{Z}_p^\wedge)$, [8, Proposition II.2.8]. We may therefore assume that Y is p-complete and that $G = \pi_1 Y$ is a finite p-group, see [13, Proposition 11.14] or [7, Section 5].

If Y is 1-connected, then [2, Proposition 5.7] applies and $H^n(Y;\mathbb{Z}_p^\wedge)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for every n. For the general situation, let us consider the universal cover fibration for Y, $\tilde{Y} \to Y \to BG$. We prove by induction that $H^n(\tilde{Y};\mathbb{F}_p)$ is finite dimensional for any n. The induction starts with the trivial case $n = 0$. Assume thus that $H^m(\tilde{Y};\mathbb{F}_p)$ is finite for all $m < n$. Then, in the second page of the Serre spectral sequence in mod p cohomology, all groups $E_2^{i,j} = H^i(BG, H^j(\tilde{Y};\mathbb{F}_p))$ on the lines $j = 0, \dots, n-1$ are finite. As $E_\infty^{0,n}$ is finite as well, it follows that $E_2^{0,n} = H^n(\tilde{Y};\mathbb{F}_p)^G$ is finite dimensional. Since G is a finite p-group, finiteness of $H^n(\tilde{Y};\mathbb{F}_p)^G$ implies finiteness of $H^n(\tilde{Y};\mathbb{F}_p)$ by Lemma 1.2.

We can now apply the 1-connected case to conclude that $H^n(\tilde{Y};\mathbb{Z}_p^\wedge)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for any n. The Evens-Venkov Theorem [14, Theorem 8.1] now shows that the E_2-term of the Serre spectral sequence with p-adic coefficients consists of finitely generated $\mathbb{Z}_p^\wedge$-modules. Thus so must be $H^n(Y;\mathbb{Z}_p^\wedge)$ for any n.

The second part of the assertion now follows easily. The first part of the proposition and the universal coefficient formula imply that $H^n(\tilde{Y};M)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for every n. We then use the Serre spectral sequence for cohomology with twisted coefficients. The only reference we know is [21, Theorem 3.2] where it is done equivariantly; we need the case of the trivial group action. ∎

2. Cohomology with Trivial coefficients

We now turn to finite generation of the cohomology algebras $H^*(Y;\mathbb{Z}_p^\wedge)$ and $H^*(Y;\mathbb{F}_p)$, where trivial coefficients are understood. This section should thus be no more than a warm up, because it seems enough to gain some control on torsion to draw conclusion from the universal coefficient theorem.

Let R be either the ring $\mathbb{Z}_p^\wedge$, or the field $\mathbb{F}_p$, and note that both are Noetherian rings. The cohomology $H^*(Y;R)$ of any connected space is a commutative graded algebra, which is a Noetherian R-algebra if and only if it is finitely generated as an R-algebra [19, Theorem 13.1].

Lemma 2.1. *Let Y be a connected space. If the $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ is Noetherian, then $H^*(Y;\mathbb{F}_p)$ is a finitely generated module over the algebra $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$.*

Proof. The ideal $\mathrm{Tor}(H^*(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p)$ of elements annihilated by p is a finitely generated ideal of $H^*(Y;\mathbb{Z}_p^\wedge)$ by assumption. It is therefore also finitely generated as an $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$-module. The conclusion follows from Remark 1.1 on the universal coefficient exact sequence. ∎

To be able to compare Noetherianity of the mod p and the p-adic cohomology, we need to analyze the p-torsion in $H^*(Y;\mathbb{Z}_p^\wedge)$. Let us denote by $\mathrm{T}_pH^*(Y;\mathbb{Z}_p^\wedge)$ the graded submodule of p-torsion elements. The key assumption in the main theorem of this section is that the order of the p-torsion is bounded. This implies that ρ is "uniformly power surjective", a strong form of integrality.

Lemma 2.2. *Let Y be a connected space and let d be an integer such that $p^d\cdot\mathrm{T}_pH^*(Y;\mathbb{Z}_p^\wedge)=0$. If $u\in H^*(Y;\mathbb{F}_p)$, then u^{p^d} belongs to the image of $\rho: H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p\to H^*(Y;\mathbb{F}_p)$.*

Proof. Following the elementary proof of [6, Lemma 4.4], we start with the observation that for any element $x\in H^*(Y;\mathbb{Z}/p^k)$ the p-th power x^p lies in the image of the reduction map $H^*(Y;\mathbb{Z}/p^{k+1})\to H^*(Y;\mathbb{Z}/p^k)$. The argument is as follows: If p is odd and the degree of x is odd, $x^p=0$ and the conclusion follows. Otherwise, $\delta(x^p)=p\delta(x)\cdot x^{p-1}=0$, because the Bockstein δ coming from the short exact sequence $\mathbb{Z}/p\to\mathbb{Z}/p^{k+1}\to\mathbb{Z}/p^k$ is a derivation with respect to the cup product pairing $H^*(Y;\mathbb{Z}/p^k)\otimes H^*(Y;\mathbb{Z}/p)\to H^*(Y;\mathbb{Z}/p)$. Therefore u^{p^d} lies in the image of the reduction $H^*(Y;\mathbb{Z}/p^{d+1})\to H^*(Y;\mathbb{F}_p)$.

The diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p^{d+1}"] \arrow[d, "\cdot p^d"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z}/p^{d+1} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

induces the commutative diagram of exact rows:

$$\begin{tikzcd}
0\to\mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p^{d+1}) \arrow[r] \arrow[d] & H^{*+1}(Y;\mathbb{Z}_p^\wedge) \arrow[r, "\cdot p^{d+1}"] \arrow[d, "\cdot p^d"] & H^{*+1}(Y;\mathbb{Z}_p^\wedge) \arrow[d, equal] \\
0\to\mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p) \arrow[r] & H^{*+1}(Y;\mathbb{Z}_p^\wedge) \arrow[r, "\cdot p"] & H^{*+1}(Y;\mathbb{Z}_p^\wedge)
\end{tikzcd}$$

Since $p^d\cdot\mathrm{T}_pH^*(Y;\mathbb{Z}_p^\wedge)=0$, the left vertical morphism is zero. Consider now the two universal coefficient sequences relating the cohomology of Y with coefficients in $\mathbb{F}_p$, respectively in $\mathbb{Z}/p^{d+1}$, to the cohomology of Y with coefficients in $\mathbb{Z}_p^\wedge$:

$$\begin{tikzcd}
0\to H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{Z}/p^{d+1} \arrow[r] \arrow[d] & H^*(Y;\mathbb{Z}/p^{d+1}) \arrow[r] \arrow[d] & \mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p^{d+1})\to 0 \arrow[d] \\
0\to H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p \arrow[r, "\rho"] & H^*(Y;\mathbb{F}_p) \arrow[r, "\partial"] & \mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p)\to 0
\end{tikzcd}$$

where the vertical morphisms are induced by the mod p reduction $\mathbb{Z}/p^{d+1} \to \mathbb{Z}/p$. The element u^{p^d} lies in the image of the mod p reduction and we have shown that the morphism between the torsion groups on the right is zero. Therefore $\partial(u^{p^d}) = 0$, which implies that u^{p^d} is in $\operatorname{Im}\rho$. ∎

Lemma 2.3. *Let Y be a connected space. If the graded $\mathbb{F}_p$-algebra $H^*(Y;\mathbb{F}_p)$ is Noetherian and if $H^*(Y;\mathbb{Z}_p^\wedge)$ has bounded torsion, then $H^*(Y;\mathbb{F}_p)$ is a finitely generated module over $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$.*

Proof. This is clear since Lemma 2.2 implies that $H^*(Y;\mathbb{F}_p)$ is integral over $\operatorname{Im}\rho$. Explicitely, let us choose homogeneous generators $w_1, \cdots, w_n$ of the graded algebra $H^*(Y;\mathbb{F}_p)$ and consider the finite set W of monomials of the form $w_1^{r_1}\cdots w_n^{r_n}$ with $0 \le r_i < p^d$. We show that the set W generates $H^*(Y;\mathbb{F}_p)$ as a module over $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$. For, consider any monomial $m = w_1^{s_1}\cdots w_n^{s_n}$ in $H^*(Y;\mathbb{F}_p)$. Writing the exponents $s_i = r_i + p^d t_i$ with $0 \le r_i < p^d$, we express $m = x^{p^d}\cdot w$ for a monomial w in W and an homogeneous element x. By Lemma 2.2, x^{p^d} lifts to an element a in $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$ and $m = \rho(a)\cdot w$. ∎

Theorem 2.4. *Let Y be a connected space with finite fundamental group. Then, the graded $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ is Noetherian if and only if the graded $\mathbb{F}_p$-algebra $H^*(Y;\mathbb{F}_p)$ is Noetherian and the torsion in $H^*(Y;\mathbb{Z}_p^\wedge)$ is bounded.*

Proof. Assume first that $H^*(Y;\mathbb{Z}_p^\wedge)$ is a Noetherian $\mathbb{Z}_p^\wedge$-algebra. By Lemma 2.1, $H^*(Y;\mathbb{F}_p)$ is a finitely generated module over $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$. Since $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$ is a Noetherian $\mathbb{F}_p$-algebra, it follows from [5, Proposition 7.2] that $H^*(Y;\mathbb{F}_p)$ is also a Noetherian $\mathbb{F}_p$-algebra. The torsion part $\mathrm{T}_pH^*(Y;\mathbb{Z}_p^\wedge)$ is an ideal of the Noetherian algebra $H^*(Y;\mathbb{Z}_p^\wedge)$, hence is finitely generated. The order of the torsion is thus bounded by the order of its generators.

Suppose now that $H^*(Y;\mathbb{F}_p)$ is a Noetherian $\mathbb{F}_p$-algebra and that the torsion in $H^*(Y;\mathbb{Z}_p^\wedge)$ is bounded. Then, by Lemma 2.3, $H^*(Y;\mathbb{F}_p)$ is a finitely generated module over $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$. As a consequence of the graded version of the so-called Eakin-Nagata Theorem, see Proposition A.1, we infer then that the graded subring $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$ of $H^*(Y;\mathbb{F}_p)$ is also Noetherian. Since $H^*(Y;\mathbb{F}_p)$ is finitely generated, Proposition 1.3 shows that $H^n(Y;\mathbb{Z}_p^\wedge)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module, hence Hausdorff, in each degree. Thus $H^*(Y;\mathbb{Z}_p^\wedge)$ is a Noetherian $\mathbb{Z}_p^\wedge$-algebra by Corollary A.3. ∎

We end this section with an example which shows that Theorem 2.4 does not hold without the condition on torsion.

Example 2.5. Aguadé, Broto, and Notbohm constructed in [2] spaces $X_k(r)$ for any odd prime p with $r|p-1$ and $k \ge 0$ satisfying : $H^*(X_k(r);\mathbb{F}_p) \cong \mathbb{F}_p[x_{2r}]\otimes E(\beta^{(k+1)}x_{2r})$ where $\beta^{(k+1)}$ denotes the Bockstein of order $k+1$. Observe that $H^*(X_k(r);\mathbb{F}_p)$ is a Noetherian $\mathbb{F}_p$-algebra. The torsion of $H^*(X_k(r);\mathbb{Z}_p^\wedge)$ is unbounded by [2, Remark 5.8]. Theorem 2.4 shows that the algebra $H^*(X_k(r);\mathbb{Z}_p^\wedge)$ is not Noetherian.

3. Cohomology with twisted coefficients

In this section we work over a ring R which is either $\mathbb{Z}_p^\wedge$ or $\mathbb{F}_p$. Let Y be a connected space whose fundamental group is a finite p-group. Let M be a $R[\pi_1 Y]$-module which is a finitely generated

R-module. We aim to show that the cohomology with twisted coefficients $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;R)$ if $H^*(Y,R)$ is Noetherian. We shall deal separately with the field of p elements and with the ring of p-adic integers.

We start with a standard Noetherianity result.

Lemma 3.1. *Let $R = \mathbb{Z}_p^\wedge$ or $\mathbb{F}_p$. Let Y be a space and let $0 \to N \to M \to Q \to 0$ be a short exact sequence of $R[\pi_1 Y]$-modules. If both $H^*(Y;N)$ and $H^*(Y;Q)$ are Noetherian modules over $H^*(Y;R)$, then so is $H^*(Y;M)$.*

Proof. The long exact sequence in cohomology induced by the short exact sequence of modules is one of $H^*(Y;R)$-modules. It exhibits $H^*(Y;M)$ as an extension of a submodule of $H^*(Y;Q)$ by a quotient of $H^*(Y;N)$. ∎

3.1. **The case of $\mathbb{F}_p$-vector spaces.** To prove the next result we follow Minh and Symonds' approach for profinite groups, [20, Lemma 1].

Theorem 3.2. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group and let M be a finite $\mathbb{F}_p[\pi_1 Y]$-module. If the graded $\mathbb{F}_p$-algebra $H^*(Y;\mathbb{F}_p)$ is Noetherian, then $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{F}_p)$.*

Proof. We use induction on $\dim_{\mathbb{F}_p} M$. Since $G = \pi_1(Y)$ is a finite p-group, the invariant submodule M^G is not trivial when M is not trivial. The induction step follows by applying Lemma 3.1 to the short exact sequence $0 \to M^G \to M \to M/M^G \to 0$. ∎

3.2. **The case of $\mathbb{Z}_p^\wedge$-modules.** We consider in this section the cohomology with twisted coefficients $H^*(Y;M)$ of a connected space Y where M is a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module which is finitely generated over $\mathbb{Z}_p^\wedge$. In a first step, let M be a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module which is finite (meaning finite as a set).

Lemma 3.3. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group. Let M be a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module which is finite. If the graded $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ is Noetherian, then $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{Z}_p^\wedge)$.*

Proof. The module M being finite, M is a finite abelian p-group. We perform an induction on the exponent e of M. When $e = 1$, the module M has the structure of an $\mathbb{F}_p$-vector space. As $H^*(Y;\mathbb{F}_p)$ is a Noetherian $\mathbb{F}_p$-algebra by Theorem 2.4, we know from Theorem 3.2 that $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{F}_p)$. The Noetherian $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ acts on $H^*(Y;M)$ through $H^*(Y;\mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$. By Lemma 2.1, $H^*(Y;\mathbb{F}_p)$ is finitely generated as a $H^*(Y;\mathbb{Z}_p^\wedge)$-module. Therefore $H^*(Y;M)$ is a Noetherian module over $H^*(Y;\mathbb{Z}_p^\wedge)$.

Let us now assume that $e > 1$ and consider the short exact sequence $0 \to M_p \to M \to Q \to 0$ where M_p is the submodule of M consisting of elements of order 1 or p. The induction step follows from Lemma 3.1. ∎

Remark 3.4. In the case of trivial coefficient modules our main tool was the universal coefficient exact sequence, but this does not exist in general for twisted coefficients. One basic counter

example is given by the module $M = \mathbb{F}_p[G]$ for a finite group G whose order is divisible by p. Then $H^*(BG; M)$ is zero in positive degrees and the universal coefficient formula does not hold.

In a second step we consider, as coefficient of the cohomology, a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module M, which is free of finite rank over $\mathbb{Z}_p^\wedge$.

Lemma 3.5. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group. Let M be a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module which is free of finite rank over $\mathbb{Z}_p^\wedge$. If the graded $\mathbb{Z}_p^\wedge$-algebra $H^*(Y; \mathbb{Z}_p^\wedge)$ is Noetherian, then $H^*(Y; M)$ is Noetherian as a module over $H^*(Y; \mathbb{Z}_p^\wedge)$.*

Proof. The short exact sequence $0 \to M \xrightarrow{\cdot p} M \to M \otimes \mathbb{F}_p \to 0$ induces in cohomology a long exact sequence of $H^*(Y; \mathbb{Z}_p^\wedge)$-modules. We see that $H^*(Y; M) \otimes \mathbb{F}_p$ is a sub-$H^*(Y; \mathbb{Z}_p^\wedge)$-module of $H^*(Y; M \otimes \mathbb{F}_p)$. Since the action of $H^*(Y; \mathbb{Z}_p^\wedge)$ on both $H^*(Y; M \otimes \mathbb{F}_p)$ and $H^*(Y; M) \otimes \mathbb{F}_p$ factors through $H^*(Y; \mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$, it follows that $H^*(Y; M) \otimes \mathbb{F}_p$ is a sub-$H^*(Y; \mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$-module of $H^*(Y; M \otimes \mathbb{F}_p)$.

This takes us back to the world of $\mathbb{F}_p$-vector spaces. We know by Theorem 3.2 that $H^*(Y; M \otimes \mathbb{F}_p)$ is a Noetherian module over $H^*(Y; \mathbb{F}_p)$, a Noetherian algebra by Theorem 2.4. As the latter is a finitely generated module over $H^*(Y; \mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$ by Lemma 2.1, we infer that $H^*(Y; M \otimes \mathbb{F}_p)$ is a Noetherian module over $H^*(Y; \mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$. Therefore $H^*(Y; M) \otimes \mathbb{F}_p$ is a Noetherian module over $H^*(Y; \mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$ as well, and since $H^*(Y; \mathbb{Z}_p^\wedge)$ acts on $H^*(Y; M) \otimes \mathbb{F}_p$ via $H^*(Y; \mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$, it is a Noetherian module over $H^*(Y; \mathbb{Z}_p^\wedge)$.

Set $A^* = H^*(Y; \mathbb{Z}_p^\wedge)$ and $N^* = H^*(Y; M)$. Both are finitely generated $\mathbb{Z}_p^\wedge$-modules in each degree by Proposition 1.3, thus also Hausdorff and complete. We then conclude by applying Proposition A.2. ∎

We now prove our main theorem.

Theorem 3.6. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group. Let M be a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module, which is finitely generated over $\mathbb{Z}_p^\wedge$. If the graded $\mathbb{Z}_p^\wedge$-algebra $H^*(Y; \mathbb{Z}_p^\wedge)$ is Noetherian, then $H^*(Y; M)$ is Noetherian as a module over $H^*(Y; \mathbb{Z}_p^\wedge)$.*

Proof. Let TM be the torsion submodule of M and consider the short exact sequence of $\mathbb{Z}_p^\wedge[\pi_1 Y]$-modules $0 \to TM \to M \to Q \to 0$. We know from Lemma 3.3 that $H^*(Y; TM)$ is a Noetherian $H^*(Y; \mathbb{Z}_p^\wedge)$-module and from Lemma 3.5 that so is $H^*(Y; Q)$. We conclude by Lemma 3.1. ∎

Remark 3.7. Our main theorem makes no assumption, except that the fundamental group be a p-group. One could try to relax it with transfer arguments, requiring a version of the transfer with twisted coefficients. However, recent work of Levi and Ragnarsson, in the context of p-local finite group theory, provides [18, Proposition 3.1] an example showing that such a transfer might not have, in general, the properties we need when the fundamental group of the space is not a p-group.

4. The case of p-compact groups and p-local finite groups

We arrive at the promised application to p-compact groups and p-local finite groups. By definition, a p-compact group is a mod p finite loop space $X = \Omega BX$, where the "classifying space" BX is p-complete, [13].

Lemma 4.1. *Let X be a p-compact group. Then the p-torsion in $H^*(BX;\mathbb{Z}_p^\wedge)$ is bounded.*

Proof. By [13, Proposition 9.9], any p-compact group admits a maximal toral p-compact subgroup S such that $\iota\colon BS \to BX$ is a monomorphism and the Euler characteristic χ of the homotopy fibre is prime to p (see [13, Proof of 2.4, page 431]). The Euler characteristic is the alternating sum of the ranks of the $\mathbb{F}_p$-homology groups. Dwyer constructed a transfer map $\tau\colon \Sigma^\infty BX \to \Sigma^\infty BS$ in [12] such that $\iota\circ\tau$ induces multiplication by χ on mod p cohomology. This is an isomorphism, so that the homotopy cofiber C of $\iota\circ\tau$ has trivial mod p cohomology.

Moreover both BX and BS have finite mod p cohomology in each degree and finite fundamental group, [13, Lemma 2.1]. Proposition 1.3 applies and in any degree, the p-adic cohomology modules of BX and BS are finitely generated over $\mathbb{Z}_p^\wedge$. The long exact sequence in cohomology associated to a cofibration then shows that the $\mathbb{Z}_p^\wedge$-modules $H^n(C;\mathbb{Z}_p^\wedge)$ are finitely generated for all n. Since $H^*(C;\mathbb{F}_p)$ is trivial, it follows from the universal coefficient exact sequence (1) that $H^*(C;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$ is trivial as well. We conclude, by the Nakayama lemma, that $H^*(C;\mathbb{Z}_p^\wedge)$ is trivial, i.e. $\iota\circ\tau$ induces also an isomorphism in cohomology with p-adic coefficients. Therefore $\iota^*\colon H^*(BX;\mathbb{Z}_p^\wedge)\to H^*(BS;\mathbb{Z}_p^\wedge)$ is a monomorphism. We are reduced to show that $H^*(BS;\mathbb{Z}_p^\wedge)$ has bounded torsion.

Now, a toral p-compact group S can be constructed, up to p-completion, as an extension of a finite p-group P and a discrete torus $H=\bigoplus\mathbb{Z}_{p^\infty}$. The fibration $BH_p^\wedge \simeq K(\bigoplus\mathbb{Z}_p^\wedge,2)\to BS\to BP$ yields a finite covering $BH_p^\wedge\to BS$ and a classical transfer argument shows then that multiplication by $|P|$ on $H^*(BS;\mathbb{Z}_p^\wedge)$ factors through the torsion free module $H^*(BH_p^\wedge;\mathbb{Z}_p^\wedge)$. ∎

Theorem 4.2. *Let X be a p-compact group, let M be a finite $\mathbb{F}_p[\pi_1BX]$-module, and let N be a $\mathbb{Z}_p^\wedge[\pi_1BX]$-module which is finitely generated over $\mathbb{Z}_p^\wedge$. Then*

1. *the $\mathbb{Z}_p^\wedge$-algebra $H^*(BX;\mathbb{Z}_p^\wedge)$ is Noetherian;*
2. *the module $H^*(BX;M)$ is Noetherian over $H^*(BX;\mathbb{F}_p)$;*
3. *the module $H^*(BX;N)$ is Noetherian over $H^*(BX;\mathbb{Z}_p^\wedge)$.*

Proof. The main theorem of Dwyer and Wilkerson, [13, Theorem 2.4], asserts that $H^*(BX;\mathbb{F}_p)$ is Noetherian. Lemma 4.1 allows us to apply Theorem 2.4 to prove the first claim. The second claim follows then from Theorem 3.2 because π_1BX is a finite p-group, [13, Lemma 2.1]. Finally Theorem 3.6 implies the third claim. ∎

Remark 4.3. Let us consider the case of $BO(n)$ at the prime 2 (the fundamental group is cyclic of order 2). E.H. Brown made in [10] an explicit computation of the integral cohomology. He actually proves that the square of any even Stiefel-Whitney class w_{2i}^2 belongs to the image of ρ and the technique we use in Lemma 2.3 is somewhat inspired by his computations. Even though the relations in the mod p cohomology of an arbitrary p-compact group (one which is not p-torsion free) make it difficult to exhibit explicit generators for the p-adic cohomology, Theorem 4.2 (1) gains qualitative control on it.

As for twisted coefficients, let $\mathbb{Z}^\vee$ be a free abelian group of rank one, endowed with the sign action of the fundamental group C_2. In [11, Theorem 1] Čadek exhibits an explicit *finite* set of

generators of $H^*(BO(n);\mathbb{Z}^\vee)$, as a module over $H^*(BO(n);\mathbb{Z})$. This is one of the few available explicit computations illustrating our results.

Broto, Levi, and Oliver defined in [9] the concept of p-local finite group. It consists of a triple $(S,\mathcal{F},\mathcal{L})$ where S is a finite p-group and, $\mathcal{F}$ and $\mathcal{L}$ are two categories whose objects are subgroups of S. The category $\mathcal{F}$ models abstract conjugacy relations among the subgroups of S, and $\mathcal{L}$ is an extension of $\mathcal{F}$ with enough information to define a classifying space $|\mathcal{L}|_p^\wedge$ which behaves like the p-completed classifying space of a finite group. In fact, to any finite group G corresponds a p-local finite group with $|\mathcal{L}|_p^\wedge \simeq (BG)_p^\wedge$, but there are also other "exotic" p-local finite groups.

Lemma 4.4. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. The p-torsion in $H^*(|\mathcal{L}|_p^\wedge;\mathbb{Z}_p^\wedge)$ is then bounded.*

Proof. In [9, p. 815] Broto, Levi and Oliver show the suspension spectrum $\Sigma^\infty\left(|\mathcal{L}|_p^\wedge\right)$ is a retract of $\Sigma^\infty BS$ following an idea due to Linckelmann and Webb (see also [23]) . Since the order of S annihilates all cohomology groups of BS, the same holds for $H^*(|\mathcal{L}|_p^\wedge;\mathbb{Z}_p^\wedge)$. ∎

Theorem 4.5. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group, let M be a finite $\mathbb{F}_p[\pi_1\left(|\mathcal{L}|_p^\wedge\right)]$-module, and let N be a $\mathbb{Z}_p^\wedge[\pi_1\left(|\mathcal{L}|_p^\wedge\right)]$-module which is finitely generated over $\mathbb{Z}_p^\wedge$. Then*

1. *the $\mathbb{Z}_p^\wedge$-algebra $H^*(|\mathcal{L}|_p^\wedge;\mathbb{Z}_p^\wedge)$ is Noetherian;*
2. *the module $H^*(|\mathcal{L}|_p^\wedge;M)$ is Noetherian over $H^*(|\mathcal{L}|_p^\wedge;\mathbb{F}_p)$;*
3. *the module $H^*(|\mathcal{L}|_p^\wedge;N)$ is Noetherian over $H^*(|\mathcal{L}|_p^\wedge;\mathbb{Z}_p^\wedge)$.*

Proof. We follow the same steps we took for p-compact groups in Theorem 4.2. The first ingredient is the stable elements theorem [9, Theorem B], which also shows that $H^*(|\mathcal{L}|_p^\wedge;\mathbb{F}_p)$ is Noetherian. We just proved that the torsion in $H^*(|\mathcal{L}|_p^\wedge;\mathbb{Z}_p^\wedge)$ is bounded. Moreover, the fundamental group of $|\mathcal{L}|_p^\wedge$ is a finite p-group by [9, Proposition 1.12]. ∎

Appendix A

This short appendix deals with Noetherianity in the graded case over the p-adics. We start however with a more general result, probably well-known to the experts: the graded Eakin-Nagata Theorem. The non-graded version can be found for example in Matsumura's book [19, Theorem 3.7(i)].

Proposition A.1. *Let A^* be a graded subring of B^*. Assume that B^* is Noetherian as a ring and finitely generated as an A^*-module. Then A^* is also a Noetherian ring.*

Proof. By [19, Theorem 13.1], B^0 is Noetherian and B^* is a finitely generated B^0-algebra. Moreover, B^0 is a finitely generated A^0-module and therefore B^* is a finitely generated A^0-algebra. Also, A^0 is Noetherian by the classical Eakin-Nagata theorem [19, Theorem 3.7(i)]. Applying [5, Proposition 7.8] to the inclusions $A^0 \subset A^* \subset B^*$ we obtain that A^* is a finitely generated A^0-algebra. Again, by [19, Theorem 13.1], A^* is a Noetherian ring. ∎

The following technical proposition allows us to deduce Noetherianity over the p-adics from the Noetherianity of the mod p reduction.

Proposition A.2. *Let A^* be a graded $\mathbb{Z}_p^\wedge$-algebra such that in each degree A^k is complete for the p-adic topology. Let N^* be a graded A^*-module such that for all k, N^k is Hausdorff for the p-adic topology. If $N^* \otimes \mathbb{F}_p$ is a Noetherian A^*-module, then so is N^*.*

Proof. Let us choose homogeneous elements $\nu_1, ..., \nu_t \in N^*$ such that $\nu_1 \otimes 1, ..., \nu_t \otimes 1$ generate $N^* \otimes \mathbb{F}_p$ as A^*-module. We claim that $\nu_1, ..., \nu_t$ generate N^* as an A^*-module. Given $n \in N^*$ we may write $n \otimes 1 = \sum a_i^0(\nu_i \otimes 1)$ for some $a_i^0 \in A^*$. Define $n_0 = \sum a_i^0 \nu_i$ and notice that $n - n_0 \in pN^*$. Thus, there exists an element $m_1 \in N^*$, homogeneous of degree $\leq \deg n$, such that $n - n_0 = pm_1$. We iterate the procedure and find elements $a_i^1 \in A^*$ such that $m_1 \otimes 1 = \sum a_i^1(\nu_i \otimes 1)$. We define $n_1 = n_0 + p\sum a_i^1 \nu_i = \sum(a_i^0 + pa_i^1)\nu_i$. In this way we construct, for any i, Cauchy sequences of coefficients $(a_i^0 + pa_i^1 + \cdots + p^k a_i^k)_k$ in A^*. By completeness this sequence converges to some $a_i \in A^*$. Since N^* is Hausdorff, the element $\sum a_i \nu_i$ is equal to n. ∎

In the following corollary, the assumption that A^* be connected, i.e. $A^0 = \mathbb{Z}_p^\wedge$, is important.

Corollary A.3. *Let A^* be a graded connected Hausdorff $\mathbb{Z}_p^\wedge$-algebra. If $A^* \otimes \mathbb{F}_p$ is a Noetherian $\mathbb{F}_p$-algebra, then A^* is a Noetherian $\mathbb{Z}_p^\wedge$-algebra.*

Proof. Since $\mathbb{Z}_p^\wedge$ is Noetherian and A^* is connected, A^* is a Noetherian $\mathbb{Z}_p^\wedge$-algebra if and only if A^* is a finitely generated $\mathbb{Z}_p^\wedge$-algebra, [19, Theorem 13.1]. Note that $A^* \otimes \mathbb{F}_p$ is also a Noetherian $\mathbb{Z}_p^\wedge$-algebra via the mod p reduction $\mathbb{Z}_p^\wedge \to \mathbb{F}_p$. Let us choose homogeneous elements $\gamma_1, ..., \gamma_n \in A^*$ such that $\gamma_1 \otimes 1, ..., \gamma_n \otimes 1$ generate $A^* \otimes \mathbb{F}_p$ as a $\mathbb{Z}_p^\wedge$-algebra. For a fixed $k \geq 0$, $A^k \otimes \mathbb{F}_p$ is generated as a $\mathbb{Z}_p^\wedge$-module by the monomials $(\gamma_1 \otimes 1)^{e_1} \cdots (\gamma_n \otimes 1)^{e_n}$ with $\sum_{i=1}^n |\gamma_i| e_i = k$. Since A^k is a Hausdorff $\mathbb{Z}_p^\wedge$-module, the proof of Proposition A.2 shows that A^k is generated by the monomials $\gamma_1^{e_1} \cdots \gamma_n^{e_n}$ with $\sum_{i=1}^n |\gamma_i| e_i = k$. This shows that A^* is generated as a $\mathbb{Z}_p^\wedge$-algebra by the elements $\gamma_1, ..., \gamma_n \in A^*$ and therefore A^* is a Noetherian $\mathbb{Z}_p^\wedge$-algebra. ∎

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